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Positive solutions to a singular fourth-order two-point boundary value problem

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Abstract. This paper studies the existence of multiple positive solutions to a nonlinear fourth-order two-point boundary value problem, where the nonlinear term may be singular with respect to both time and space variables. In order to estimate the growth of the nonlinear term, we introduce new control functions. By applying the Hammerstein integral equation and the Guo–Krasnosel'skiĭ fixed point theorem of cone expansion-compression type, several local existence theorems are proved.

1. Introduction. This paper studies the positive solutions of the following nonlinear fourth-order two-point boundary value problem:

$$(P1)\qquad \begin{cases} u^{(4)}(t) = h(t)f(t,u(t)) + \zeta(t,u(t)), & 0 < t < 1,\\ u(0) = u'(0) = u(1) = u''(1) = 0. \end{cases}$$

Here, the function u^* is called a positive solution of $(P1)$ if u^* is a solution of $(P1)$ and $u^*(t) > 0,\ 0 < t < 1$. In mechanics, problem $(P1)$ models the deflection of an elastic beam rigidly fixed on the left and simply supported on the right.

Throughout this paper, $0 < \alpha < \beta < 1$, $q(t) = \frac{2}{3}t^2(1-t)$ and

- (H1) $h \in L^1[0,1]$ is a nonnegative function and $\int_\alpha^\beta h(t)\,dt > 0$.
- (H2) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function.
- (H3) $\zeta : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous and, for every pair of positive numbers $0 < r_1 < r_2$, there exists a nonnegative function $j_{r_1}^{r_2} \in L^1[0,1] \cap C(0,1)$ such that $\zeta(t,u) \le j_{r_1}^{r_2}(t)$ for $0 \le t \le 1$ and $r_1 q(t) \le u \le r_2$.

Thus, we allow the nonlinear term $h(t)f(t,u) + \zeta(t,u)$ to be singular in the time variable at $t = 0,\ t = 1$ and in the space variable at $u = 0$. In this

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paper, the functions $f(t,u)$ and $\zeta(t,u)$ are called the *continuous part* and *singular part* of problem $(P1)$ respectively.

A typical example is

$$(P2)\qquad \begin{cases} u^{(4)}(t) = \gamma u^{\lambda}(t) + \dfrac{\eta}{u^{\tau}(t)}, & 0 < t < 1, \\ u(0) = u'(0) = u(1) = u''(1) = 0, \end{cases}$$

where λ, τ, γ, η are positive constants. In problem $(P2)$, the nonlinear term $\gamma u^{\lambda} + \eta/u^{\tau}$ is singular at $u = 0$.

When $\zeta(t,u) \equiv 0$ and $h(t) \equiv 1$, the existence and multiplicity of solutions and positive solutions of problem $(P1)$ have been studied by several authors (see [1, 2, 5, 6, 9–11]). Very recently, we proved in [11] the following local existence theorem.

THEOREM 1.1. *Assume that $h(t) \equiv 1$, $\zeta(t,u) \equiv 0$ and there exist positive numbers $a < b$ such that one of the following conditions is satisfied*:

(a1) $\varphi(a) \le 48a$, $\psi(b) \ge \frac{486b}{\beta^3(4-3\beta)-\alpha^3(4-3\alpha)}$.

(a2) $\psi(a) \ge \frac{486a}{\beta^3(4-3\beta)-\alpha^3(4-3\alpha)}$, $\varphi(b) \le 48b$.

Then problem $(P1)$ has a positive solution $u^* \in K$ such that $a \le \|u^*\| \le b$.

In Theorem 1.1, we introduced the following control functions and cone:

$$\varphi(r) = \max\{f(t,u) : 0 \le t \le 1,\ rq(t) \le u \le r\},$$
$$\psi(r) = \min\{f(t,u) : \alpha \le t \le \beta,\ rq(t) \le u \le r\},$$
$$K = \{u \in C[0,1] : u(t) \ge \|u\| q(t),\ 0 \le t \le 1\},$$

where $\|u\| = \max_{0\le t\le 1} |u(t)|$ is the norm in the Banach space $C[0,1]$. Geometrically, $\varphi(r)$ is the maximal height of $f(t,u)$ on the swallow-tailed domain $\{(t,u) : 0 \le t \le 1,\ rq(t) \le u \le r\}$, and $\psi(r)$ is the minimal height of $f(t,u)$ on the domain $\{(t,u) : \alpha \le t \le \beta,\ rq(t) \le u \le r\}$. We will also use these symbols in this paper.

The solvability of other singular boundary value problems has been extensively discussed in the literature in the past ten years: see for example [3, 4, 7, 8, 12]. However, to the best of our knowledge, the existence of positive solutions has not been studied previously when the problem $(P1)$ has singular nonlinear term with respect to the space variable.

The purpose of this paper is to extend Theorem 1.1 to the general singular problem $(P1)$ under the assumptions (H1)–(H3). In order to achieve this aim, we will construct new control functions defined on the swallow-tailed domain $\{(t,u) : 0 \le t \le 1,\ rq(t) \le u \le r\}$. By using new control functions, we will consider not only the height of the continuous part $f(t,u)$, but also the integral of the singular part $\zeta(t,u)$. By applying the Guo–Krasnosel'skiĭ fixed point theorem of cone expansion-compression type, we will prove the existence of finitely and infinitely many positive solutions for problem $(P1)$.

Furthermore, we will verify the existence of one and two positive solutions for problem $(P2)$. Finally, we will illustrate that our method is different from ones in [3, 4, 7, 8, 12] by two examples. In particular, we will give an example with infinitely many positive solutions.

2. Preliminaries. It is easy to check that K is a cone of nonnegative functions in $C[0,1]$. Write $K[r_1,r_2] = \{u \in K : r_1 \le \|u\| \le r_2\}$.

Let $G(t,s)$ be the Green function of the homogeneous linear problem

$$u^{(4)}(t) = 0, \quad 0 \le t \le 1, \quad u(0) = u'(0) = u(1) = u''(1) = 0.$$

From [2], the exact expression of $G(t,s)$ is

$$G(t,s) = \begin{cases} \frac{1}{12}(1-t)s^2[3(1-s) - (1-t)^2(3-s)], & 0 \le s \le t \le 1, \\ \frac{1}{12}t^2(1-s)[3(1-t) - (1-s)^2(3-t)], & 0 \le t \le s \le 1. \end{cases}$$

Obviously, $G : [0,1] \times [0,1] \to [0,1]$ is continuous and $G(t,s) > 0$ for $0 < t, s < 1$.

Let $H(s) = \frac{1}{4}s^2(1-s)$. By Lemma 2.1 in [11], we have

Lemma 2.1. *$q(t)H(s) \le G(t,s) \le H(s)$ for $0 \le t, s \le 1$.*

Define the operator T as follows:

$$(Tu)(t) = \int_0^1 G(t,s)[h(s)f(s,u(s)) + \zeta(s,u(s))]\,ds,\ 0 \le t \le 1,\ u \in K \setminus \{0\}.$$

Lemma 2.2. *$T : K[r_1,r_2] \to K$ is a completely continuous operator for any $0 < r_1 < r_2$.*

Proof. Let $j_{r_1}^{r_2}(t)$ be as in (H3) and

$$\tilde{\zeta}(t,u) = \begin{cases} \zeta(t, r_1 q(t)), & 0 < t < 1,\ 0 \le u \le r_1 q(t), \\ \zeta(t,u), & 0 < t < 1,\ r_1 q(t) \le u \le r_2, \\ \zeta(t, r_2), & 0 < t < 1,\ r_2 \le u < +\infty; \end{cases}$$

$$(T_1 u)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))\,ds, \quad 0 \le t \le 1,$$

$$(T_2 u)(t) = \int_0^1 G(t,s)\tilde{\zeta}(s,u(s))\,ds, \quad 0 \le t \le 1.$$

Obviously, $\tilde{\zeta} : (0,1) \times [0,+\infty) \to [0,+\infty)$ is continuous and $\tilde{\zeta}(t,u) \le j_{r_1}^{r_2}(t)$, $(t,u) \in (0,1) \times [0,+\infty)$. By the Arzelà–Ascoli theorem, $T_1, T_2 : K \to C[0,1]$ are completely continuous operators.

If $u \in K[r_1,r_2]$, then $r_1 \le \|u\| \le r_2$ and $r_1 q(t) \le u(t) \le r_2$ for $0 \le t \le 1$. So, for any $u \in K[r_1,r_2]$, $Tu = T_1 u + T_2 u$. Therefore, $T : K[r_1,r_2] \to C[0,1]$ is completely continuous.

On the other hand, by Lemma 2.1, for $0 \le t \le 1$,

$$
\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)[h(s)f(s,u(s)) + \zeta(s,u(s))]\,ds \\
&\ge q(t)\int_0^1 H(s)[h(s)f(s,u(s)) + \zeta(s,u(s))]\,ds \\
&\ge q(t)\max_{0\le t\le 1}\int_0^1 G(t,s)[h(s)f(s,u(s)) + \zeta(s,u(s))]\,ds = \|Tu\|q(t).
\end{aligned}
$$

Hence, $T : K[r_1, r_2] \to K$. ∎

LEMMA 2.3 (Guo–Krasnosel'skiĭ). *Let X be a Banach space, and K be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $F : \overline{\Omega}_2 \setminus \Omega_1 \to K$ is a completely continuous operator such that either*

(1) $\|F(x)\| \le \|x\|$ *for* $x \in \partial\Omega_1$ *and* $\|F(x)\| \ge \|x\|$ *for* $x \in \partial\Omega_2$, *or*
(2) $\|F(x)\| \ge \|x\|$ *for* $x \in \partial\Omega_1$ *and* $\|F(x)\| \le \|x\|$ *for* $x \in \partial\Omega_2$.

Then F has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

In this paper, we use the following constants:

$$
A = \max_{0\le t\le 1}\int_0^1 G(t,s)h(s)\,ds, \qquad B = \max_{\alpha\le t\le \beta} q(t)\int_\alpha^\beta H(s)h(s)\,ds,
$$
$$
M = \max_{0\le t,s\le 1} G(t,s), \qquad m = \min_{\alpha\le t,s\le \beta} G(t,s).
$$

Direct computations give

$$
\begin{aligned}
M &= G(2-\sqrt{2}, 2-\sqrt{2}) = \frac{17-12\sqrt{2}}{3} \approx \frac{1}{101.9117}, \\
m &= G(\alpha,\beta) = \frac{1}{12}\alpha^2(1-\beta)[\beta(3-\alpha)(2-\beta) - 2\alpha].
\end{aligned}
$$

If $h(t) \equiv 1$, then

$$
A = \frac{1}{48}\max_{0\le t\le 1} t^2(1-t)(3-2t) = \frac{117+115\sqrt{33}}{196608} \approx \frac{1}{252.8315},
$$
$$
B = \begin{cases} \dfrac{\beta^3(4-3\beta) - \alpha^3(4-3\alpha)}{486}, & \dfrac{2}{3} \in [\alpha,\beta], \\[2ex] \dfrac{16\max\{\alpha^2(1-\alpha), \beta^2(1-\beta)\}[\beta^3(4-3\beta) - \alpha^3(4-3\alpha)]}{243}, & \dfrac{2}{3} \notin [\alpha,\beta]. \end{cases}
$$

For $r > 0$, let

$$\mu(r) = \int_0^1 \max\{\zeta(t,u) : rq(t) \le u \le r\}\, dt,$$

$$\nu(r) = \int_\alpha^\beta \min\{\zeta(t,u) : rq(t) \le u \le r\}\, dt.$$

Geometrically, $\max\{\zeta(t,u) : rq(t) \le u \le r\}$ is the maximal height of the singular part $\zeta(t,u)$ on the set $\{(t,u) : 0 < t < 1,\ rq(t) \le u \le r\}$, and $\mu(r)$ is the integral of the maximal height function on $[0,1]$. For $\nu(r)$, the geometric meaning is similar.

In order to study the singular problem $(P1)$, we introduce the following two control functions:

$$A\varphi(r) + M\mu(r), \qquad B\psi(r) + m\nu(r).$$

We will use them to estimate the growth of the nonlinear term $h(t)f(t,u) + \zeta(t,u)$ on the swallow-tailed domain

$$\{(t,u) : 0 < t < 1,\ rq(t) \le u \le r\}.$$

3. Main results. We obtain the following local existence theorems.

THEOREM 3.1. *Suppose that there exist positive numbers $a < b$ such that one of the following conditions is satisfied*:

(b1) $A\varphi(a) + M\mu(a) \le a,\ B\psi(b) + m\nu(b) \ge b$.
(b2) $B\psi(a) + m\nu(a) \ge a,\ A\varphi(b) + M\mu(b) \le b$.

Then problem $(P1)$ has a positive solution $u^* \in K$ such that $a \le \|u^*\| \le b$.

Proof. We prove only the case (b1). The proof of the case (b2) is similar.

Let $\Omega(r) = \{u \in K : \|u\| < r\}$. Then $K[a,b] = \overline{\Omega(b)} \setminus \Omega(a)$.

If $u \in \partial\Omega(a)$, then $aq(t) \le u(t) \le a$ for $0 \le t \le 1$. Thus $f(t,u(t)) \le \varphi(a)$ for $0 \le t \le 1$ and $\int_0^1 \zeta(t,u(t))\,dt \le \mu(a)$. It follows that

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le 1} \int_0^1 G(t,s)[h(s)f(s,u(s)) + \zeta(s,u(s))]\, ds \\
&\le \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)f(s,u(s))\, ds + \max_{0\le t\le 1} \int_0^1 G(t,s)\zeta(s,u(s))\, ds \\
&\le \varphi(a) \max_{0\le t\le 1} \int_0^1 G(t,s)h(s)\, ds + M \int_0^1 \zeta(s,u(s))\, ds \\
&\le A\varphi(a) + M\mu(a) \le a = \|u\|.
\end{aligned}$$

If $u \in \partial\Omega(b)$, then $bq(t) \le u(t) \le b$ for $0 \le t \le 1$. Thus $f(t,u(t)) \ge \psi(b)$ for $\alpha \le t \le \beta$ and $\int_\alpha^\beta \zeta(t,u(t))\,dt \ge \nu(b)$. Applying Lemma 2.1, we obtain

$$\begin{aligned}\|Tu\| &\ge \max_{\alpha\le t\le\beta}\int_\alpha^\beta G(t,s)[h(s)f(s,u(s))+\zeta(s,u(s))]\,ds\\ &\ge \max_{\alpha\le t\le\beta}\int_\alpha^\beta G(t,s)h(s)f(s,u(s))\,ds+\min_{\alpha\le t\le\beta}\int_\alpha^\beta G(t,s)\zeta(s,u(s))\,ds\\ &\ge \psi(b)\max_{\alpha\le t\le\beta} q(t)\int_\alpha^\beta H(s)h(s)\,ds+\min_{\alpha\le t,s\le\beta} G(t,s)\int_\alpha^\beta \zeta(s,u(s))\,ds\\ &\ge B\psi(b)+m\nu(b)\ge b=\|u\|.\end{aligned}$$

By Lemmas 2.2 and 2.3, there exists $u^* \in K[a,b]$ such that $Tu^* = u^*$. In other words, $u^* \in K$, $a \le \|u^*\| \le b$ and

$$u^*(t)=\int_0^1 G(t,s)[h(s)f(s,u^*(s))+\zeta(s,u^*(s))]\,ds,\quad 0\le t\le 1.$$

Direct checks give

$$(u^*)^{(4)}(t)=h(t)f(t,u^*(t))+\zeta(t,u^*(t)),\quad 0<t<1,$$

and $u^*(0)=u^*(1)=(u^*)'(0)=(u^*)''(1)=0$. Therefore, u^* is a solution of problem $(P1)$. Since $u^*(t)\ge aq(t)>0$, $0<t<1$, u^* is a positive solution. ■

THEOREM 3.2. *Assume that there exist positive numbers $a<b<c$ such that one of the following conditions is satisfied*:

(c1) $A\varphi(a)+M\mu(a)\le a$, $B\psi(b)+m\nu(b)>b$, $A\varphi(c)+M\mu(c)\le c$.
(c2) $B\psi(a)+m\nu(a)\ge a$, $A\varphi(b)+M\mu(b)<b$, $B\psi(c)+m\nu(c)\ge c$.

Then problem $(P1)$ has positive solutions $u_1^*, u_2^* \in K$ such that $a\le\|u_1^*\|< b<\|u_2^*\|\le c$.

Proof. By the assumptions (H1)–(H3), we can prove that $\varphi,\psi,\mu,\nu : (0,+\infty)\to[0,+\infty)$ are continuous.

If $B\psi(b)+m\nu(b)>b$, then there exist $a<b_1<b<b_2<c$ such that $B\psi(b_1)+m\nu(b_1)\ge b_1$ and $B\psi(b_2)+m\nu(b_2)\ge b_2$. By Theorem 3.1, problem $(P1)$ has positive solutions $u_1^*,u_2^*\in K$ satisfying $a\le\|u_1^*\|\le b_1<b<b_2\le \|u_2^*\|\le c$. Similarly, we can prove the case (c2). ■

Generally, we have the following theorem on existence of n positive solutions, where $[c]$ is the integer part of c.

THEOREM 3.3. *Assume that there exist positive numbers $a_1<\cdots< a_{n+1}$ such that one of the following conditions is satisfied*:

(d1) $A\varphi(a_{2k-1}) + M\mu(a_{2k-1}) < a_{2k-1}$ *for* $k = 1, \ldots, [(n+2)/2]$ *and* $B\psi(a_{2k}) + m\nu(a_{2k}) > a_{2k}$ *for* $k = 1, \ldots, [(n+1)/2]$.
(d2) $B\psi(a_{2k-1}) + m\nu(a_{2k-1}) > a_{2k-1}$ *for* $k = 1, \ldots, [(n+2)/2]$ *and* $A\varphi(a_{2k}) + M\mu(a_{2k}) < a_{2k}$ *for* $k = 1, \ldots, [(n+1)/2]$.

Then problem $(P1)$ *has positive solutions* $u_k^* \in K$, $k = 1, \ldots, n$, *such that* $a_k < \|u_k^*\| < a_{k+1}$.

REMARK 3.1. If $\zeta(t,u) \equiv 0$ and $h(t) \equiv 1$, then $\mu(r) = \nu(r) \equiv 0$ for any $r > 0$, and

$$A < \frac{1}{48}, \qquad B \leq \frac{\beta^3(4-3\beta) - \alpha^3(4-3\alpha)}{486}.$$

From this, we see that Theorem 1.1 is a simple corollary of Theorem 3.1.

4. Results concerning growth limits. In this section, we study some cases involving the growth limits of the nonlinear term. Let

$$\begin{aligned}
\underline{\varphi}_0 &= \liminf_{r\to 0} \varphi(r)/r, & \underline{\varphi}_\infty &= \liminf_{r\to+\infty} \varphi(r)/r,\\
\overline{\psi}_0 &= \limsup_{r\to+0} \psi(r)/r, & \overline{\psi}_\infty &= \limsup_{r\to+\infty} \psi(r)/r,\\
\underline{\mu}_0 &= \liminf_{r\to 0} \mu(r)/r, & \underline{\mu}_\infty &= \liminf_{r\to+\infty} \mu(r)/r,\\
\overline{\nu}_0 &= \limsup_{r\to 0} \nu(r)/r, & \overline{\nu}_\infty &= \limsup_{r\to+\infty} \nu(r)/r.
\end{aligned}$$

THEOREM 4.1. *Assume that one of the following conditions is satisfied*:

(e1) $A\underline{\varphi}_0 + M\underline{\mu}_0 < 1$ *and* $B\overline{\psi}_\infty + m\overline{\nu}_\infty > 1$.
(e2) $B\overline{\psi}_0 + m\overline{\nu}_0 > 1$ *and* $A\underline{\varphi}_\infty + M\underline{\mu}_\infty < 1$.

Then problem $(P1)$ *has a positive solution* $u^* \in K$.

Proof. We prove only the case (e1).

Since $A\underline{\varphi}_0 + M\underline{\mu}_0 < 1$, there is $a > 0$ such that $A\varphi(a)/a + M\mu(a)/a < 1$. Thus, $A\varphi(a) + M\mu(a) < a$.

Since $B\overline{\psi}_\infty + m\overline{\nu}_\infty > 1$, there is $b > a$ such that $B\psi(b)/b + m\nu(b)/b > 1$. Thus, $B\psi(b) + m\nu(b) > b$.

By Theorem 3.1, the proof is complete. ∎

THEOREM 4.2. *Assume that the following conditions are satisfied*:

(f1) *There exist* $\bar{r}, \bar{L} > 0$ *such that* $\zeta(t,u) \geq \bar{L}$ *for* $(t,u) \in [\alpha,\beta] \times (0,\bar{r}]$.
(f2) $\lim_{u\to+\infty} \max_{0\leq t\leq 1} f(t,u)/u < A^{-1}$.
(f3) *There exists* $r_0 > 0$ *and a nonnegative function* $J_{r_0} \in L^1[0,1]$ *such that*

$$\zeta(t,u) \leq J_{r_0}(t), \quad (t,u) \in (0,1) \times [r_0, +\infty).$$

Then problem $(P1)$ *has a positive solution* $u^* \in K$.

Proof. By (f1), for any $0 < r \le \bar{r}$, $\nu(r) \ge \int_\alpha^\beta \bar{L}\,dt = L(\beta-\alpha)$. Thus, $\overline{\nu}_0 = \underline{\nu}_0 = +\infty$. Let $j_{r_0}^{2r_0}(t)$ be as in (H3). By (f3), for any $r \ge r_0$, we have

$$\zeta(t,u) \le j_{r_0}^{2r_0}(t) + J_{r_0}(t), \qquad 0 < t < 1,\, rq(t) \le u \le r.$$

Thus, $\mu(r) \le \int_0^1 j_{r_0}^{2r_0}(t)\,dt + \int_0^1 J_{r_0}(t)\,dt < +\infty$ and $\underline{\mu}_\infty = \overline{\mu}_\infty = 0$.

Let

$$\varepsilon = \frac{1}{3}[A^{-1} - \lim_{u\to+\infty} \max_{0\le t\le 1} f(t,u)/u].$$

By (f2), $\varepsilon > 0$ and there exists $r_1 > 0$ such that

$$\max_{0\le t\le 1} f(t,u) < (A^{-1} - 2\varepsilon)u, \quad u \in [r_1, +\infty).$$

Let

$$r_2 = \max\{r_1 + 1, \varepsilon^{-1} \max\{\max_{0\le t\le 1} f(t,u) : 0 \le u \le r_1\}\}.$$

Then, for any $r \ge r_2$,

$$\begin{aligned}
\varphi(r) &\le \max\{\max_{0\le t\le 1} f(t,u) : 0 \le u \le r\} \\
&\le \max\{\max_{0\le t\le 1} f(t,u) : 0 \le u \le r_1\} + \max\{\max_{0\le t\le 1} f(t,u) : r_1 \le u \le r\} \\
&\le \varepsilon r_2 + (A^{-1} - 2\varepsilon)r \le (A^{-1} - \varepsilon)r.
\end{aligned}$$

So, $\underline{\varphi}_\infty \le \overline{\varphi}_\infty < A^{-1}$.

Therefore, $B\overline{\psi}_0 + m\overline{\nu}_0 = +\infty > 1$ and $A\underline{\varphi}_\infty + M\underline{\mu}_\infty < 1$. By Theorem 4.1(e2), the proof is complete. ∎

Similarly, we have the following multiplicity result.

THEOREM 4.3. *Assume that one of the following conditions is satisfied*:

(g1) $A\underline{\varphi}_0 + M\underline{\mu}_0 < 1$, $A\underline{\varphi}_\infty + M\underline{\mu}_\infty < 1$ *and there exists* $d > 0$ *such that* $B\psi(d) + m\nu(d) > d$.

(g2) $B\overline{\psi}_0 + m\overline{\nu}_0 > 1$, $B\overline{\psi}_\infty + m\overline{\nu}_\infty > 1$ *and there exists* $d > 0$ *such that* $A\varphi(d) + M\mu(d) < d$.

Then problem $(P1)$ *has positive solutions* $u_1^*, u_2^* \in K$ *such that* $0 < \|u_1^*\| < d < \|u_2^*\|$.

Furthermore, we have the following result on the existence of infinitely many positive solutions.

THEOREM 4.4. *Assume that* $A\underline{\varphi}_\infty + M\underline{\mu}_\infty < 1$ *and* $B\overline{\psi}_\infty + m\overline{\nu}_\infty > 1$. *Then problem* $(P1)$ *has a sequence of positive solutions* u_k^*, $k = 1, 2, \ldots,$ *such that* $\|u_k^*\| \to +\infty$.

Proof. By assumption, there exist sequences of positive numbers $a_k \to +\infty$, $b_k \to +\infty$ such that

$$A\varphi(a_k) + M\mu(a_k) < a_k, \quad B\psi(b_k) + m\nu(b_k) > b_k.$$

Without loss of generality, we can assume that $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k < \cdots$. By Theorem 3.3, for each $k = 1, 2, \ldots$, there exists a positive solution $u_k^* \in K$ with $a_k \le \|u_k^*\| \le b_k$. ■

5. On problem $(P2)$**.** In this section, we use the notation of problem $(P2)$.

Let

$$h(t) \equiv 1, \qquad f(t,u) = f(u) = \gamma u^\lambda, \qquad \zeta(t,u) = \zeta(u) = \eta/u^\tau.$$

If $0 < r_1 < r_2$, then

$$\zeta(u) = \frac{\eta}{u^\tau} \le \frac{\eta}{[r_1 q(t)]^\tau}, \qquad r_1 q(t) \le u \le r_2.$$

Let $0 < \tau < 1/2$ and

$$j_{r_1}^{r_2}(t) = \frac{\eta}{[r_1 q(t)]^\tau} = \frac{\eta}{[\frac{2}{3} r_1 t^2 (1-t)]^\tau}.$$

Then $j_{r_1}^{r_2} \in L^1[0,1] \cap C(0,1)$. This shows that condition (H3) is satisfied if $0 < \tau < 1/2$.

Theorem 5.1. *Assume that one of the following conditions is satisfied*:

(h1) $0 < \lambda < 1$, $0 < \tau < 1/2$.
(h2) $\lambda = 1$, $0 < \tau < 1/2$ *and* $\gamma < A^{-1}$.

Then problem $(P2)$ *has a positive solution* $u^* \in K$.

Proof. We have $\zeta(u) \ge \eta$ for $0 < u \le 1$, and $\zeta(u) \le \eta$ for $1 \le u < +\infty$. If (h1) holds, then $\lim_{u\to+\infty} f(u)/u = 0 < A^{-1}$; if (h2) holds, then $\lim_{u\to+\infty} f(u)/u = \gamma < A^{-1}$. By Theorem 4.2, problem $(P2)$ has a positive solution $u^* \in K$. ■

Theorem 5.2. *Assume* $\lambda > 1$ and $0 < \tau < 1/2$.

(j1) *Let* $\eta > 0$ *be fixed. Then problem* $(P2)$ *has two positive solutions* $u_1^*, u_2^* \in K$ *for any* $0 < \gamma \le \frac{1}{2A}\big[\frac{2^\tau(1-2\tau)}{3^{1+\tau}\eta M}\big]^{(\lambda-1)/(1+\tau)}$.
(j2) *Let* $\gamma > 0$ *be fixed. Then problem* $(P2)$ *has two positive solutions* $u_1^*, u_2^* \in K$ *for any* $0 < \eta \le \frac{2^\tau(1-2\tau)}{3^{1+\tau}M}\big[\frac{1}{2A\gamma}\big]^{(1+\tau)/(\lambda-1)}$.

Proof. Obviously, $B\underline{\psi}_0 + m\underline{\nu}_0 = +\infty > 1$ and $B\underline{\psi}_\infty + m\underline{\nu}_\infty = +\infty > 1$. If $\eta > 0$ is fixed, let

$$a = \left[\frac{3^{1+\tau}\eta M}{2^\tau(1-2\tau)}\right]^{1/(1+\tau)} > 0, \quad \text{so} \quad \frac{3^\tau \eta M}{2^\tau a^\tau (1-2\tau)} = \frac{a}{3}.$$

Direct computations give $\varphi(a) = \gamma a^\lambda$ and

$$\mu(a) = \int_0^1 \frac{\eta\,dt}{(aq(t))^\tau} = \int_0^1 \frac{\eta\,dt}{(\frac{2}{3} a t^2 (1-t))^\tau} \le \frac{3^\tau \eta}{2^\tau a^\tau}\int_0^1 \frac{dt}{t^{2\tau}} = \frac{3^\tau \eta}{2^\tau a^\tau (1-2\tau)}.$$

Therefore,

$$A\varphi(a) + M\mu(a) \le A\gamma a^{\lambda} + \frac{3^{\tau}\eta M}{2^{\tau}a^{\tau}(1-2\tau)}.$$

Let

$$\gamma^* = \frac{1}{2A}\left[\frac{2^{\tau}(1-2\tau)}{3^{1+\tau}\eta M}\right]^{(\lambda-1)/(1+\tau)} = \frac{1}{2Aa^{\lambda-1}}, \quad \text{so} \quad A\gamma^* a^{\lambda} = \frac{a}{2}.$$

It follows that, for any $0 < \gamma \le \gamma^*$,

$$A\varphi(a) + M\mu(a) \le A\gamma a^{\lambda} + \frac{3^{\tau}\eta M}{2^{\tau}a^{\tau}(1-2\tau)} \le \frac{a}{2} + \frac{a}{3} < a.$$

By Theorem 4.3(g2), the conclusion (j1) is proved.

If $\gamma > 0$ is fixed, then choose

$$a = \left[\frac{1}{2A\gamma}\right]^{1/(\lambda-1)}, \quad \text{so} \quad A\gamma a^{\lambda} = \frac{a}{2}.$$

Let

$$\eta^* = \frac{2^{\tau}(1-2\tau)}{3^{1+\tau}M}\left[\frac{1}{2A\gamma}\right]^{(1+\tau)/(\lambda-1)}, \quad \text{so} \quad \frac{3^{\tau}\eta^* M}{2^{\tau}a^{\tau}(1-2\tau)} = \frac{a}{3}.$$

It follows that, for any $0 < \eta \le \eta^*$,

$$A\varphi(a) + M\mu(a) \le A\gamma a^{\lambda} + \frac{3^{\tau}\eta M}{2^{\tau}a^{\tau}(1-2\tau)} \le \frac{a}{2} + \frac{a}{3} < a.$$

By Theorem 4.3(g2), the conclusion (j2) is proved. ∎

6. Two examples. In this section, we illustrate our improvements by two examples.

Example 6.1. The example shows that our method is different from those used in [8–12].

Consider the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = (1+\sin u(t))\left[\sqrt[3]{\dfrac{u(t)}{t(1-t)}} + \dfrac{1}{\sqrt[3]{u(t)}}\right], & 0 < t < 1, \\ u(0) = u'(0) = u(1) = u''(1) = 0. \end{cases}$$

In this problem $h(t) = 1/\sqrt{t(1-t)}$, $f(t,u) = f(u) = (1+\sin u)\sqrt[3]{u}$, $\zeta(t,u) = \zeta(u) = (1+\sin u)/\sqrt[3]{u}$. So assumptions (H1)–(H3) are satisfied and $\zeta(u)$ is singular at $u = 0$.

We have $\zeta(u) \ge 1$ for $0 < u \le 1$, $\zeta(u) \le 2$ for $1 \le u < +\infty$, and $\lim_{u\to+\infty} f(u)/u = 0$. By Theorem 4.2, the problem has a positive solution $u^* \in K$.

If $u = (2k+1)\pi$, then $f(u) = \zeta(u) = 0$. So $\inf_{u>r}[h(t)f(u) + \zeta(u)] = 0$ for any $r > 0$ and $0 < t < 1$. Moreover, $\zeta(u)$ is not nonincreasing in u.

Therefore, the conclusion cannot be derived by applying the methods in [3, 4, 7, 8, 12].

Example 6.2. The example shows that the existence of infinitely many positive solutions is possible.

Let $\alpha = \frac{1}{2},\ \beta = \frac{2}{3},\ C = \frac{13}{B\sin\frac{\pi}{16}}$. Consider the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = C\max\{0, \sin\sqrt{\max\{0, \ln u(t)\}}\}u(t) + \dfrac{1}{\sqrt[3]{u(t)}}, & 0 < t < 1, \\ u(0) = u'(0) = u(1) = u''(1) = 0. \end{cases}$$

In this problem $h(t) \equiv 1,\ f(t,u) = f(u) = C\max\{0, \sin\sqrt{\max\{0, \ln u\}}\}u$, $\zeta(t,u) = \zeta(u) = 1/\sqrt[3]{u}$. So assumptions (H1)–(H3) are satisfied and $\zeta(u)$ is singular at $u = 0$.

Let $\sigma = \min_{1/2\le t\le 2/3} q(t) = \frac{1}{12}$. Then $\sigma > e^{-\frac{7}{8}\pi} \approx \frac{1}{15.6253}$.

For arbitrary $k = 1, 2, \dots$, we have

$$\begin{aligned} &\sin\sqrt{\ln u} \ge 0, \quad f(u) \ge 0, \quad e^{(2k\pi)^2} \le u \le e^{[(2k+1)\pi]^2}, \\ &\sin\sqrt{\ln u} \le 0, \quad f(u) = 0, \quad e^{[(2k+1)\pi]^2} \le u \le e^{[(2k+2)\pi]^2}. \end{aligned}$$

Since $\sigma > e^{-\frac{7}{8}\pi}$, we have $\sigma e^{(2k+\frac{15}{16})\pi} \ge e^{(2k+\frac{1}{16})\pi}$. So

$$\sigma e^{[(2k+\frac{15}{16})\pi]^2} \ge e^{[(2k+\frac{1}{16})\pi]^2}.$$

Hence, we get

$$\begin{aligned} \varphi(e^{[(2k+2)\pi]^2}) &= C\max\{\max\{0, \sin\sqrt{\max\{0, \ln\}}\}u : 0 \le u \le e^{[(2k+2)\pi]^2}\} \\ &= C\max\{\sin\sqrt{\ln u}\,u : e^{[2k\pi]^2} \le u \le e^{[(2k+1)\pi]^2}\} \\ &\le Ce^{[(2k+1)\pi]^2}, \\ \psi(e^{[2k+\frac{15}{16}\pi]^2}) &\ge C\min\{\sin\sqrt{\ln u}\,u : \sigma e^{[(2k+\frac{15}{16})\pi]^2} \le u \le e^{[(2k+\frac{15}{16})\pi]^2}\} \\ &\ge C\sin\sqrt{\ln(e^{[(2k+\frac{15}{16})\pi]^2})} \cdot \sigma e^{[(2k+\frac{15}{16})\pi]^2} \\ &\ge C\sigma e^{[(2k+\frac{15}{16})\pi]^2}\sin\frac{\pi}{16}. \end{aligned}$$

It follows that

$$\begin{aligned} \underline{\varphi}_\infty &\le \lim_{k\to\infty} \frac{Ce^{[(2k+1)\pi]^2}}{e^{[(2k+2)\pi]^2}} = \lim_{k\to\infty} \frac{C}{e^{(4k+3)\pi^2}} = 0, \\ \overline{\psi}_\infty &\ge \lim_{k\to\infty} \frac{C\sigma e^{[(2k+\frac{15}{16})\pi]^2}\sin\frac{\pi}{16}}{e^{[(2k+\frac{15}{16})\pi]^2}} = C\sigma\sin\frac{\pi}{16} = \frac{13}{12B}. \end{aligned}$$

Obviously, $\underline{\mu}_\infty = 0$ and $\overline{\nu}_\infty = 0$.

Therefore, $A\underline{\varphi}_\infty + M\underline{\mu}_\infty = 0 < 1$ and $B\overline{\psi}_\infty + m\overline{\nu}_\infty = \frac{13}{12} > 1$. By Theorem 4.4, the problem has a sequence of positive solutions $u_k^*,\ k = 1, 2, \dots$, such that $\|u_k^*\| \to +\infty$.

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