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On the geometry of tangent bundles with the metric $II + III$

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Abstract. The main purpose of this paper is to investigate some relations between the flatness or locally symmetric property on the tangent bundle TM equipped with the metric $II + III$ and the same property on the base manifold M and study geodesics by means of the adapted frame on TM.

1. Introduction. Let M be an n-dimensional manifold and TM its tangent bundle. We denote by $\Im_s^r(M)$ the set of all tensor fields of type (r,s) on M. Similarly, we denote by $\Im_s^r(TM)$ the corresponding set on TM.

Tangent bundles of differentiable manifolds are of great importance in many areas of mathematics and physics. The geometry of tangent bundles goes back to the fundamental paper [11] of Sasaki published in 1958. He uses a given Riemannian metric g on a differentiable manifold M to construct a metric $\tilde{g}$ on the tangent bundle TM of M. Today this metric is a standard notion in the differential geometry called the Sasaki metric (or the metric $I + III$). Its construction is based on a natural splitting of the tangent bundle TTM of TM into its vertical and horizontal subbundles by means of the Levi-Civita connection ∇ on (M,g). The Sasaki metric is defined by

$$\begin{aligned}\tilde{g}(X^H, Y^H) &= g_x(X,Y),\\ \tilde{g}(X^H, Y^V) &= \tilde{g}(X^V, Y^H) = 0,\\ \tilde{g}(X^V, Y^V) &= g_x(X,Y),\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$ and $x \in M$. The Sasaki metric has been extensively studied by several authors, including Yano and Davies [12], Kowalski [9], Musso and Tricerri [10], and Aso [1]. Kowalski [9] calculated the Levi-Civita connection $\tilde{\nabla}$ of the Sasaki metric on TM and its Riemannian curvature

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tensor $\tilde{R}$. With this in hand Kowalski, Aso [1], Musso and Tricerri [10] derived interesting connections between the geometric properties of (M, g) and $(TM, \tilde{g})$.

Given a Riemannian metric g on a differentiable manifold M, other well known classical Riemannian metrics on TM, which are not necessarily positive definite, are as follows.

(a) The metric II is defined by

$$\begin{aligned}&\tilde{g}(X^H, Y^H) = 0,\\ &\tilde{g}(X^H, Y^V) = \tilde{g}(X^V, Y^H) = g_x(X, Y),\\ &\tilde{g}(X^V, Y^V) = 0,\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$ and $x \in M$.

(b) The metric $I + II$ is defined by

$$\begin{aligned}&\tilde{g}(X^H, Y^H) = g_x(X, Y),\\ &\tilde{g}(X^H, Y^V) = \tilde{g}(X^V, Y^H) = g_x(X, Y),\\ &\tilde{g}(X^V, Y^V) = 0,\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$ and $x \in M$. The metric $I + II$ was introduced by Yano and Ishihara [13, pp. 147–155]. Also, they proved that the tangent bundle TM with the metric $I + II$ or the metric II has vanishing scalar curvature. In [4], Eni considered a pseudo-Riemannian metric on the tangent bundle over a Riemannian manifold, which is a generalization of the metric $I + II$, depending on a symmetric tensor field on the base manifold and on four real-valued smooth functions defined on $[0, \infty]$ and studied the conditions under which the pseudo-Riemannian manifold has constant sectional curvature.

(c) The metric $II + III$ is defined by

$$\begin{aligned}&\tilde{g}(X^H, Y^H) = 0,\\ &\tilde{g}(X^H, Y^V) = \tilde{g}(X^V, Y^H) = g_x(X, Y),\\ &\tilde{g}(X^V, Y^V) = g_x(X, Y),\end{aligned}$$

for all $X, Y \in \Im_0^1(M)$ and $x \in M$ [13, p. 138]. Hasegawa and Yamauchi [6, 7] investigated infinitesimal projective transformations on the tangent bundle TM with the metric $II + III$. In this paper, we study some properties of the curvature tensor of the metric $II + III$ and geodesics by means of the adapted frame on TM.

2. Basic formulas on the tangent bundle. Let ∇ be the Levi-Civita connection of g. Then the tangent space of TM at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇: $(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}$.

If $(x,u) \in TM$ is given, then for any vector $X \in \Im_0^1(M)$ there exists a unique vector $X^H \in H_{(x,u)}$ such that $\pi_* X^H = X$, where $\pi : TM \to M$ is the natural projection. We call X^H the *horizontal lift* of X to the point $(x,u) \in TM$. The *vertical lift* of a vector $X \in \Im_0^1(M)$ to $(x,u) \in TM$ is a vector $X^V \in V_{(x,u)}$ such that $X^V(df) = Xf$ for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e. $df(x,u) = uf$). Note that the map $X \mapsto X^H$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \to X^V$ is an isomorphism between the vector spaces M_x and $V_{(x,u)}$. Obviously each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^H + Y^V$, where $X, Y \in M_x$ are uniquely determined vectors.

If ϕ is a smooth function on M, then

$$X^H(\phi \circ \pi) = (X\phi) \circ \pi \quad \text{and} \quad X^V(\phi \circ \pi) = 0 \tag{2.1}$$

for every vector field X on M.

A system of local coordinates $\{(U; x^i, i = 1, \dots, n)\}$ in M induces on TM a system of local coordinates $\{(\pi^{-1}(U); x^i, u^i, i = 1, \dots, n)\}$. Let $X = \sum X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the horizontal lift X^H and the vertical lift X^V of X are given, in the induced coordinates, by

$$X^H = \sum X^i \frac{\partial}{\partial x^i} - \sum \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i} \tag{2.2}$$

and

$$X^V = \sum X^i \frac{\partial}{\partial u^i} \tag{2.3}$$

respectively, where Γ^i_{jk} denote the Christoffel symbols of ∇.

Now, let r be the norm of a vector u. Then, for any smooth function f from $\mathbb{R}$ to $\mathbb{R}$, we have

$$X^H_{(x,u)}(f(r^2)) = 0, \tag{2.4}$$

$$X^V_{(x,u)}(f(r^2)) = 2f'(r^2) g_x(X_x, u), \tag{2.5}$$

and in particular,

$$X^H_{(x,u)}(r^2) = 0, \tag{2.6}$$

$$X^V_{(x,u)}(r^2) = 2g_x(X_x, u). \tag{2.7}$$

Let X, Y and Z be any vector fields on M. If F_Y is the function on TM defined by $F_Y(x,u) = g_x(Y_x, u)$ for all $(x,u) \in TM$, then

$$X^H_{(x,u)}(F_Y) = g_x((\nabla_X Y)_x, u) = F_{\nabla_X Y}(x,u), \tag{2.8}$$

$$X^V_{(x,u)}(F_Y) = g_x(X,Y), \tag{2.9}$$

$$X^H_{(x,u)}(g(Y,Z)\circ\pi) = X_x(g(Y,Z)), \tag{2.10}$$

$$X^V_{(x,u)}(g(Y,Z)\circ\pi) = 0. \tag{2.11}$$

The formulas (2.4)–(2.9) follow from (2.1) and

$$X^H u^i = -\sum X^\lambda u^\mu \Gamma^i_{\lambda\mu} \quad \text{and} \quad X^V u^i = X^i,$$

and the relations (2.10) and (2.11) follow from (2.1) [2].

Suppose that $F \in \Im^1_1(M)$. Using (2.2) and (2.3), we define vector fields $(F(u))^V$ and $(F(u))^H$ on the tangent bundle TM by

$$(F(u))^V = \sum F^i_m u^m \frac{\partial}{\partial u^i},$$
$$(F(u))^H = \sum F^i_m u^m \frac{\partial}{\partial x^i} - \sum \Gamma^i_{jk} u^j F^k_m u^m \frac{\partial}{\partial u^i},$$

for any $u \in TM$.

Explicit expressions for the Lie bracket $[\,,\,]$ of the tangent bundle TM are given by Dombrowski in [3]. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{cases} [X^H, Y^H]_{(x,u)} = [X,Y]^H_{(x,u)} - (R(X_x,Y_x)u)^V, \\ [X^H, Y^V]_{(x,u)} = (\nabla_X Y)^V_{(x,u)}, \\ [X^H, Y^V]_{(x,u)} = 0, \end{cases} \tag{2.12}$$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Finally, the following Koszul formula holds:

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])$$

for all vector fields X, Y and Z on M [8, p. 160].

3. Levi-Civita connection on TM. Let (M,g) be a Riemannian manifold. The metric $II+III$ is a well defined Riemannian metric on the tangent bundle TM of M by the identities:

$$\tilde g_{(x,u)}(X^H, Y^H) = 0,$$
$$\tilde g_{(x,u)}(X^H, Y^V) = \tilde g_{(x,u)}(X^V, Y^H) = g_x(X,Y),$$
$$\tilde g_{(x,u)}(X^V, Y^V) = g_x(X,Y),$$

for all vector fields $X, Y \in \Im^1_0(TM)$ and $x \in M$.

THEOREM 3.1. *Let (M,g) be a Riemannian manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the tangent bundle $(TM,\tilde{g})$ equipped with the metric $II+III$. Then*

$$\begin{aligned}
&\text{(i)} && (\tilde{\nabla}_{X^H}Y^H)_{(x,u)} = (\nabla_X Y)^H_{(x,u)} - \tfrac{1}{2}(R_x(u,X)Y + R_x(u,Y)X)^H \\
& && \qquad + (R_x(u,X)Y)^V,\\
&\text{(ii)} && (\tilde{\nabla}_{X^H}Y^V)_{(x,u)} = -\tfrac{1}{2}(R_x(u,Y)X)^H + (\nabla_X Y)^V_{(x,u)} \\
& && \qquad + \tfrac{1}{2}(R_x(u,Y)X)^V,\\
&\text{(iii)} && (\tilde{\nabla}_{X^V}Y^H)_{(x,u)} = -\tfrac{1}{2}(R_x(u,X)Y)^H + \tfrac{1}{2}(R_x(u,X)Y)^V,\\
&\text{(iv)} && (\tilde{\nabla}_{X^V}Y^V)_{(x,u)} = 0,
\end{aligned}$$

for all vector fields $X,Y \in \Im^1_0(M)$, where R is the Riemannian curvature of ∇.

Since the horizontal and the vertical lifts to TM of vector fields on M generate the $C^\infty(TM,\mathbb{R})$-module of vector fields on TM, formulas (i)–(iv) above completely determine the Levi-Civita connection $\tilde{\nabla}$ of the metric $II+III$ on TM.

Proof. The statement is a direct consequence of usual calculations using the Koszul formula. ∎

4. Curvature tensor on TM. Let G be a tensor field of type $(1,2)$ on M. Then we define vector fields $(G(u,v))^V$ and $(G(u,v))^H$ on the tangent bundle TM by

$$\begin{aligned}
(G(u,v))^V &= \sum G^k_{ij}u^i v^j \,\frac{\partial}{\partial u^k},\\
(G(u,v))^H &= \sum G^k_{ij}u^i v^j \,\frac{\partial}{\partial x^k} - \sum \Gamma^k_{sl}u^s G^l_{ij}u^i v^j \,\frac{\partial}{\partial u^k},
\end{aligned}$$

for any $u,v \in TM$.

We now turn to the Riemannian curvature tensor $\tilde{R}$ of the tangent bundle TM equipped with the metric $II+III$. For this we need the following useful lemma:

LEMMA 4.1. *Let (M,g) be a Riemannian manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the tangent bundle $(TM,\tilde{g})$ with the metric $II+III$. Let $F: TM \to TM$ be a smooth bundle endomorphism. Then*

$$\begin{aligned}
\tilde{\nabla}_{X^V}(F(u))^V &= F(X)^V,\\
\tilde{\nabla}_{X^V}(F(u))^H &= F(X)^H - \tfrac{1}{2}(R(u,X)F(u))^H + \tfrac{1}{2}(R(u,X)F(u))^V,
\end{aligned}$$

$$\tilde{\nabla}_{X^H}(F(u))^V = ((\nabla_X F)(u))^V + \tfrac{1}{2}(R(u,F(u))X)^V - \tfrac{1}{2}(R(u,F(u))X)^H,$$
$$\tilde{\nabla}_{X^H}(F(u))^H = (R(u,X)F(u))^V + ((\nabla_X F)(u))^H - \tfrac{1}{2}(R(u,X)F(u) + R(u,F(u))X)^H,$$
$$\tilde{\nabla}_{(F(u))^V} X^V = 0,$$
$$\tilde{\nabla}_{(F(u))^V} X^H = \tfrac{1}{2}(R(u,F(u))X)^V - \tfrac{1}{2}(R(u,F(u))X)^H,$$

for any $X \in \Im_0^1(M)$ *and* $u \in TM$ *(for natural metrics, see* [5]*).*

Proof. The statement is a direct consequence of Theorem 3.1. ∎

THEOREM 4.2. *Let* (M,g) *be a Riemannian manifold and* $\tilde{R}$ *be the Riemannian curvature tensor of the tangent bundle* $(TM,\tilde{g})$ *equipped with the metric* $II + III$*. Then*

(i) $\tilde{R}_{(x,u)}(X^V,Y^V)Z^V = 0,$

(ii) $\tilde{R}_{(x,u)}(X^V,Y^V)Z^H =$
$[R(X,Y)Z + \frac{1}{4}R(u,Y)(R(u,X)Z) - \frac{1}{4}R(u,X)(R(u,Y)Z)]_x^V$
$+ [-R(X,Y)Z + \frac{1}{4}R(u,X)(R(u,Y)Z) - \frac{1}{4}R(u,Y)(R(u,X)Z)]_x^H,$

(iii) $\tilde{R}_{(x,u)}(X^H,Y^V)Z^V = [-\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)(R(u,Z)X)]_x^V$
$+ [\frac{1}{2}R(Y,Z)X - \frac{1}{4}R(u,Y)(R(u,Z)X)]_x^H,$

(iv) $\tilde{R}_{(x,u)}(X^H,Y^V)Z^H = [R(X,Y)Z + \frac{1}{2}(\nabla_x R)(u,Y)Z$
$+ \frac{1}{4}R(u,Y)(R(u,X)Z) + \frac{1}{4}R(u,Y)(R(u,Z)X)$
$+ \frac{1}{4}R(u,R(u,Y)Z)X - \frac{1}{2}R(u,X)(R(u,Y)Z)]_x^V$
$+ [\frac{1}{2}R(Y,X)Z + \frac{1}{2}R(Y,Z)X$
$- \frac{1}{2}(\nabla_X R)(u,Y)Z + \frac{1}{4}R(u,X)(R(u,Y)Z) - \frac{1}{4}R(u,Y)(R(u,X)Z)$
$- \frac{1}{4}R(u,Y)(R(u,Z)X)]_x^H,$

(v) $\tilde{R}_{(x,u)}(X^H,Y^H)Z^V =$
$[R(X,Y)Z + \frac{1}{2}(\nabla_X R)(u,Z)Y - \frac{1}{2}(\nabla_Y R)(u,Z)X$
$+ \frac{1}{4}R(u,R(u,Z)Y)X - \frac{1}{4}R(u,R(u,Z)X)Y + \frac{1}{2}R(u,Y)(R(u,Z)X)$
$- \frac{1}{2}R(u,X)(R(u,Z)Y)]_x^V + [\frac{1}{2}(\nabla_Y R)(u,Z)X - \frac{1}{2}(\nabla_X R)(u,Z)Y$
$+ \frac{1}{4}R(u,X)(R(u,Z)Y) - \frac{1}{4}R(u,Y)(R(u,Z)X)]_x^H,$

(vi) $\tilde{R}_{(x,u)}(X^H,Y^H)Z^H =$
$[(\nabla_X R)(u,Y)Z - (\nabla_Y R)(u,X)Z + \frac{1}{2}R(u,Y)(R(u,X)Z)$
$+ \frac{1}{2}R(u,Y)(R(u,Z)X) - \frac{1}{2}R(u,X)(R(u,Y)Z)$

$$
\begin{aligned}
&-\tfrac{1}{2}R(u,X)(R(u,Z)Y)+\tfrac{1}{2}R(u,R(u,Y)Z)X\\
&+\tfrac{1}{2}R(u,R(X,Y)u)Z-\tfrac{1}{2}R(u,R(u,X)Z)Y]^V_x\\
&+[R(X,Y)Z+\tfrac{1}{2}(\nabla_Y R)(u,X)Z+\tfrac{1}{2}(\nabla_Y R)(u,Z)X\\
&-\tfrac{1}{2}(\nabla_X R)(u,Y)Z-\tfrac{1}{2}(\nabla_X R)(u,Z)Y+\tfrac{1}{4}R(u,X)(R(u,Y)Z)\\
&+\tfrac{1}{4}R(u,X)(R(u,Z)Y)-\tfrac{1}{4}R(u,Y)(R(u,X)Z)\\
&-\tfrac{1}{4}R(u,Y)(R(u,Z)X)+\tfrac{1}{4}R(u,R(u,Z)Y)X\\
&+\tfrac{1}{4}R(u,R(u,X)Z)Y-\tfrac{1}{4}R(u,R(u,Y)Z)X\\
&-\tfrac{1}{4}R(u,R(u,Z)X)Y-\tfrac{1}{2}R(u,R(X,Y)u)Z]^H_x,
\end{aligned}
$$

for vectors $X, Y, Z \in \Im^1_0(M)$.

Proof. (i) The result follows directly from Theorem 3.1 and (2.12).

(iii) Let $F : TM \to TM$ be the bundle endomorphism given by

$$F : u \mapsto \tfrac{1}{2}R(u,Z)X.$$

Applying Theorem 3.1 and Lemma 4.1 we see that

$$\tilde{\nabla}_{Y^V}(F(u))^H = F(Y)^H - \tfrac{1}{2}(R(u,Y)F(u))^H + \tfrac{1}{2}(R(u,Y)F(u))^V.$$

This implies that

$$
\begin{aligned}
\tilde{R}(X^H,Y^V)Z^V &= \tilde{\nabla}_{X^H}\tilde{\nabla}_{Y^V}Z^V - \tilde{\nabla}_{Y^V}\tilde{\nabla}_{X^H}Z^V - \tilde{\nabla}_{[X^H,Y^V]}Z^V\\
&= -\tilde{\nabla}_{Y^V}((\nabla_X Z)^V + \tfrac{1}{2}(R(u,Z)X)^V - \tfrac{1}{2}(R(u,Z)X)^H) - \tilde{\nabla}_{(\nabla_X Y)^V}Z^V\\
&= -\tilde{\nabla}_{Y^V}(F(u))^V + \tilde{\nabla}_{Y^V}(F(u))^H\\
&= -F(Y)^V + F(Y)^H - \tfrac{1}{2}(R(u,Y)F(u))^H + \tfrac{1}{2}(R(u,Y)F(u))^V\\
&= [-\tfrac{1}{2}R(Y,Z)X + \tfrac{1}{4}R(u,Y)(R(u,Z)X)]^V\\
&\quad + [\tfrac{1}{2}R(Y,Z)X - \tfrac{1}{4}R(u,Y)(R(u,Z)X)]^H.
\end{aligned}
$$

By the calculations similar to those in (i) and (iii), the proofs of (ii) and (iv)–(vi) are obtained easily. ∎

We shall now compare the geometries of the manifold (M,g) and its tangent bundle $(TM,\tilde{g})$ with the metric $II + III$.

THEOREM 4.3. *Let (M,g) be a Riemannian manifold and $(TM,\tilde{g})$ be its tangent bundle with the metric $II+III$. Then TM is flat if and only if M is flat.*

Proof. From Theorem 4.2 it is clear that (M,g) is flat, then $(TM,\tilde{g})$ is also flat. Conversely, if we assume $\tilde{R} = 0$ and calculate the Riemannian curvature tensor for three horizontal vector fields at $(x,0)$ we get

$$R_x(X,Y)Z = \tilde{R}_{(x,0)}(X^H,Y^H)Z^H = 0.$$

Hence (M,g) is flat. ∎

THEOREM 4.4. *Let (M,g) be a Riemannian manifold and $(TM,\tilde{g})$ be its tangent bundle with the metric $II+III$. If $(TM,\tilde{g})$ is locally symmetric, then (M,g) is also locally symmetric.*

Proof. We begin by calculating $(\tilde{\nabla}_{W^H}\tilde{R})(X^H,Y^H)Z^H$ for all $X,Y,Z\in\Im^1_0(M)$. If we extend X,Y,Z to vectors on TM, then we can write

$$(\tilde{\nabla}_{W^H}\tilde{R})(X^H,Y^H)Z^H=\tilde{\nabla}_{W^H}(\tilde{R}(X^H,Y^H)Z^H)-\tilde{R}(\tilde{\nabla}_{W^H}X^H,Y^H)Z^H -\tilde{R}(X^H,\tilde{\nabla}_{W^H}Y^H)Z^H-\tilde{R}(X^H,Y^H)\tilde{\nabla}_{W^H}Z^H.$$

Using Theorems 3.1(i) and 4.2(vi), we deduce that

$$\begin{aligned}(4.1)\qquad &(\tilde{\nabla}_{W^H}\tilde{R})(X^H,Y^H)Z^H=\tilde{\nabla}_{W^H}[((\nabla_X R)(u,Y)Z-(\nabla_Y R)(u,X)Z\\ &+\tfrac{1}{2}R(u,Y)(R(u,X)Z)+\tfrac{1}{2}R(u,Y)(R(u,Z)X)-\tfrac{1}{2}R(u,X)(R(u,Y)Z)\\ &-\tfrac{1}{2}R(u,X)(R(u,Z)Y)+\tfrac{1}{2}R(u,R(u,Y)Z)X+\tfrac{1}{2}R(u,R(X,Y)u)Z\\ &-\tfrac{1}{2}R(u,R(u,X)Z)Y)^V_x+(R(X,Y)Z+\tfrac{1}{2}(\nabla_Y R)(u,X)Z+\tfrac{1}{2}(\nabla_Y R)(u,Z)X\\ &-\tfrac{1}{2}(\nabla_X R)(u,Y)Z-\tfrac{1}{2}(\nabla_X R)(u,Z)Y+\tfrac{1}{4}R(u,X)(R(u,Y)Z)\\ &+\tfrac{1}{4}R(u,X)(R(u,Z)Y)-\tfrac{1}{4}R(u,Y)(R(u,X)Z)-\tfrac{1}{4}R(u,Y)(R(u,Z)X)\\ &+\tfrac{1}{4}R(u,R(u,Z)Y)X+\tfrac{1}{4}R(u,R(u,X)Z)Y-\tfrac{1}{4}R(u,R(u,Y)Z)X\\ &-\tfrac{1}{4}R(u,R(u,Z)X)Y-\tfrac{1}{2}R(u,R(X,Y)u)Z)^H_x]-\tilde{R}((\nabla_W X)^H_{(x,u)},Y^H)Z^H\\ &+\tilde{R}(\tfrac{1}{2}(R_x(u,W)X+R_x(u,X)W)^H,Y^H)Z^H-\tilde{R}((R_x(u,W)X)^V,Y^H)Z^H\\ &-\tilde{R}(X^H,(\nabla_W Y)^H_{(x,u)})Z^H+\tilde{R}(X^H,\tfrac{1}{2}(R_x(u,W)Y+R_x(u,Y)W)^H)Z^H\\ &-\tilde{R}(X^H,(R_x(u,W)Y)^V)Z^H-\tilde{R}(X^H,Y^H)(\nabla_W Z)^H_{(x,u)}\\ &-\tilde{R}(X^H,Y^H)(R_x(u,W)Z)^V+\tfrac{1}{2}\tilde{R}(X^H,Y^H)(R_x(u,W)Z+R_x(u,Z)W)^H.\end{aligned}$$

If we restrict ourselves to the zero section of TM which is the base manifold M, then from (4.1) we can write

$$\begin{aligned}&[(\tilde{\nabla}_{W^H}\tilde{R})(X^H,Y^H)Z^H]_{(x,0)}\\ &\quad=\tilde{\nabla}_{W^H}[R(X,Y)Z]^H_{(x,0)}-\tilde{R}_{(x,0)}((\nabla_W X)^H,Y^H)Z^H\\ &\qquad-\tilde{R}_{(x,0)}(X^H,(\nabla_W Y)^H)Z^H-\tilde{R}_{(x,0)}(X^H,Y^H)(\nabla_W Z)^H.\end{aligned}$$

By Theorem 3.1(i), we have

$$\tilde{\nabla}_{W^H}[R(X,Y)Z]^H_{(x,0)}=[\nabla_W(R(X,Y)Z)]^H_{(x,0)},\tag{4.2}$$

$$\tilde{R}_{(x,0)}((\nabla_W X)^H,Y^H)Z^H=[R(\nabla_W X,Y)Z]^H_{(x,0)},\tag{4.3}$$

$$\tilde{R}_{(x,0)}(X^H,(\nabla_W Y)^H)Z^H=[R(X,\nabla_W Y)Z]^H_{(x,0)},\tag{4.4}$$

$$\tilde{R}_{(x,0)}(X^H,Y^H)(\nabla_W Z)^H=[R(X,Y)\nabla_W Z]^H_{(x,0)}.\tag{4.5}$$

By substituting (4.2)–(4.5) to the above formula, we conclude that

$$[(\tilde{\nabla}_{W^H}\tilde{R})(X^H,Y^H)Z^H]_{(x,0)} = [\nabla_W(R(X,Y)Z)]^H_{(x,0)} - [R(\nabla_W X,Y)Z]^H_{(x,0)} - [R(X,\nabla_W Y)Z]^H_{(x,0)} - [R(X,Y)\nabla_W Z]^H_{(x,0)}.$$

It follows that

$$(4.6)\qquad [(\tilde{\nabla}_{W^H}\tilde{R})(X^H,Y^H)Z^H]_{(x,0)} = [(\nabla_W R)(X,Y)Z]^H_{(x,0)}$$

for all $X,Y,Z,W \in \Im^1_0(M)$. Hence, if we suppose that $(TM,\tilde{g})$ is locally symmetric, i.e. $\tilde{\nabla}\tilde{R}=0$ identically, then by (4.6), $\nabla R = 0$ identically. ■

5. Geodesics on the tangent bundle with the metric $II + III$**.** Let (M,g) be a Riemannian manifold, ∇ the Riemannian connection of g, and Γ^a_{ji} the coefficients of ∇, i.e. $\nabla_{\partial_j}\partial_i = \Gamma^a_{ji}\partial_a$ with respect to the natural frame $\{\partial_h\}$. The curvature tensor R of ∇ has components R^h_{kji}. The indices $i,j,h,\ldots$ range in $\{1,\ldots,n\}$ while the indices $\alpha,\beta,\lambda,\ldots$ range in $\{1,\ldots,n;n+1,\ldots,2n\}$. We put $\bar{i} = n+i$. Summation over repeated indices is always implied.

With the Riemannian connection ∇ given on M, we can introduce on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM a frame field which is very useful in our computation. In each local chart $U(x^h)$ of M, we put

$$X_{(j)} = \frac{\partial}{\partial x^j} = \delta^h_j \frac{\partial}{\partial x^h} \in \Im^1_0(M).$$

We now define $2n$ local vector fields $X^H_{(j)}$ and $X^V_{(j)}$ which form a basis of the tangent space $T_{\tilde{P}}TM$ at each point $\tilde{P} \in \pi^{-1}(P)$. Their components are given respectively by

$$X^H_{(j)} = \delta^h_j\partial_h - y^s\Gamma^h_{sj}\partial_{\bar{h}}, \qquad X^V_{(j)} = \delta^h_j\partial_{\bar{h}}$$

with respect to the natural frame $\{\partial/\partial x^H\} = \{\partial/\partial x^h, \partial/\partial x^h\}$ on TM, where δ^j_i is the Kronecker delta and $y^s = x^{\bar{s}}$. These $2n$ vector fields are linearly independent and generate, respectively, the horizontal distribution of ∇ and the vertical distribution of TM. We call the set $\{X^H_{(j)}, X^V_{(j)}\}$ the *frame adapted to the affine connection* ∇ in $\pi^{-1}(U) \subset TM$. On putting $e_{(j)} = X^H_{(j)}$, $e_{(\bar{j})} = X^V_{(j)}$, we write the adapted frame as $\{e_\beta\} = \{e_{(j)}, e_{(\bar{j})}\}$.

We now consider local 1-forms ω^α defined by

$$\omega^\alpha = \tilde{A}^\alpha{}_B\, dx^B$$

in $\pi^{-1}(U)$, where

$$\tilde{A}^\alpha{}_B = \begin{pmatrix} \tilde{A}^h{}_j & \tilde{A}^h{}_{\bar{j}} \\ \tilde{A}^{\bar{h}}{}_j & \tilde{A}^{\bar{h}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ y^s\Gamma^h_{sj} & \delta^h_j \end{pmatrix}$$

is the inverse matrix of the matrix

$$A_\beta{}^A = \begin{pmatrix} A_j{}^h & A_{\bar{j}}{}^h \\ A_j{}^{\bar{h}} & A_{\bar{j}}{}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} \delta_j^h & 0 \\ -y^s \Gamma_{sj}^h & \delta_j^h \end{pmatrix}$$

of frame changes $e_\beta = A_\beta{}^A \, \partial_A$. These $2n$ 1-forms ω^α are linearly independent on TM. We call the set $\{\omega^\alpha\}$ the *dual adapted co-frame*.

For various types of indices, we have

$$\begin{cases} e_j = A_j{}^A \, \partial_A = \partial_j - y^s \Gamma_{sj}^h \partial_{\bar{h}}, \\ e_{\bar{j}} = A_{\bar{j}}{}^A \, \partial_A = \partial_{\bar{j}}, \end{cases}$$

and

$$\begin{cases} \omega^j = \tilde{A}^j{}_B \, dx^B = dx^j, \\ \omega^{\bar{j}} = \tilde{A}^{\bar{j}}{}_B \, dx^B = \delta y^h, \end{cases} \tag{5.1}$$

where $\delta y^h = dy^h + y^b \Gamma_{ba}^h dx^a$.

Let $\tilde{\Gamma}_{\alpha\beta}^\gamma$ denote the components of the Riemannian connection $\tilde{\nabla}$ determined by the metric $II + III$. If we take e_j and $e_{\bar{j}}$ instead of X^H and X^V in Theorem 3.1, then we get

$$\begin{cases} \tilde{\Gamma}_{ji}^h = \Gamma_{ji}^h - \frac{1}{2} y^b (R_{bji}^h + R_{bij}^h), \quad \tilde{\Gamma}_{ji}^{\bar{h}} = y^b R_{bji}^h, \quad \tilde{\Gamma}_{\bar{j}\bar{i}}^{\bar{h}} = 0, \\ \tilde{\Gamma}_{\bar{j}\bar{i}}^h = 0, \quad \tilde{\Gamma}_{j\bar{i}}^{\bar{h}} = \Gamma_{ji}^h + \frac{1}{2} y^b \Gamma_{bij}^h, \quad \tilde{\Gamma}_{j\bar{i}}^h = -\frac{1}{2} y^b R_{bij}^h, \\ \tilde{\Gamma}_{\bar{j}i}^h = \frac{1}{2} y^b R_{bji}^h, \quad \tilde{\Gamma}_{\bar{j}i}^h = -\frac{1}{2} y^b R_{bji}^h, \end{cases} \tag{5.2}$$

with respect to the adapted frame, where Γ_{ji}^h denote the Levi-Civita connection components constructed with g on M with respect to the natural frame $\{\partial_i\}$ (see also [6, 7]).

Let $\tilde{\gamma} = \tilde{\gamma}(t)$ be a curve on TM and suppose that $\tilde{\gamma}$ is locally expressed by $x^R = x^R(t)$, i.e. $x^r = x^r(t)$, $y^r = X^r(t)$ with respect to the natural frame $\{\partial/\partial x^I\} = \{\partial/\partial x^i, \partial/\partial x^{\bar{i}}\}$, t being the arc length of $\tilde{\gamma}$. Then the curve $\gamma = \pi \circ \tilde{\gamma}$ on M is called the projection of the curve $\tilde{\gamma}$ and denoted by $\pi\tilde{\gamma}$; it is expressed locally by $x^r = x^r(t)$.

Let ∇ be a Riemannian connection on M. Then a curve $\tilde{\gamma}$ is, by definition, a geodesic on TM with respect to $\tilde{\nabla}$ if and only if it satisfies the differential equations

$$\frac{\delta^2 x^R}{dt^2} = \frac{d^2 x^R}{dt^2} + \tilde{\Gamma}_{CB}^R \frac{dx^C}{dt} \frac{dx^B}{dt} = 0. \tag{5.3}$$

We find it more convenient to refer equations (5.3) to the adapted frame. Using (5.1), we now put

$$\frac{\omega^r}{dt} = \frac{dx^r}{dt}, \qquad \frac{\omega^{\bar{r}}}{dt} = \frac{\delta y^r}{dt} \tag{5.4}$$

along a curve $\tilde{\gamma}$. The equation (5.3) can be transformed, using (5.4), into

$$\frac{d}{dt}\left(\frac{\omega^{\varepsilon}}{dt}\right)+\tilde{\Gamma}^{\varepsilon}_{\alpha\beta}\frac{\omega^{\alpha}}{dt}\frac{\omega^{\beta}}{dt}=0 \tag{5.5}$$

with respect to the adapted frame.

By means of (5.2), (5.5) reduces to

$$\frac{d^2x^r}{dt^2}+\Gamma^r_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}-\frac{1}{2}y^b(R^r_{bji}+R^r_{bij})\frac{dx^j}{dt}\frac{dx^i}{dt}-y^bR^r_{bji}\frac{dx^i}{dt}\frac{\delta y^j}{dt}=0, \tag{5.6}$$

$$\frac{d}{dt}\left(\frac{\delta y^r}{dt}\right)+\Gamma^r_{ij}\frac{dx^i}{dt}\frac{\delta y^j}{dt}+y^bR^r_{bji}\frac{dx^j}{dt}\frac{dx^i}{dt}+y^bR^r_{bji}\frac{dx^i}{dt}\frac{\delta y^j}{dt}=0. \tag{5.7}$$

Let now $\tilde{\gamma}$ be a geodesic of $\tilde{\nabla}$. If $\tilde{\gamma}$ lies on a fibre $\pi^{-1}(P)=T(P)$, $P=P(x^h)$ given by $x^h=c^h=\text{const}$, then (5.7) reduces to

$$\frac{d^2y^r}{dt^2}=0\quad\left(\frac{dx^h}{dt}=0\right),$$

from which we have

$$x^{\bar{r}}=a^{\bar{r}}t+b^{\bar{r}},\qquad \bar{r}=n+1,\dots,2n$$

$a^{\bar{r}}$ and $b^{\bar{r}}$ being constant. Hence we have

THEOREM 5.1. *If a geodesic $\tilde{\gamma}$ lies on a fibre of TM with respect to the metric $II+III$, then the geodesic is expressed by linear equations*

$$\begin{cases} x^h=c^h,\\ x^{\bar{h}}=a^{\bar{h}}t+b^{\bar{h}},\end{cases}$$

with respect to the natural frame, where c^h, $a^{\bar{h}}$ and $b^{\bar{h}}$ are constant.

Next, let γ be a curve on M expressed locally by $x^h=x^h(t)$ and $X^h(t)$ be a vector field along γ. Then, on the tangent bundle TM over the Riemannian manifold M, we define a curve γ^H by

$$\begin{cases} x^h=x^h(t),\\ x^{\bar{h}}=X^h(t).\end{cases}$$

If the curve γ^H satisfies at all points the relation

$$\frac{\delta X^h}{dt}=0,$$

i.e. $X^h(t)$ is a parallel vector field along γ, then the curve γ^H is said to be a *horizontal lift* of γ. From (5.6) and (5.7), we easily deduce

THEOREM 5.2. *The horizontal lift of a geodesic on M need not be a geodesic on TM with respect to the connection $\tilde{\nabla}$.*

The natural lift of the curve γ having the local expression $x^h = x^h(t)$ is defined by

$$\tilde{\gamma} : \begin{cases} x^h = x^h(t), \\ x^{\bar{h}} = \dfrac{dx^h}{dt}(t). \end{cases}$$

For the natural lift of the curve γ, from (5.6) and (5.7), we obtain

$$\frac{\delta^2 x^r}{dt^2} - R^r_{bji} \frac{dx^i}{dt} \frac{\delta^2 x^j}{dt^2} \frac{dx^b}{dt} = 0, \tag{5.8}$$

$$\frac{\delta^3 x^r}{dt^3} + R^r_{bji} \frac{dx^i}{dt} \frac{\delta^2 x^j}{dt^2} \frac{dx^b}{dt} = 0, \tag{5.9}$$

which shows that the natural lift of the curve γ is a geodesic if and only if the equations (5.8) and (5.9) hold.

Let now γ be a geodesic on M. Then

$$\frac{\delta^2 x^r}{dt^2} = \frac{d^2 x^r}{dt^2} + \Gamma^r_{ji} \frac{dx^j}{dt} \frac{dx^i}{dt} = 0. \tag{5.10}$$

Substituting (5.10) into (5.8) and (5.9), we have

THEOREM 5.3. *The natural lift of any geodesic on M is a geodesic on TM with the metric $II + III$.*

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