ANNALES
POLONICI MATHEMATICI
97.1 (2010)

Uniqueness of entire functions and fixed points

by Xiao-Guang Qi and Lian-Zhong Yang (Jinan)

Abstract. Let f and g be entire functions, n, k and m be positive integers, and λ, μ be complex numbers with $|\lambda| + |\mu| \neq 0$. We prove that $(f^n(z)(\lambda f^m(z) + \mu))^{(k)}$ must have infinitely many fixed points if $n \geq k+2$; furthermore, if $(f^n(z)(\lambda f^m(z) + \mu))^{(k)}$ and $(g^n(z)(\lambda g^m(z)+\mu))^{(k)}$ have the same fixed points with the same multiplicities, then either $f \equiv cg$ for a constant c, or f and g assume certain forms provided that $n > 2k + m^* + 4$, where m^* is an integer that depends only on λ.

1. Introduction and main results. In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations of value distribution theory [10]: $T(r,f), m(r,f), N(r,f), \overline{N}(r,f)$, etc. We denote by $S(r,f)$ any function that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measure.

We say that two meromorphic functions f and g *share a small function* $a(z)$ *IM* (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If f and g have the same zeros with the same multiplicities, then we say that f and g *share* $a(z)$ *CM* (counting multiplicities).

Let p be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, 1/(f-a))$ the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$. We say that a finite value z_0 is a *fixed point* of f if $f(z_0) = z_0$.

In 1959, Hayman [4] proved the following result.

Theorem A. *Let f be a transcendental entire function, and $n \geq 1$ be a positive integer. Then $f^n f' - 1$ has infinitely many zeros.*

Wang [8] extended Theorem A, and proved the next result.

Theorem B. *Let f be a transcendental meromorphic function, and n, k be positive integers with $n \geq k+1$. Then $(f^n)^{(k)} - 1$ has infinitely many zeros.*

2010 *Mathematics Subject Classification*: 30D35, 30D20.
Key words and phrases: uniqueness, fixed point, sharing value, entire solutions.

DOI: 10.4064/ap97-1-7

It is of interest to establish uniqueness theorems corresponding to the above results. Fang and Hua [2], Yang and Hua [9] obtained the following results.

THEOREM C. *Let f and g be nonconstant entire functions, and $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

THEOREM D. *Let f and g be nonconstant entire functions, and n and k be positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = -1$, or $f = tg$ for a constant t such that $t^n = 1$.*

In [1], Fang also obtained the following results.

THEOREM E. *Let f be a transcendental entire function, and n and k be positive integers with $n \geq k+2$. Then $(f^n(f-1))^{(k)} - 1$ has infinitely many zeros.*

THEOREM F. *Let f and g be nonconstant entire functions, and n, k be positive integers with $n \geq 2k + 8$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 CM, then $f = g$.*

Corresponding to the above results, some authors considered uniqueness of entire functions that have fixed points (see Fang and Qiu [3], Lin and Yi [7]). In the present paper, we consider the existence of fixed points of $(f^n(\lambda f^m + \mu))^{(k)}$ and the corresponding uniqueness theorems, where n, m and k are positive integers, and we obtain the following results which generalize the above theorems.

THEOREM 1. *Let $f(z)$ be a transcendental entire function, n, k and m be positive integers, and λ, μ be complex numbers satisfying $|\lambda| + |\mu| \neq 0$. Then*

$$(n-k-1)T(r,f) \leq \overline{N}\left(r, \frac{1}{(f^n(z)(\lambda f^m(z) + \mu))^{(k)} - z}\right) + S(r,f).$$

COROLLARY. *Let $f(z)$ be a transcendental entire function, n, k and m be positive integers with $n \geq k+2$, and λ, μ be complex numbers such that $|\lambda| + |\mu| \neq 0$. Then $(f^n(z)(\lambda f^m(z) + \mu))^{(k)}$ has infinitely many fixed points.*

REMARK 1. It is easy to see that a polynomial $P(z)$ with degree $n \geq 1$ has exactly n fixed points (counting multiplicities), but a transcendental entire function may have no fixed points. For example, the function $f = e^{\alpha(z)} + z$ has no fixed points, where $\alpha(z)$ is an entire function.

We define an integer m^*, corresponding to the differential polynomials $(f^n(z)(\lambda f^m(z)+\mu))^{(k)}$ and $(g^n(z)(\lambda g^m(z)+\mu))^{(k)}$ in Theorem 2, by

$$m^* = \begin{cases} m, & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$$

THEOREM 2. *Let $f(z)$ and $g(z)$ be transcendental entire functions, n, m and k be positive integers, and λ and μ be constants that satisfy $|\lambda|+|\mu| \neq 0$. Suppose that $n > 2k+m^*+4$. If $(f^n(z)(\lambda f^m(z)+\mu))^{(k)}$ and $(g^n(z)(\lambda g^m(z) + \mu))^{(k)}$ share z CM, then the following conclusions hold:*

(i) *If $\lambda\mu \neq 0$, then $f^d(z) \equiv g^d(z)$, where $d = \mathrm{GCD}(n, m)$; in particular, $f(z) \equiv g(z)$ when $d = 1$.*

(ii) *If $\lambda\mu = 0$, then either $f = cg$ for a constant c that satisfies $c^{n+m^*} = 1$, or $k = 1$ and $f(z) = b_1 e^{bz^2}$, $g(z) = b_2 e^{-bz^2}$ for some constants b_1, b_2 and b that satisfy $4(\lambda+\mu)^2(b_1b_2)^{n+m^*}((n+m^*)b)^2 = -1$.*

2. Some lemmas

LEMMA 1 ([10]). *Let f be a nonconstant meromorphic function, and a_0, $a_1, \dots, a_n$ be finite complex numbers such that $a_n \neq 0$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

LEMMA 2 ([6]). *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, 1/f^{(k)}) \le T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 1/f) + S(r, f), \tag{2.1}$$

$$N_p(r, 1/f^{(k)}) \le k\overline{N}(r, f) + N_{p+k}(r, 1/f) + S(r, f). \tag{2.2}$$

LEMMA 3 ([11]). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.3}$$

where F and G are nonconstant meromorphic functions. If F and G share 1 CM and $H \not\equiv 0$, then

$$\begin{aligned} T(r, F) + T(r, G) \le 2(&N_2(r, 1/F) + N_2(r, 1/G) \\ &+ N_2(r, F) + N_2(r, G)) + S(r, F) + S(r, G). \end{aligned} \tag{2.4}$$

LEMMA 4 ([10]). *Let $f(z)$ be a nonconstant meromorphic function, and $a_1(z)$, $a_2(z)$ and $a_3(z)$ be distinct small functions of $f(z)$. Then*

$$T(r, f) < \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

LEMMA 5. *Let f and g be nonconstant entire functions, n, m and k be positive integers, and let*

$$F = (f^n(z)(\lambda f^m(z) + \mu))^{(k)}, \qquad G = (g^n(z)(\lambda g^m(z) + \mu))^{(k)},$$

where $\lambda\mu \neq 0$. *If there exist nonzero constants* a_1 *and* a_2 *such that*

$$\overline{N}\left(r, \frac{1}{F-a_1}\right) = \overline{N}\left(r, \frac{1}{G}\right), \qquad \overline{N}\left(r, \frac{1}{G-a_2}\right) = \overline{N}\left(r, \frac{1}{F}\right),$$

then $n \leq 2k+2+m$.

Proof. By the second fundamental theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-a_1}\right) + S(r,F) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r,F) \\ &\leq N_1\left(r, \frac{1}{F}\right) + N_1\left(r, \frac{1}{G}\right) + S(r,F). \end{aligned} \tag{2.5}$$

From 2.5, Lemma 1 and Lemma 2, we obtain

$$\begin{aligned} T(r,F) \leq\ & T(r,F) - T(r, f^n(z)(\lambda f^m(z)+\mu)) \\ &+ N_{k+1}\left(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\right) \\ &+ N_{k+1}\left(r, \frac{1}{g^n(z)(\lambda g^m(z)+\mu)}\right) + S(r,f) + S(r,g). \end{aligned}$$

Hence

$$\begin{aligned} (n+m)T(r,f) &\leq N_{k+1}\left(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\right) \\ &\quad + N_{k+1}\left(r, \frac{1}{g^n(z)(\lambda g^m(z)+\mu)}\right) + S(r,f) + S(r,g) \\ &\leq (k+1)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) \\ &\quad + m(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \end{aligned} \tag{2.6}$$

By a similar reasoning, we have

$$\begin{aligned} (n+m)T(r,g) \leq\ & (k+1)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) \\ &+ m(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \end{aligned} \tag{2.7}$$

From 2.6 and 2.7, we have

$$(n-2k-2-m)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which implies that $n \leq 2k+2+m$. Lemma 5 is thus proved.

Lemma 6. *Suppose that* F *and* G *are given by Lemma* 5. *If* $n > 2k+m$ *and* $F = G$, *then* $f^d(z) \equiv g^d(z)$, *where* $d = \mathrm{GCD}(n,m)$.

Proof. From $F = G$, we get

$$(f^n(z)(\lambda f^m(z)+\mu))^{(k)} = (g^n(z)(\lambda g^m(z)+\mu))^{(k)}.$$

By integration, we have

$$(f^n(z)(\lambda f^m(z)+\mu))^{(k-1)} = (g^n(z)(\lambda g^m(z)+\mu))^{(k-1)} + a_{k-1},$$

where a_{k-1} is a constant. If $a_{k-1} \neq 0$, Lemma 5 yields $n \leq 2k+m$, which is a contradiction. Hence $a_{k-1}=0$. Repeating the same process $k-1$ times, we obtain

$$f^n(z)(\lambda f^m(z)+\mu) = g^n(z)(\lambda g^m(z)+\mu). \tag{2.8}$$

Now we suppose that $h = f/g$. By 2.8, we get

$$\lambda g^m(h^{n+m}-1) = \mu(1-h^n).$$

When $h^{n+m}=1$, the above equation yields $h^n=1$, that is, $f^n=g^n$ and $f^m=g^m$, so $f^d(z)\equiv g^d(z)$, where $d=\mathrm{GCD}(n,m)$. When $h^{n+m}\not\equiv 1$, by substituting $f=gh$ into 2.8, we have

$$g^m = -\frac{\mu}{\lambda}\cdot\frac{1+h+\cdots+h^{n-1}}{1+h+\cdots+h^{n+m-1}} = -\frac{\mu}{\lambda}\cdot\frac{\prod_{i=1}^{n-1}(h-\zeta_i)}{\prod_{i=1}^{n+m-1}(h-\eta_i)},$$

where $\zeta_i \neq 1$, $\zeta_i^n = 1$, and $\eta_i \neq 1$, $\eta_i^{n+m}=1$. Since g is an entire function, we know that every zero of $h^{n+m}-1$ has to be a zero of h^n-1. Noting that $n > 2k+m$, we deduce that h is a constant. Hence, g is a constant, which is a contradiction. Therefore, $f^d(z)\equiv g^d(z)$, where $d=\mathrm{GCD}(n,m)$.

Lemma 7. *Let f and g be transcendental entire functions, n,m and k be positive integers, and $F=(f^n(z)(\lambda f^m(z)+\mu))^{(k)}$, $G=(g^n(z)(\lambda g^m(z)+\mu))^{(k)}$, where $\lambda\mu\neq 0$. If $FG=z^2$, then $n\leq k+2$.*

Proof. Suppose $n>k+2$. From $FG=z^2$, we have

$$(f^n(z)(\lambda f^m(z)+\mu))^{(k)}(g^n(z)(\lambda g^m(z)+\mu))^{(k)} = z^2. \tag{2.9}$$

Suppose that z_0 is a p-fold zero of f. Since $\lambda\mu\neq 0$, we know that z_0 must be an $(np-k)$-fold zero of $(f^n(z)(\lambda f^m(z)+\mu))^{(k)}$. As g is an entire function and $n>k+2$, it follows from 2.9 that z_0 is a zero of z^2 of order at least 3, which is impossible. Thus f has no zeros. Let $f(z)=e^{\beta(z)}$, where $\beta(z)$ is a nonconstant entire function. Then

$$(f^{m+n})^{(k)} = (e^{(m+n)\beta})^{(k)} = P_1(\beta',\beta'',\ldots,\beta^{(k)})e^{(m+n)\beta}, \tag{2.10}$$

$$(f^n)^{(k)} = (e^{n\beta})^{(k)} = P_2(\beta',\beta'',\ldots,\beta^{(k)})e^{n\beta}, \tag{2.11}$$

where P_1 and P_2 are differential polynomials in $\beta',\beta'',\ldots,\beta^{(k)}$. It is easy to see that $P_1\not\equiv 0$, $P_2\not\equiv 0$, $T(r,P_1)=S(r,f)$ and $T(r,P_2)=S(r,f)$. From 2.9, 2.10 and 2.11 we obtain

$$N\left(r,\frac{1}{\lambda P_1 e^{m\beta}+\mu P_2}\right) = S(r,f).$$

By Lemmas 4 and 1, we have

$$\begin{aligned} mT(r,f) &= T(r, P_1e^{m\beta}) + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{\lambda P_1 e^{m\beta} + \mu P_2}\right) + \overline{N}\left(r, \frac{1}{P_1e^{m\beta}}\right) + S(r,f) \\ &= S(r,f), \end{aligned}$$

which is a contradiction. Thus $n \le k+2$. This completes the proof of Lemma 7.

Lemma 8. *Let f and g be nonconstant entire functions, n, m and k be positive integers, and $F = (f^n(z)(\lambda f^m(z)+\mu))^{(k)}$, $G = (g^n(z)(\lambda g^m(z)+\mu))^{(k)}$, where $|\lambda| + |\mu| \ne 0$, and $\lambda\mu = 0$. If there exist nonzero constants a_1 and a_2 such that $\overline{N}(r, 1/(F-a_1)) = \overline{N}(r, 1/G)$ and $\overline{N}(r, 1/(G-a_2)) = \overline{N}(r, 1/F)$, then $n \le 2k+2-m^*$.*

Proof. If $\lambda \ne 0$, by the same arguments as in the proof of Lemma 5, we have

$$(n-2k-2+m)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g),$$

which implies that $n \le 2k+2-m^*$.

If $\lambda = 0$, a similar argument gives

$$nT(r,f) \le (k+1)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + S(r,f) + S(r,g), \tag{2.12}$$

$$nT(r,g) \le (k+1)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + S(r,f) + S(r,g). \tag{2.13}$$

Hence

$$(n-2k-2)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g),$$

and we deduce that $n \le 2k+2$.

By the arguments much similar to the proof of Lemma 6, we have the following lemma.

Lemma 9. *Suppose that F and G are given by Lemma 8. If $n > 2k - m^*$ and $F = G$, then $f = cg$ for a constant c that satisfies $c^{n+m^*} = 1$.*

Proof. Suppose that $\lambda \ne 0$. By using the same arguments as in the proof of Lemma 6, we have $\lambda f^{m+n} = \lambda g^{m+n}$ if $n > 2k - m$. If $\lambda = 0$, then we have $\mu f^n = \mu g^n$. Thus we obtain the conclusion of Lemma 9.

Lemma 10 ([5]). *Suppose that f is a nonconstant meromorphic function, and $k \ge 2$ is an integer. If*

$$N(r,f) + N(r,1/f) + N(r,1/f^{(k)}) = S(r, f'/f),$$

then $f = e^{az+b}$, where $a \ne 0$, b are constants.

3. Proofs of theorems

Proof of Theorem 1. Set $F = f^n(z)(\lambda f^m(z) + \mu)$. By Lemma 4, we have

$$T(r, F^{(k)}) \le \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f). \tag{3.1}$$

Case 1: $\lambda \neq 0$. By (3.1) and Lemma 2 with $p = 1$, we obtain

$$\begin{aligned} T(r, F^{(k)}) &\le N_1\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f) \\ &\le T(r, F^{(k)}) - T(r, F) + N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f), \end{aligned} \tag{3.2}$$

and so

$$\begin{aligned} T(r, F) &\le N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f) \\ &\le N_{k+1}\left(r, \frac{1}{f^n}\right) + N_{k+1}\left(r, \frac{1}{\lambda f^m(z) + \mu}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f) \\ &\le (k + 1 + m)T(r, f) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f). \end{aligned}$$

Since $T(r, F) = (m + n)T(r, f) + S(r, f)$, we have

$$(n - k - 1)T(r, f) \le \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f).$$

Hence, the conclusion of Theorem 1 holds in this case.

Case 2: $\lambda = 0$. Since $|\lambda| + |\mu| \neq 0$, we know that $\mu \neq 0$. By using the same arguments as above, we have

$$\begin{aligned} T(r, F) &\le N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f) \\ &\le N_{k+1}\left(r, \frac{1}{\mu f^n}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f) \\ &\le (k + 1)T(r, f) + \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f). \end{aligned}$$

Noting that $T(r, F) = nT(r, f) + S(r, f)$, we obtain

$$(n - k - 1)T(r, f) \le \overline{N}\left(r, \frac{1}{F^{(k)} - z}\right) + S(r, f).$$

Theorem 1 follows.

Proof of Theorem 2. We consider the following two cases.

(i) $\lambda\mu \neq 0$. Let

$$F = \frac{(f^n(z)(\lambda f^m(z) + \mu))^{(k)}}{z}, \qquad G = \frac{(g^n(z)(\lambda g^m(z) + \mu))^{(k)}}{z}. \tag{3.3}$$

Then F and G are transcendental meromorphic functions that share 1 CM. Let H be given by (2.3). If $H \not\equiv 0$, by Lemma 3 we know that (2.4) holds. From Lemma 2, we have

$$\begin{aligned}(3.4)\qquad N_2(r,1/F) &\le N_2\Big(r, \frac{1}{(f^n(z)(\lambda f^m(z)+\mu))^{(k)}}\Big) + S(r,f)\\ &\le T(r,(f^n(z)(\lambda f^m(z)+\mu))^{(k)}) - (m+n)T(r,f)\\ &\quad + N_{k+2}\Big(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\Big) + S(r,f)\\ &= T(r,F) - (m+n)T(r,f)\\ &\quad + N_{k+2}\Big(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\Big) + S(r,f).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(3.5)\qquad N_2(r,1/G) &\le T(r,G) - (m+n)T(r,g)\\ &\quad + N_{k+2}\Big(r, \frac{1}{g^n(z)(\lambda g^m(z)+\mu)}\Big) + S(r,g).\end{aligned}$$

From (3.4) and (3.5), we obtain

$$(3.6)\qquad N_2(r,1/F) \le N_{k+2}\Big(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\Big) + S(r,f),$$

$$(3.7)\qquad N_2(r,1/G) \le N_{k+2}\Big(r, \frac{1}{g^n(z)(\lambda g^m(z)+\mu)}\Big) + S(r,g).$$

Again, from (3.4) and (3.5), we have

$$\begin{aligned}(m+n)(T(r,f)+T(r,g)) &\le T(r,F)+T(r,G) - N_2(r,1/F) - N_2(r,1/G)\\ &\quad + N_{k+2}\Big(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\Big) + N_{k+2}\Big(r, \frac{1}{g^n(z)(\lambda g^m(z)+\mu)}\Big)\\ &\quad + S(r,f) + S(r,g).\end{aligned}$$

Combining (3.6), (3.7) and Lemma 3, we get

$$\begin{aligned}(3.8)\qquad (m+n)(T(r,f)+T(r,g)) &\le 2N_{k+2}\Big(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\Big)\\ &\quad + 2N_{k+2}\Big(r, \frac{1}{g^n(z)(\lambda g^m(z)+\mu)}\Big) + S(r,f) + S(r,g)\\ &\le (2k+4)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + 2N_{k+2}\Big(r, \frac{1}{\lambda f^m(z)+\mu}\Big)\\ &\quad + 2N_{k+2}\Big(r, \frac{1}{\lambda g^m(z)+\mu}\Big) + S(r,f) + S(r,g).\end{aligned}$$

Thus, we deduce that

$$(m+n-2k-4-2m)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g),$$

which contradicts the assumption that $n > 2k+4+m$. Therefore $H \equiv 0$. Integrating twice, we deduce from (2.3) that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.9}$$

where A ($\neq 0$) and B are constants. From (3.9) we have

$$F = \frac{(B+1)G+(A-B-1)}{BG+(A-B)}, \qquad G = \frac{(B-A)F+(A-B-1)}{BF-(B+1)}. \tag{3.10}$$

We consider the following three cases.

Case 1: $B \neq 0, -1$. From (3.10) we have $\overline{N}\big(r, \frac{1}{F-\frac{B+1}{B}}\big) = \overline{N}(r,G)$. From the second fundamental theorem,

$$\begin{aligned} T(r,F) &\le \overline{N}(r,1/F) + \overline{N}\left(r, \frac{1}{F-\frac{B+1}{B}}\right) + S(r,F) \\ &= \overline{N}(r,1/F) + \overline{N}(r,G) + S(r,F) \le \overline{N}(r,1/F) + S(r,F). \end{aligned} \tag{3.11}$$

By (3.11) and the same reasoning as in the proof of (3.4), we obtain

$$\begin{aligned} T(r,F) &\le N_1(r,1/F) + S(r,f) \\ &\le T(r,F) - (m+n)T(r,f) + N_{k+1}\left(r, \frac{1}{f^n(z)(\lambda f^m(z)+\mu)}\right) + S(r,f). \end{aligned}$$

Hence

$$\begin{aligned} (m+n)T(r,f) &\le (k+1)\overline{N}(r,1/f) + N_{k+1}\left(r, \frac{1}{\lambda f^m(z)+\mu}\right) + S(r,f) \\ &\le (k+m+1)T(r,f) + S(r,f), \end{aligned}$$

which contradicts $n > 2k+4+m$.

Case 2: $B = 0$. From (3.10) we have

$$F = \frac{G+(A-1)}{A}, \qquad G = AF-(A-1). \tag{3.12}$$

If $A \neq 1$, we infer from (3.12) that

$$\overline{N}\left(r, \frac{1}{F-\frac{A-1}{A}}\right) = \overline{N}(r,1/G), \qquad \overline{N}(r,1/F) = \overline{N}\left(r, \frac{1}{G+(A-1)}\right).$$

By Lemma 5, we have $n \le 2k+2+m$. This contradicts the assumption that $n > 2k+4+m$. Thus $A = 1$ and $F = G$. By Lemma 6, we have $f^d(z) \equiv g^d(z)$, where $d = \mathrm{GCD}(n,m)$ in this case.

Case 3: $B = -1$. From (3.10) we obtain

$$F = \frac{A}{-G+(A+1)}, \qquad G = \frac{(A+1)F-A}{F}. \tag{3.13}$$

If $A \neq -1$, we deduce from (3.13) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A}{A+1}}\right) = \overline{N}(r, 1/G), \quad \overline{N}(r, F) = \overline{N}\left(r, \frac{1}{G - A - 1}\right).$$

By the same reasoning as in Cases 1 and 2, we get a contradiction. Hence $A = -1$. From (3.13), we have $FG = 1$, that is,

$$(f^n(z)(\lambda f^m(z) + \mu))^{(k)}(g^n(z)(\lambda g^m(z) + \mu))^{(k)} = z^2$$

by Lemma 7, which is impossible.

(ii) $\lambda\mu = 0$. Since $|\lambda| + |\mu| \neq 0$, we distinguish two cases.

Case A: $\mu = 0$, $\lambda \neq 0$. In this case, we have $F = (\lambda f^{n+m}(z))^{(k)}$ and $G = (\lambda g^{n+m}(z))^{(k)}$. Let

$$F_1 = \frac{(\lambda f^{n+m}(z))^{(k)}}{z}, \quad G_1 = \frac{(\lambda g^{n+m}(z))^{(k)}}{z}.$$

Then F_1 and G_1 share 1 CM. By similar arguments to those in the proof of (i), we have $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$. If $F_1 \equiv G_1$, then Lemma 9 yields $f \equiv cg$, where c is a constant that satisfies $c^{n+m} = 1$. Now we assume that $F_1G_1 = 1$.

If $k = 1$, then

$$\lambda^2(f^{n+m})'(g^{n+m})' = z^2. \tag{3.14}$$

Since f and g are entire functions and $n > 2k + m + 4$, by using similar arguments to the proof of Lemma 7 we deduce from (3.14) that f and g have no zeros. Let $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are nonconstant entire functions. Set

$$h(z) = \frac{1}{f(z)g(z)}; \tag{3.15}$$

we know that $h(z) = e^{\gamma(z)}$, where $\gamma(z)$ is an entire function.

We claim that $\gamma(z)$ is a constant. In fact, suppose $\gamma(z)$ is a nonconstant entire function. Then $h(z)$ is a transcendental entire function. From (3.14), we get

$$(m+n)^2\lambda^2(f^{n+m-1})f'(g^{n+m-1})g' = z^2. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\left(\frac{g'}{g} + \frac{1}{2}\frac{h'}{h}\right)^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2h^{m+n}}{(m+n)^2\lambda^2}. \tag{3.17}$$

Let $\xi = \frac{g'}{g} + \frac{1}{2}\frac{h'}{h}$. Then (3.17) becomes

$$\xi^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2h^{m+n}}{(m+n)^2\lambda^2}. \tag{3.18}$$

If $\xi \equiv 0$, from (3.18), we get

$$h^{m+n} = \frac{(m+n)^2\lambda^2}{4z^2}\left(\frac{h'}{h}\right)^2. \tag{3.19}$$

Since $h(z) = e^{\gamma(z)}$, from (3.19) we obtain

$$\begin{aligned}(m+n)T(r,h) &= (m+n)m(r,h) + O(1)\\ &\le m\left(r, \frac{1}{4z^2}\right) + 2m\left(r, \frac{h'}{h}\right) + O(1) = S(r,h).\end{aligned}$$

Hence h is a constant, which is a contradiction. Therefore $\xi \not\equiv 0$. Differentiating (3.18), we have

$$\begin{aligned}2\xi\xi' &= \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' - \frac{2z}{\lambda^2(m+n)^2}\,h^{m+n} - \frac{1}{\lambda^2(m+n)}\,z^2h^{m+n-1}h'\\ &= \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' - \frac{1}{\lambda^2(m+n)^2}\,h^{m+n-1}(2zh + (m+n)z^2h').\end{aligned} \tag{3.20}$$

From (3.18) and (3.20), we obtain

$$\begin{aligned}\frac{1}{\lambda^2(m+n)^2}\,h^{m+n}&\left(2z + (m+n)z^2\,\frac{h'}{h} - 2z^2\,\frac{\xi'}{\xi}\right)\\ &= \frac{1}{2}\,\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\,\frac{\xi'}{\xi}\right).\end{aligned} \tag{3.21}$$

If $2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi} \equiv 0$, then we deduce from (3.21) that either $\frac{h'}{h} \equiv 0$ or $(\frac{h'}{h})' - \frac{h'}{h}\frac{\xi'}{\xi} \equiv 0$. If $\frac{h'}{h} \equiv 0$, then h is a constant, which is a contradiction. If $\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\xi'}{\xi} \equiv 0$, we have

$$\frac{h'}{h} = \frac{\xi}{d}, \tag{3.22}$$

where d ($\ne 0$) is a constant. Thus from (3.18) and (3.22) we get

$$\frac{z^2h^{m+n}}{\lambda^2(m+n)^2} = \left(\frac{1}{4} - d^2\right)\left(\frac{h'}{h}\right)^2. \tag{3.23}$$

Hence, $(m+n)T(r,h) = S(r,h)$, which is also a contradiction.

Now we assume that $2z + (m+n)z^2\frac{h'}{h} - 2z^2\frac{\xi'}{\xi} \not\equiv 0$. Since $h = e^{\gamma(z)}$ and $\xi = \frac{g'}{g} + \frac{1}{2}\frac{h'}{h}$, from (3.18) and (3.21) we have

$$N(r, h'/h) = S(r,h), \qquad N(r,\xi) = S(r,h),$$

and

$$
\begin{aligned}
(3.24)\qquad (m+n)T(r,h) &= (m+n)m(r,h)\\
&\le m\left(r, \frac{1}{2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}}\right)\\
&\quad + m\left(r, \frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\xi'}{\xi}\right)\right) + O(1)\\
&\le m\left(r, \frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\xi'}{\xi}\right)\right) + m\left(r, 2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\right)\\
&\quad + N\left(r, 2z+(m+n)z^2\frac{h'}{h}-2z^2\frac{\xi'}{\xi}\right) + O(1)\\
&\le N(r,\xi'/\xi) + S(r,h) + S(r,\xi)\\
&\le T(r,\xi) + S(r,h) + S(r,\xi).
\end{aligned}
$$

Noting that $h = e^{\gamma(z)}$ is a transcendental entire function, from (3.18) we get

$$
\begin{aligned}
(3.25)\qquad 2T(r,\xi) &= T(r,\xi^2) + S(r,\xi)\\
&= T\left(r, \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2h^{m+n}}{\lambda^2}\right) + S(r,\xi)\\
&= N\left(r, \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2h^{m+n}}{\lambda^2(m+n)^2}\right)\\
&\quad + m\left(r, \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \frac{z^2h^{m+n}}{\lambda^2(m+n)^2}\right) + S(r,\xi)\\
&\le (m+n)m(r,h) + N\left(r, \left(\frac{h'}{h}\right)^2\right) + S(r,h) + S(r,\xi)\\
&\le (m+n)T(r,h) + S(r,h) + S(r,\xi).
\end{aligned}
$$

Combining this with (3.24), we have

$$\frac{m+n}{2}\,T(r,h) = S(r,h),$$

which is a contradiction. Thus, $\gamma(z)$ is a constant, and so $h(z) = e^{\gamma(z)}$ is also a constant. From (3.15), we obtain

$$f(z)g(z) = e^{\alpha(z)}e^{\beta(z)} = C, \tag{3.26}$$

where C $(\neq 0)$ is a constant. So we have

$$\beta(z) = -\alpha(z) + c_1 \tag{3.27}$$

for a constant c_1. Substituting $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$ into (3.16), we infer from

(3.26) and (3.27) that

$$f(z) = b_1 e^{bz^2}, \quad g(z) = b_2 e^{-bz^2},$$

where b_1, b_2 and b are constants that satisfy $4\lambda^2(b_1b_2)^{n+m}((m+n)b)^2 = -1$.

If $k \geq 2$, then

$$\lambda^2(f^{n+m})^{(k)}(g^{n+m})^{(k)} = z^2. \tag{3.28}$$

Since f and g are entire functions and $n > 2k+m+4$, by using the arguments similar to the proof of Lemma 7, we deduce from (3.14) that f and g have no zeros. Let

$$f = e^{\alpha(z)}, \quad g = e^{\beta(z)}, \tag{3.29}$$

where $\alpha(z)$, $\beta(z)$ are nonconstant entire functions. By (3.28), we have

$$N(r, 1/(f^{m+n})^{(k)}) \leq N(r, 1/z^2) = O(\log r). \tag{3.30}$$

Combining (3.29) and (3.30), we obtain

$$N(r, f^{m+n}) + N(r, 1/f^{m+n}) + N(r, 1/(f^{m+n})^{(k)}) = O(\log r).$$

By (3.29), $T(r, (f^{m+n})'/f^{m+n}) = T(r, (m+n)\alpha')$. If α is transcendental, we know from Lemma 10 that $f = e^{\alpha} = e^{az+b}$ for some constants $a \neq 0$ and b. This is impossible. Hence α must be a polynomial, and so β is also a polynomial. Let $\deg(\alpha) = p$ and $\deg(\beta) = q$. If $p = q = 1$, we have

$$f = e^{Az+B}, \quad g = e^{Cz+D}, \tag{3.31}$$

where A, B, C and D are constants that satisfy $AC \neq 0$. Substituting (3.31) into (3.28), we obtain

$$\lambda^2(m+n)^{2k}(AC)^k e^{(m+n)(A+C)z+(m+n)(B+D)} = z^2,$$

which is impossible. Thus $\max\{p, q\} > 1$. We can assume that $p > 1$. Then $(f^{m+n})^{(k)} = Pe^{(m+n)\alpha}$, where P is a polynomial of degree $kp - k \geq k \geq 2$. From (3.28), we have $p = k = 2$ and $q = 1$. Suppose that

$$f^{m+n} = e^{(m+n)(A_1z^2+B_1z+C_1)}, \quad g^{m+n} = e^{(m+n)(D_1z+E_1)},$$

where A_1, B_1, C_1, D_1, E_1 are constants such that $A_1D_1 \neq 0$. Then

$$(f^{m+n})'' = (m+n)(4(m+n)A_1^2z^2 + 4(m+n)A_1B_1z + (m+n)B_1^2 + 2A_1)e^{(m+n)(A_1z^2+B_1z+C_1)}, \tag{3.32}$$

$$(g^{m+n})'' = (m+n)^2 D_1^2 e^{(m+n)(D_1z+E_1)}. \tag{3.33}$$

Substituting (3.32) and (3.33) into (3.28), we have

$$Q(z)e^{(m+n)(A_1z^2+(B_1+D_1)z+C_1+E_1)} = z^2,$$

where $Q(z)$ is a polynomial of degree 2. Since $A_1 \neq 0$, we get a contradiction.

Case B: $\lambda = 0$, $\mu \neq 0$. In this case, by similar arguments to those in Case A, f and g must satisfy $f(z) = b_1e^{bz^2}, g(z) = b_2e^{-bz^2}$ or $f = cg$, where

b_1, b_2, b and c are constants that satisfy $4\mu^2(b_1b_2)^n(nb)^2 = -1$ and $c^n = 1$. This completes the proof of Theorem 2.

Acknowledgements. The authors would like to thank the referee for his/her helpful suggestions and comments.

This work was supported by the NNSF of China (No. 10671109).

References

[1] M. L. Fang, *Uniqueness and value-sharing of entire functions*, Comput. Math. Appl. 44 (2002), 823–831.

[2] M. L. Fang and X. H. Hua, *Entire functions that share one value*, Nanjing Daxue Xuebao Shuxue Bannian Kan 13 (1996), 44–48.

[3] M. L. Fang and H. L. Qiu, *Meromorphic functions that share fixed-points*, J. Math. Anal. Appl. 268 (2000), 426–439.

[4] W. K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. of Math. 70 (1959), 9–42.

[5] —, *Meromorphic Functions*, Oxford Univ. Press, 1964.

[6] W. C. Lin and H. X. Yi, *Uniqueness theorems for meromorphic functions*, Indian J. Pure Appl. Math. 35 (2004), 121–132.

[7] —, —, *Uniqueness theorems for meromorphic functions concerning fixed-points*, Complex Var. Theory Appl. 49 (2004), 793–806.

[8] Y. F. Wang, *On Mues' conjecture and Picard values*, Sci. China 36 (1993), 28–35.

[9] C. C. Yang and X. H. Hua, *Uniqueness and value-sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. 22 (1997), 395–406.

[10] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer, 2003.

[11] J. L. Zhang and L. Z. Yang, *Some results related to a conjecture of R. Brück*, J. Inequal. Pure Appl. Math. 8 (2007), no. 1, art. 18, 11 pp.

Xiao-Guang Qi, Lian-Zhong Yang
School of Mathematics
Shandong University
Jinan, Shandong, 250100, P.R. China
E-mail: xiaogqi@mail.sdu.edu.cn
lzyang@sdu.edu.cn

Received 15.2.2009
and in final form 10.4.2009 (1979)