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Some convergence theorems for HK-integral in locally convex spaces

by Sokol Bush Memetaj (Elbasan)

Abstract. We present some convergence theorems for the HK-integral of functions taking values in a locally convex space. These theorems are based on the concept of HK-equiintegrability.

1. Introduction. We define the HK-integral of functions defined on a compact subinterval S of $\mathbb{R}^m$, $m \geq 1$, and taking values in a locally convex space V. In the case of functions taking values in a Banach space, our definition is equivalent to [10, Definition III.2.2], and for the case when $m = 1$ and $S = [0, 1]$, it is equivalent to [7, Definition 4].

We present three convergence theorems for HK-integral which are based on the notion of HK-equiintegrability of a sequence $f_n : S \to V$ that converges pointwise to a function $f : S \to V$ in the standard topology or in the weak topology of the locally convex space V. These are Theorems 2.1, 2.5 and 2.6. For functions taking values in a Banach space, our Definition 1.3 of HK-equiintegrability is equivalent to [10, Definition III.5.1]. This notion was first introduced in [6] for real valued functions and permitted the proof of a convergence theorem for a pointwise convergent sequence of Henstock–Kurzweil integrable functions (see also [1], [10], [3], [4] and [2]). Another convergence theorem of this sort for functions taking values in a complete locally convex space has been shown in [8, Theorem 5].

The differences between Theorems 2.1, 2.5 and 2.6 consist in the completeness assumptions on V (sequential completeness, completeness and weak sequential completeness), and the topology on the space V (standard or weak). In Theorem 2.1 we assume that a sequence (f_n) converges pointwise to a function f in the standard topology of V (V is a sequentially complete locally convex space). In Theorems 2.5 and 2.6 we assume that

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(f_n) converges pointwise to f in the weak topology of V, under different completeness assumptions on V.

Theorem 2.1 is a generalization of [10, Theorem III.5.2]. For the case when $m = 1$ and $S = [0,1]$, Theorem 2.5 was proved in [8, Theorem 5], by a different approach. It can be proved that Theorem 2.1 follows from Lemma 2.3 (the Banach version of Theorem 2.5).

Theorem 2.1 and Lemma 2.3 are used in the proof of Theorem 2.6. Theorem 2.6 is useful for the case when V is weakly sequentially complete but not complete. In this case we use Theorem 2.6 instead of Theorem 2.5. Do locally convex spaces of this type exist? According to [5, p. 159], there exist locally convex spaces that are reflexive but not complete. Since a reflexive space is semi-reflexive (see [9, p. 144]), we deduce by [9, Theorem IV.5.5] that a reflexive locally convex space is weakly quasi-complete. Since a quasi-complete space is sequentially complete or semi-complete, a reflexive locally convex space is weakly sequentially complete. Consequently, there exist locally convex spaces that are weakly sequentially complete but not complete.

Throughout this paper, V is a locally convex space with its topology τ and topological dual V'. By P we denote the family of all continuous seminorms in V; for every $p \in P$, $\widetilde{V}^p$ denotes the quotient vector space of V with respect to the equivalence relation $x \sim^p y \Leftrightarrow p(x-y) = 0$; the map $\phi_p : V \to \widetilde{V}^p$ is the canonical quotient map, thus $\phi_p(x)$ is the equivalence class of $x \in V$ with respect to "$\sim^p$"; the quotient normed space $(\widetilde{V}^p, \widetilde{p})$ is called the *normed component* of V, where $\widetilde{p}(\phi_p(x)) = p(x)$ for each $x \in V$; the Banach space $(\overline{V}^p, \overline{p})$ which is the completion of $(\widetilde{V}^p, \widetilde{p})$ is called the *Banach component* of V; V'_p, $\widetilde{V}'_p$ and $\overline{V}'_p$ are the topological duals of (V, p), $(\widetilde{V}^p, \widetilde{p})$ and $(\overline{V}^p, \overline{p})$, respectively; $\sigma(V, V')$ is the weak topology of V. It is easy to see that

$$V' = \{\widetilde{v}'_p \circ \phi_p : \widetilde{v}'_p \in \widetilde{V}'_p,\ p \in P\}, \tag{1.1}$$

because for every $v' \in V'$, we have $|v'(\cdot)| \in P$.

For every $p, q \in P$ such that $p \le q$, we define the map

$$\widetilde{g}_{pq} : \widetilde{V}^q \to \widetilde{V}^p, \qquad \widetilde{g}_{pq}(w_q) = w_p, \qquad w_q \in \widetilde{V}^q,$$

where $w_p = \phi_p(x)$ for some $x \in w_q$. Since for every $y \in w_q$ we have $\phi_p(y) = w_p$, the map $\widetilde{g}_{pq}$ is well defined. It is easily proved that $\widetilde{g}_{pq}$ is a continuous linear map. We also define the map $\overline{g}_{pq} : \overline{V}^q \to \overline{V}^p$ as the continuous linear extension of $\widetilde{g}_{pq}$, for every $p, q \in P$ such that $p \le q$. By $\varprojlim((\overline{V}^p, \overline{p}), \overline{g}_{pq})$, we denote the projective limit of the family $\{(\overline{V}^p, \overline{p}) : p \in P\}$ with respect to the mappings $\overline{g}_{pq}$ $(p, q \in P,\ p \le q)$. For the definition of the projective limit see [9, p. 52].

Assume that a compact subinterval $S = [a_1, b_1] \times \cdots \times [a_m, b_m]$ of $\mathbb{R}^m$, $m \geq 1$, and a function $f : S \to V$ are given. Let $D = \{(s_i, J_i) : i = 1, \ldots, n\}$ be a set such that $s_i \in S$ and J_i is a compact subinterval of S for $i = 1, \ldots, n$; then D is called an *HK-partition* of S if:

1. $\bigcup_{i=1}^{n} J_i = S$,
2. $s_i \in J_i$ for $i = 1, \ldots, n$,
3. $(J_i)_{i=1}^{n}$ is a finite sequence of pairwise non-overlapping intervals (i.e., with pairwise disjoint interiors).

A function $\delta : S \to (0, +\infty)$ is called a *gauge* on S; an HK-partition $D = \{(s_i, J_i) : i = 1, \ldots, n\}$ of S is called *δ-fine* (written $D \ll \delta$) if

$$J_i \subset B(s_i, \delta(s_i)),$$

where $B(s_i, \delta(s_i))$ is the ball in $\mathbb{R}^m$ centered at s_i with radius $\delta(s_i)$, for $i = 1, \ldots, n$. We set

$$S(f, D) = \sum_{i=1}^{n} f(s_i)\mu_L(J_i),$$

where μ_L is the Lebesgue outer measure in S.

Definition 1.1. A function $f : S \to V$ is called *HK-integrable in V* if there exists a vector $I_f \in V$ with the following property: for every $p \in P$ and $\epsilon > 0$ there exists a gauge $\delta_\epsilon^{(p)}$ on S such that

$$p(S(f, D) - I_f) < \epsilon$$

for every HK-partition D of S such that $D \ll \delta_\epsilon^{(p)}$. Since the family P is separated, the vector I_f is uniquely determined and it is called the *HK-integral* of f on S in V, and denoted

$$(\mathrm{HK})\int_S f = I_f.$$

From Definition 1.1, we obtain Theorem 1.2 below. This theorem guarantees a simple and important relation between HK-integral in a locally convex space and HK-integral in the normal components of this locally convex space.

Theorem 1.2. *A function $f : S \to V$ is HK-integrable in V if and only if there exists a vector $I_f \in V$ such that for every $p \in P$ the function $\phi_p \circ f$ is HK-integrable in the normed component $(\widetilde{V}^p, \widetilde{p})$ and*

$$(\mathrm{HK})\int_S \phi_p \circ f = \phi_p(I_f).$$

Definition 1.3. A family $\mathcal{M}$ of functions $f : S \to V$ is called *HK-equiintegrable* in V if every $f \in \mathcal{M}$ is HK-integrable in V and for every

$p \in P$ and $\epsilon > 0$ there exists a gauge $\delta_\epsilon^{(p)}$ on S such that

$$p\Big(S(f,D) - (\mathrm{HK})\int_S f\Big) < \epsilon,$$

for every HK-partition D of S such that $D \ll \delta_\epsilon^{(p)}$ and for all $f \in \mathcal{M}$.

2. Some convergence theorems for the HK-integral. In the first convergence theorem we assume that a sequence (f_n) converges pointwise to a function f in V with respect to the standard topology.

THEOREM 2.1. *Let V be a sequentially complete locally convex space. If a sequence (f_n), where $f_n : S \to V$, is HK-equiintegrable in V and converges to a function $f : S \to V$ in V, then f is HK-integrable in V and*

$$\lim_{n\to\infty} (\mathrm{HK})\int_S f_n = (\mathrm{HK})\int_S f$$

in V.

Proof. The sequence (f_n) converges to f in V if and only if for each $p \in P$ the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$. Therefore, (f_n) converges to f in V if and only if $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$, for each $p \in P$.

By Definition 1.3, for each $p \in P$ the sequence $(\phi_p \circ f_n)$ is HK-equiintegrable in the Banach component $(\overline{V}^p, \overline{p})$.

Thus, for each $p \in P$ the conditions of Theorem III.5.2 in [10] are satisfied. Hence, for each $p \in P$ the function $\phi_p \circ f$ is HK-integrable in the Banach component $(\overline{V}^p, \overline{p})$ and

$$\lim_{n\to\infty} (\mathrm{HK})\int_S \phi_p \circ f_n = (\mathrm{HK})\int_S \phi_p \circ f \tag{2.1}$$

in $(\overline{V}^p, \overline{p})$.

According to Theorem 1.2 we have

$$(\mathrm{HK})\int_S \phi_p \circ f_n = \phi_p\Big((\mathrm{HK})\int_S f_n\Big) \in \widetilde{V}^p$$

for each $n \in \mathbb{N}$ and $p \in P$. Therefore, for each $p \in P$ the sequence $(\phi_p((\mathrm{HK})\int_S f_n))$ is a Cauchy sequence in $(\widetilde{V}^p, \widetilde{p})$. Hence $((\mathrm{HK})\int_S f_n)$ is a Cauchy sequence in the sequentially complete locally convex space V and so it converges to $I_f \in V$ in V. By (2.1), this implies that

$$\phi_p(I_f) = (\mathrm{HK})\int_S \phi_p \circ f$$

for every $p \in P$. Consequently, by Theorem 1.2, the function f is HK-integrable in V and

$$\lim_{n\to\infty} (\mathrm{HK}) \int_S f_n = (\mathrm{HK}) \int_S f$$

in V. ■

In Theorems 2.5 and 2.6 below we assume that (f_n) converges pointwise to f in V with respect to the weak topology $\sigma(V, V')$. The following lemmas prepare the proof of Theorem 2.5.

LEMMA 2.2. *Assume that a locally convex space L is given and let $T: V \to L$ be a continuous linear function. If a function $f: S \to V$ is HK-integrable in V, then the function $T(f)$ is HK-integrable in L and*

$$(\mathrm{HK}) \int_S T(f) = T\Big((\mathrm{HK}) \int_S f\Big).$$

Proof. Let Q be the family of all continuous seminorms in L. Let q be an arbitrary element of Q. Since T is a continuous linear function, for a given $\epsilon > 0$ there exist $p \in P$ and $\epsilon' > 0$ such that

$$p(x) < \epsilon' \;\Rightarrow\; q(T(x)) < \epsilon, \qquad x \in V. \tag{2.2}$$

According to Definition 1.1, there exists a gauge $\delta_{\epsilon'}^{(p)}$ on S such that

$$p(S(f, D) - I_f) < \epsilon'$$

for every HK-partition D of S such that $D \ll \delta_{\epsilon'}^{(p)}$. Therefore, by (2.2),

$$q(S(T(f), D) - T(I_f)) < \epsilon$$

for every such D. Because of the arbitrariness of q the function $T(f)$ is HK-integrable in L and

$$(\mathrm{HK}) \int_S T(f) = T\Big((\mathrm{HK}) \int_S f\Big). \;■$$

In the next lemma, $(X, \|\cdot\|)$ is a Banach space, X' is its topological dual and $B(X')$ is the closed unit ball in X'.

LEMMA 2.3. *If the sequence of functions $f_n : S \to X$ is HK-equiintegrable in $(X, \|\cdot\|)$ and converges to $f : S \to X$ in the weak topology $\sigma(X, X')$, then f is HK-integrable in $(X, \|\cdot\|)$ and*

$$\lim_{n\to\infty} (\mathrm{HK}) \int_S f_n = (\mathrm{HK}) \int_S f$$

in the weak topology $\sigma(X, X')$.

Proof. According to the definition of HK-equiintegrability in Banach spaces, for a given $\epsilon > 0$ there exists a gauge δ_ϵ on S such that

$$\Big\| S(f_n, D) - (\mathrm{HK}) \int_S f_n \Big\| < \epsilon$$

for every HK-partition D of S such that $D \ll \delta_\epsilon$ and for all $n \in \mathbb{N}$. According to Lemma 2.2, for every $x' \in X'$ the inequality

$$\Big|S(x'(f_n), D) - (\mathrm{HK}) \int_S x'(f_n)\Big| = \Big|x'\Big(S(f_n, D) - (\mathrm{HK}) \int_S f_n\Big)\Big| \le \|x'\|\epsilon \tag{2.3}$$

holds for every D as above and for all $n \in \mathbb{N}$. Therefore, for every $x' \in X'$ the sequence $(x'(f_n))$ is HK-equiintegrable in $(\mathbb{R}, |\cdot|)$. Thus, for every $x' \in X'$ this sequence is HK-equiintegrable in $(\mathbb{R}, |\cdot|)$ and converges to $x'(f)$ in $(\mathbb{R}, |\cdot|)$. Then, according to [10, Theorem III.5.2], the function $x'(f)$ is HK-integrable in $(\mathbb{R}, |\cdot|)$ and

$$\lim_{n\to\infty} (\mathrm{HK}) \int_S x'(f_n) = (\mathrm{HK}) \int_S x'(f) \tag{2.4}$$

for every $x' \in X'$.

If we prove that the family $\{x'(f) : x' \in B(X')\}$ is HK-equiintegrable in $(\mathbb{R}, |\cdot|)$, then by [10, Proposition III.5.4] the function f is HK-integrable in $(X, \|\cdot\|)$.

Notice that

$$\begin{aligned}
&\Big|S(x'(f), D) - (\mathrm{HK}) \int_S x'(f)\Big| \\
&\quad \le |S(x'(f), D) - S(x'(f_n), D)| + \Big|S(x'(f_n), D) - (\mathrm{HK}) \int_S x'(f_n)\Big| \\
&\qquad + \Big|(\mathrm{HK}) \int_S x'(f_n) - (\mathrm{HK}) \int_S x'(f)\Big|.
\end{aligned}$$

Assume that an arbitrary $x' \in B(X')$ and an arbitrary HK-partition D of S such that $D \ll \delta_\epsilon$ are given. Since

$$\lim_{n\to\infty} S(x'(f_n), D) = S(x'(f), D) \quad \text{and} \quad \lim_{n\to\infty} (\mathrm{HK}) \int_S x'(f_n) = (\mathrm{HK}) \int_S x'(f),$$

there exists $n_{(x',D)} \in \mathbb{N}$ such that

$$|S(x'(f_{n_{(x',D)}}), D) - S(x'(f), D)| < \epsilon$$

and

$$\Big|(\mathrm{HK}) \int_S x'(f_{n_{(x',D)}}) - (\mathrm{HK}) \int_S x'(f)\Big| < \epsilon.$$

Also, by (2.3), we have

$$\Big|S(x'(f_{n_{(x',D)}}), D) - (\mathrm{HK}) \int_S x'(f_{n_{(x',D)}})\Big| \le \epsilon.$$

Consequently, because of the arbitrariness of x' and D, we see that

$$\Big|S(x'(f), D) - (\mathrm{HK}) \int_S x'(f)\Big| < 3\epsilon$$

for every HK-partition D of S such that $D \ll \delta_\epsilon$ and for all $x' \in B(X')$. This means that $\{x'(f) : x' \in B(X')\}$ is HK-equiintegrable in $(\mathbb{R}, |\cdot|)$. Hence, f is HK-integrable in $(X, \|\cdot\|)$. According to (2.4) and Lemma 2.2, we obtain

$$\lim_{n\to\infty} x'\Big((\mathrm{HK})\int_S f_n\Big) = x'\Big((\mathrm{HK})\int_S f\Big)$$

for every $x' \in X'$. Consequently,

$$\lim_{n\to\infty} (\mathrm{HK})\int_S f_n = (\mathrm{HK})\int_S f$$

in the weak topology $\sigma(X, X')$ and the proof is finished. ∎

Lemma 2.4. *Let V be a complete locally convex space. A function $f : S \to V$ is HK-integrable in V if and only if for every $p \in P$ the function $\phi_p \circ f$ is HK-integrable in the Banach component $(\overline{V}^p, \overline{p})$. In this case,*

$$\phi_p\Big((\mathrm{HK})\int_S f\Big) = (\mathrm{HK})\int_S \phi_p \circ f$$

for every $p \in P$.

Proof. The "only if" part is easily proved by applying Theorem 1.2.

Conversely, assume that for every $p \in P$ the function $\phi_p \circ f$ is HK-integrable in $(\overline{V}^p, \overline{p})$. We set $(\mathrm{HK})\int_S \phi_p \circ f = \overline{I}_p$ for $p \in P$.

Assume that two arbitrary continuous seminorms p and q such that $p \le q$ are given. According to Lemma 2.2, we have

$$\begin{aligned}\overline{g}_{pq}\Big((\mathrm{HK})\int_S \phi_q \circ f\Big) &= (\mathrm{HK})\int_S (\overline{g}_{pq} \circ \phi_q) \circ f \\ &= (\mathrm{HK})\int_S (\widetilde{g}_{pq} \circ \phi_q) \circ f = (\mathrm{HK})\int_S \phi_p \circ f\end{aligned}$$

or $\overline{g}_{pq}(\overline{I}_q) = \overline{I}_p$. Consequently, we obtain

$$(\overline{I}_p) \in \varprojlim((\overline{V}^p, \overline{p}), \overline{g}_{pq}),$$

and by [9, Theorem II.5.4], there exists $I_f \in V$ such that $\phi_p(I_f) = \overline{I}_p$ for each $p \in P$. Hence, by Theorem 1.2, the function f is HK-integrable in V and the proof is finished. ∎

Now, we are ready to present the second convergence theorem.

Theorem 2.5. *Let V be a complete locally convex space. If a sequence (f_n), where $f_n : S \to V$, is HK-equiintegrable in V and converges to $f : S \to V$ in the weak topology $\sigma(V, V')$, then f is HK-integrable in V and*

$$\lim_{n\to\infty} (\mathrm{HK})\int_S f_n = (\mathrm{HK})\int_S f$$

in the weak topology $\sigma(V, V')$.

Proof. According to Definition 1.3, the sequence $(\phi_p \circ f_n)$ is HK-equiintegrable in the Banach component $(\overline{V}^p, \overline{p})$ for each $p \in P$.

By (1.1), for every $p \in P$ the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$ in the weak topology. Therefore, for every $p \in P$ this sequence also converges to $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$ in the weak topology.

Thus, for every $p \in P$ the conditions of Lemma 2.3 are satisfied. Hence for every $p \in P$ the function $\phi_p \circ f$ is HK-integrable in $(\overline{V}^p, \overline{p})$ and

$$\lim_{n\to\infty} \overline{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f_n\Big) = \overline{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f\Big) \tag{2.5}$$

for every $\overline{v}'_p \in \overline{V}'_p$.

We know that for every $p \in P$ the function $\phi_p \circ f$ is HK-integrable in the Banach component $(\overline{V}^p, \overline{p})$. Then, by Lemma 2.4, f is HK-integrable in V and

$$(\mathrm{HK})\int_S \phi_p \circ f = \phi_p\Big((\mathrm{HK})\int_S f\Big) \in \widetilde{V}^p \tag{2.6}$$

for every $p \in P$. Again by applying Lemma 2.4 for every f_n, we also obtain

$$(\mathrm{HK})\int_S \phi_p \circ f_n = \phi_p\Big((\mathrm{HK})\int_S f_n\Big) \in \widetilde{V}^p \tag{2.7}$$

for every $p \in P$.

Since every $\overline{v}'_p \in \overline{V}'_p$ is the continuous extension of an element $\widetilde{v}'_p \in \widetilde{V}'_p$, by (2.7), (2.6) and (2.5) it follows that for every $p \in P$, the equality

$$\lim_{n\to\infty} (\widetilde{v}'_p \circ \phi_p)\Big((\mathrm{HK})\int_S f_n\Big) = (\widetilde{v}'_p \circ \phi_p)\Big((\mathrm{HK})\int_S f\Big) \tag{2.8}$$

holds for every $\widetilde{v}'_p \in \widetilde{V}'_p$.

Now, let v' be an arbitrary element of V'. According to (1.1), there exist $p \in P$ and $\widetilde{v}'_p \in \widetilde{V}'_p$ such that $v' = \widetilde{v}'_p \circ \phi_p$. The last equality together with (2.8) implies

$$\lim_{n\to\infty} v'\Big((\mathrm{HK})\int_S f_n\Big) = v'\Big((\mathrm{HK})\int_S f\Big)$$

for every $v' \in V'$ and the proof is finished. ∎

Finally, we present the third convergence theorem.

THEOREM 2.6. *Let V be a locally convex space which is sequentially complete with respect to the weak topology $\sigma(V, V')$. If a sequence (f_n), where $f_n : S \to V$, is HK-equiintegrable in V and converges pointwise to the function $f : S \to V$ in the weak topology, then f is HK-integrable in V and*

$$\lim_{n\to\infty} (\mathrm{HK})\int_S f_n = (\mathrm{HK})\int_S f$$

in the weak topology.

Proof. The locally convex space $(V, \sigma(V, V'))$ is Hausdorff (see [11, Cor. IV.6.1, p. 107]). Denote by P' the family of all continuous seminorms in $(V, \sigma(V, V'))$. Since $P' \subset P$, the sequence (f_n) is HK-equiintegrable in $(V, \sigma(V, V'))$ and converges to f in this space. Thus, the conditions of Theorem 2.1 are satisfied. Hence there exists $I_f \in V$ such that

$$\lim_{n\to\infty} p'\Big((\mathrm{HK})\int_S f_n - I_f\Big) = 0$$

for every $p' \in P'$, and consequently

$$\lim_{n\to\infty} v'\Big((\mathrm{HK})\int_S f_n\Big) = v'(I_f) \tag{2.9}$$

for every $v' \in V'$, because $|v'(\cdot)| \in P'$.

Let p be any continuous seminorm in V. According to (1.1), the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$ with respect to the weak topology. Consequently, $(\phi_p \circ f_n)$ also converges to $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$ with respect to the weak topology. Thus, $(\phi_p \circ f_n)$ is HK-equiintegrable in $(\widetilde{V}^p, \widetilde{p})$ and converges to $\phi_p \circ f$ in $(\overline{V}^p, \overline{p})$ with respect to the weak topology. Then, by Lemma 2.3, the function $\phi_p \circ f$ is HK-integrable in $(\overline{V}^p, \overline{p})$ and

$$\lim_{n\to\infty} \overline{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f_n\Big) = \overline{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f\Big)$$

for every $\overline{v}'_p \in \overline{V}'_p$; since every $\overline{v}'_p \in \overline{V}'_p$ is the continuous extension of an element $\widetilde{v}'_p \in \widetilde{V}'_p$, it follows that

$$\lim_{n\to\infty} \widetilde{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f_n\Big) = \overline{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f\Big) \tag{2.10}$$

for every $\widetilde{v}'_p \in \widetilde{V}'_p$, where $\overline{v}'_p$ is the continuous extension of $\widetilde{v}'_p$.

By applying Lemma 2.2, for every $\phi_p \circ f_n$ we obtain

$$\widetilde{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f_n\Big) = (\mathrm{HK})\int_S \widetilde{v}'_p \circ (\phi_p \circ f_n) = (\mathrm{HK})\int_S v'_p \circ f_n,$$

where $v'_p = \widetilde{v}'_p \circ \phi_p$, and again by applying Lemma 2.2, for every n,

$$(\mathrm{HK})\int_S v'_p \circ f_n = v'_p\Big((\mathrm{HK})\int_S f_n\Big),$$

and consequently

$$\widetilde{v}'_p\Big((\mathrm{HK})\int_S \phi_p \circ f_n\Big) = v'_p\Big((\mathrm{HK})\int_S f_n\Big). \tag{2.11}$$

Hence, by (2.10) and (2.11), we get

$$\lim_{n\to\infty} v_p'\Big((\mathrm{HK})\int_S f_n\Big) = \overline{v}_p'\Big((\mathrm{HK})\int_S \phi_p\circ f\Big). \tag{2.12}$$

Also, according to (2.9), we have

$$\lim_{n\to\infty} v_p'\Big((\mathrm{HK})\int_S f_n\Big) = v_p'(I_f) = \widetilde{v}_p'(\phi_p(I_f)) \tag{2.13}$$

for every $v_p' \in V_p'$. Hence, by (2.12) and (2.13), we obtain

$$\widetilde{v}_p'(\phi_p(I_f)) = \overline{v}_p'\Big((\mathrm{HK})\int_S \phi_p\circ f\Big)$$

for every $\widetilde{v}_p' \in \widetilde{V}_p'$, where $\overline{v}_p'$ is the continuous extension of $\widetilde{v}_p'$. Consequently,

$$\overline{v}_p'(\phi_p(I_f)) = \overline{v}_p'\Big((\mathrm{HK})\int_S \phi_p\circ f\Big)$$

for every $\overline{v}_p' \in \overline{V}_p'$ and according to [11, Corollary IV.6.2], this means that

$$(\mathrm{HK})\int_S \phi_p\circ f = \phi_p(I_f) \in \widetilde{V}^p.$$

Therefore, by Theorem 1.2, the function f is HK-integrable and

$$(\mathrm{HK})\int_S f = I_f. \ \blacksquare$$

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Sokol Bush Memetaj
Mathematics Department
Science Natural Faculty
University of Elbasan
Elbasan, Albania
E-mail: alb_sokol@yahoo.co.uk

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