ANNALES
POLONICI MATHEMATICI
100.1 (2011)

A note on the number of zeros of polynomials in an annulus

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Abstract. Let $p(z)$ be a polynomial of the form

$$p(z) = \sum_{j=0}^{n} a_j z^j, \qquad a_j \in \{-1, 1\}.$$

We discuss a sufficient condition for the existence of zeros of $p(z)$ in an annulus

$$\{z \in \mathbb{C} : 1 - c < |z| < 1 + c\},$$

where $c > 0$ is an absolute constant. This condition is a combination of Carleman's formula and Jensen's formula, which is a new approach in the study of zeros of polynomials.

1. Introduction. Let p denote a polynomial of the form

$$p(z) = \sum_{j=0}^{n} a_j z^j, \qquad |a_j| \le 1, \qquad a_j \in \mathbb{C}.$$

Such polynomials and various related classes have been studied from a number of points of view. In [2]–[6] and [8], the number and location of zeros of polynomials with bounded coefficients are considered. Many problems concerning polynomials with restricted coefficients are explored in [2] and [5].

In this paper, we are concerned with one of the open problems which are listed in [5]. We try to attack Question 4 in [5] which seems to be quite interesting (see Question A below).

Many results in this direction are based on Jensen's formula. Our purpose here is to determine whether the polynomials with coefficients -1 or 1 have at least one zero in some annulus by Carleman's formula approach. We will prove that the existence of zeros for such a polynomial in an annulus $\{z \in \mathbb{C} : 1 - c < |z| < 1 + c\}$ can be determined by the averaged number of zeros in $|z| < 1 + c$ and the sine value of the zeros in $|z| < 1 - c$.

2010 *Mathematics Subject Classification*: Primary 30B30; Secondary 11C08, 30C15.
Key words and phrases: polynomials, zeros, Carleman's formula.

DOI: 10.4064/ap100-1-3

Let us consider in greater detail the question of the number of polynomials in an annulus. The following earlier result related to this question is proved in [8].

THEOREM A ([8]). *For every $n \in \mathbb{N}$ there is a polynomial p_n of the form*

$$p_n(z) = \sum_{j=0}^{n} a_{j,n} z^j, \qquad |a_{j,n}| = 1, \qquad a_j \in \mathbb{C},$$

such that p_n has no zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c \log n}{n} < |z| < 1 + \frac{c \log n}{n} \right\},$$

where $c > 0$ is an absolute constant.

Furthermore, the following conjecture is put forward in [8].

CONJECTURE A ([8]). *Every polynomial of the form*

$$p(z) = \sum_{j=0}^{n} a_j z^j, \qquad a_j \in \{-1, 1\},$$

has at least one zero in the annulus

$$\{z \in \mathbb{C} : 1 - c/n < |z| < 1 + c/n\},$$

where $c > 0$ is an absolute constant.

In the recent paper [5], the following question is presented.

QUESTION A ([5]). *Establish whether every polynomial p of degree n with coefficients in the set $\{-1, 1\}$ has at least one zero in the annulus*

$$\{z \in \mathbb{C} : 1 - c/n < |z| < 1 + c/n\},$$

where $c > 0$ is an absolute constant.

Let us present the main result of this paper. With a sequence of numbers $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots\}$, $\lambda_n \in \mathbb{C}$, we associate the *averaged counting function* ([10])

$$N_\Lambda(r) = \int_0^r \frac{n_\Lambda(t)}{t}\, dt, \qquad n_\Lambda(t) = \sum_{|\lambda_n| \le t} 1, \tag{1}$$

and

$$C_\Lambda(t) = \sum_{|\lambda_n| \le t} |\sin \theta_n|. \tag{2}$$

THEOREM 1. *Let p be a polynomial of the form*

$$p(z) = \sum_{j=0}^{n} a_j z^j, \qquad a_j \in \{-1, 1\}, \tag{3}$$

and let $\Lambda = \{b_k\}_{k=1}^{n}$ *be its zero sequence. If for some* $c > 0$,

$$N_\Lambda(1+c) - \frac{16}{9} C_\Lambda(1-c) > 0,$$

where N_Λ *and* C_Λ *are defined in* (1) *and* (2) *respectively, then* $p(z)$ *has at least one zero in the annulus* $\{z \in \mathbb{C} : 1-c < |z| < 1+c\}$.

REMARK 1. We will show the existence of a positive constant c satisfying the condition in Theorem 1.

It is easy to see that

$$\int_{1-c}^{1+c} \frac{n_\Lambda(t)}{t}\, dt \geq \int_{1-c}^{1+c} \frac{n_\Lambda(1-c)}{t}\, dt \geq \int_{1-c}^{1+c} \frac{C_\Lambda(1-c)}{t}\, dt.$$

If we choose c satisfying

$$1 > c > \frac{e^{16/9} - 1}{e^{16/9} + 1},$$

then

$$N_\Lambda(1+c) = \int_{0}^{1+c} \frac{n_\Lambda(t)}{t}\, dt \geq \int_{1-c}^{1+c} \frac{n_\Lambda(1-c)}{t}\, dt \geq \int_{1-c}^{1+c} \frac{C_\Lambda(1-c)}{t}\, dt,$$

thus, we have

$$N_\Lambda(1+c) - \frac{16}{9} C_\Lambda(1-c) > 0.$$

2. Proof of the Theorem. In contrast to previous works on the number of zeros of polynomials, we will apply Carleman's formula which is often used to describe the property of functions analytic in a half annulus.

LEMMA 1 ([7], [10]). *Let* $f(z)$ *be a function analytic on* $S = \{z : \Im z \geq 0, |z| \leq R\}$. *Then*

$$\sum_{|b_n|<R,\, 0<\theta_n<\pi} \left(\frac{1}{|b_n|} - \frac{|b_n|}{R^2}\right) \sin\theta_n = \frac{1}{\pi R} \int_{0}^{\pi} \log|f(Re^{i\theta})| \sin\theta\, d\theta$$
$$+ \frac{1}{2\pi} \int_{0}^{R} \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log|f(x)f(-x)|\, dx + \frac{1}{2} \Im f'(0)$$

where $\{b_n\}$ *is the zero of* $f(z)$ *in* S *and* $\{\theta_n\}$ *is the corresponding sequence of arguments.*

Lemma 2 ([10]). *Let $f(z)$ be a function analytic on $\{z : |z| \le R\}$, with $f(0) \neq 0$, and let Λ be the zero sequence of f. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta = N_\Lambda(R) + \log|f(0)|,$$

where N_Λ is defined in (1).

We are now ready to prove Theorem 1.

Proof of Theorem 1. Without loss of generality, we may assume that $p(1-c) \neq 0$ and $p(1+c) \neq 0$. Applying Carleman's formula of Lemma 1 to $P(z) = \frac{p(z)}{c(1+c)^{n+1}}$ on $S = \{z : \Im z \ge 0, |z| \le 1+c\}$, we have

$$\sum_{|b_k|<1+c,\,0<\theta_k<\pi} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)\sin\theta_k = \frac{1}{\pi(1+c)}\int_0^{\pi} \log|P((1+c)e^{i\theta})|\sin\theta\,d\theta + \frac{1}{2\pi}\int_0^{1+c}\left(\frac{1}{x^2} - \frac{1}{(1+c)^2}\right)\log|P(x)P(-x)|\,dx \tag{4}$$

where $\{b_k\}$ are the zeros of $f(z)$ in S and $\{\theta_k\}$ are the arguments of $\{b_k\}$. By the same reasoning on $S' = \{z : \Im z \le 0,\ |z| \le 1+c\}$, we have

$$\sum_{|b_k|<1+c,\,\pi<\theta_k<2\pi} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)\sin\theta_k = \frac{1}{\pi(1+c)}\int_\pi^{2\pi} \log|P((1+c)e^{i\theta})|\sin\theta\,d\theta + \frac{1}{2\pi}\int_0^{1+c}\left(\frac{1}{x^2} - \frac{1}{(1+c)^2}\right)\log|P(x)P(-x)|\,dx \tag{5}$$

where $\{b_k\}$ are the zeros of $f(z)$ in S' and $\{\theta_k\}$ are the arguments of $\{b_k\}$. From (4) and (5), we have

$$\sum_{|b_k|<1+c,\,0<\theta_k<\pi} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)\sin\theta_k - \sum_{|b_k|<1+c,\,\pi<\theta_k<2\pi} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)\sin\theta_k \tag{6}$$

$$= \frac{1}{\pi(1+c)} \int_0^{\pi} \log|P((1+c)e^{i\theta})| \sin\theta \, d\theta$$
$$- \frac{1}{\pi(1+c)} \int_{\pi}^{2\pi} \log|P((1+c)e^{i\theta})| \sin\theta \, d\theta.$$

Since $\log a \le 0$ for $0 < a \le 1$, from (3) it is obvious that

$$\log|P((1+c)e^{i\theta})| \le 0. \tag{7}$$

By (6) and (7), we have

$$\begin{aligned}(8)\quad \sum_{|b_k|<1-c} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)|\sin\theta_k| \\ + \sum_{1-c\le|b_k|<1+c} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)|\sin\theta_k| \\ \ge \frac{1}{\pi(1+c)} \int_0^{2\pi} \log|P((1+c)e^{i\theta})| \, d\theta.\end{aligned}$$

We claim that *all the zeros of* $P(z)$ *are located in the annulus* $c_0 < |z| < 2$ *where* c_0 *is some positive constant satisfying* $c \le c_0 < 1$. Actually, the zeros of $p(z)$ and $P(z)$ are the same. If $0 < r = |z| \le c_0$ and

$$p(z) = \sum_{j=0}^{n} a_j z^j, \qquad a_j \in \{-1, 1\},$$

then

$$\begin{aligned}|p(z)| &\ge |a_0| - |a_1 z| - \cdots - |a_n z^n| = 1 - (c_0 + c_0^2 + \cdots + c_0^n) \\ &= 1 - \frac{c_0(1-c_0^n)}{1-c_0} \ge 1 - (1 - c_0^n) > 0.\end{aligned}$$

And for $r = |z| \ge 2$,

$$\begin{aligned}|p(z)| &\ge |a_n z^n| - |a_{n-1} z^{n-1}| - \cdots - |a_1 z| - |a_0| \\ &= r^n - r^{n-1} - \cdots - r - 1 = r^n - \frac{r^n - 1}{r - 1} > 0.\end{aligned}$$

Whence, by combining (8) and Lemma 2, we have

$$\begin{aligned}(9)\quad \sum_{1-c\le|b_k|<1+c} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)|\sin\theta_k| \\ \ge \frac{2}{1+c} N_{\Lambda}(1+c) - \sum_{1/2<|b_k|<1-c} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)|\sin\theta_k|.\end{aligned}$$

Let m denote the number of zeros of $p(z)$ in $\{z \in \mathbb{C} : 1 - c \le |z| < 1 + c\}$. By (9), we have

$$\sum_{1-c\le|b_k|<1+c} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)|\sin\theta_k| \le \sum_{1-c\le|b_k|<1+c} \frac{4c}{(1-c)(1+c)^2} = m\frac{4c}{(1-c)(1+c)^2}.$$

Since $c < 1$ and

$$-\sum_{1/2<|b_k|<1-c} \left(\frac{1}{|b_k|} - \frac{|b_k|}{(1+c)^2}\right)|\sin\theta_n| \ge -\frac{16}{9}C_\Lambda(1-c),$$

if

$$N_\Lambda(1+c) - \frac{16}{9}C_\Lambda(1-c) > 0$$

then $p(z)$ has at least one zero in the annulus $\{z \in \mathbb{C} : 1 - c < |z| < 1 + c\}$.

Acknowledgments. This research was supported by YunNan Provincial Basic Research Foundation (Grant No. 2009ZC013X) and Basic Research Foundation of Education Bureau of YunNan Province (Grant No. 09Y0079).

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Received 3.10.2009
and in final form 21.7.2010 (2096)