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Entropy of distal groups, pseudogroups, foliations and laminations

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Abstract. A distality property for pseudogroups and foliations is defined. Distal foliated bundles satisfying some growth conditions are shown to have zero geometric entropy in the sense of É. Ghys, R. Langevin and P. Walczak [Acta Math. 160 (1988)].

Introduction. In [Pa], Parry proved directly that distal homeomorphisms of compact metric spaces have zero topological entropy. In the introduction there, the author says that this result can be obtained *via* the Furstenberg [Fu] structure theorem for distal flows. Later on, Ghys, Langevin and the second author [GLW] introduced a notion of entropy for foliations of Riemannian manifolds and finitely generated pseudogroups (in particular, groups) of local (in particular, global) homeomorphisms. In [CC], it was observed that this notion applies also to laminations (or foliated spaces as the authors call laminations there). The value of entropy itself depends on either the Riemannian structure or the finite generating set under consideration; however if this value is equal to zero for one such structure, then it vanishes for any other. Therefore, one can speak about zero or non-zero entropy foliations (groups, pseudogroups, laminations).

Our goal here is to provide a class of distal foliations (pseudogroups, groups, laminations) which have zero entropy in the sense of [GLW]. Even if distal groups of homeomorphisms of compact spaces admit invariant measures ([Fu, Thm. 12.3]), the lack of good variational principle for our systems (groups, foliations, etc.) forces us to use the structure theorem for distal groups. The key tool is the result (analogous to that of Bowen, [Bo, Thm. 17]) relating entropies of a given group acting on the total space, base and between the fibres.

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1. Distality. Recall that a group G of homeomorphisms of a metric space (X, d) is said to be *distal* if for any distinct points x and y of X, the distances $d(g(x), g(y))$, $g \in G$, are bounded away from zero. For compact X, this property is purely topological, i.e. independent of the distance function d. Naively, distality of a pseudogroup of local homeomorphisms of X could be defined in the same way assuming only that g ranges over all the elements of the pseudogroup which are defined at x and y. However, this approach fails immediately: even pseudogroups of local isometries are not distal in this sense.

EXAMPLE 1. If α and β are real numbers with $\alpha/\beta \notin \mathbb{Q}$ and our pseudogroup on the circle S^1 is generated by the rotations R_α and R_β by the angles α and β, respectively, and p_0 and p_1 are distinct points of S^1 splitting it into two (open) arcs I_1 and I_2, then the map $h : I_1' \cup I_2' \to S^1$ defined for some arcs $I_1' \subset I_1$ and $I_2' \subset I_2$ with the common end point, say p_0, by $f|I_1 = R_\alpha$ and $f|I_2 = R_\beta$ belongs to our pseudogroup because pseudogroups are assumed to be closed under the operation of taking unions of their elements. One can find points $x \in I_1$ and $y \in I_2$ and a sequence $n_k \to \infty$ of natural numbers such that $d(f^{n_k}x, f^{n_k}y) \to 0$ as $k \to \infty$.

Therefore, a pseudogroup Γ of local homeomorphisms of X will here be called *distal* whenever there exists a symmetric set S generating Γ, closed under composition and such that

$$\inf\{d(g(x), g(y));\ g \in S,\ x, y \in D_g\} > 0 \tag{1.1}$$

for all $x, y \in X$, $x \neq y$. (Hereafter, D_g denotes the domain of g.)

This definition is analogous to that in [AC], where such a pseudogroup is called *strongly equicontinuous* whenever (up to some simplification) there exists such a generating set S with the following property: for any $\epsilon > 0$ one can find $\delta > 0$ such that the implication

$$d(x, y) < \delta \ \Rightarrow\ d(g(x), g(y)) < \epsilon \tag{1.2}$$

holds for all $g \in S$, x and $y \in D_g$. Certainly, all strongly equicontinuous pseudogroups are distal.

One can also think about a pseudogroup Γ of local diffeomorphisms of a Riemannian manifold X to be *infinitesimally distal* whenever

$$\inf\{\|dh(v)\|; h \in \Gamma, x \in D_h\} > 0$$

for all nonzero vectors $v \in T_xX$, $x \in X$. Here, the problem caused by admitting in pseudogroups arbitrary unions of elements as in Example 1 does not appear but the notions of "standard" and infinitesimal distality are independent as has been shown for groups (or even, single transformations) by examples in [Re] (see also [MN]): (1) a diffeomorphism of S^1 with irrational rotation number which is not C^1-conjugate to a rotation is distal but not infinitesimally distal and (2) the classical horocycle flow is infinitesimally distal but not distal (even after arbitrary reparametrization).

Let G and H be pseudogroups acting on topological spaces X and Y, respectively. Following Haefliger ([Hae]) we say that an *étale morphism* $\Phi : G \to H$ is a maximal collection Φ of homeomorphisms of open subsets of X onto open subsets of Y such that:

1. if $\phi \in \Phi$, $g \in G$ and $h \in H$, then $h \circ \phi \circ g \in \Phi$,
2. the domains D_ϕ of the elements of Φ form a covering of X,
3. if $\phi, \psi \in \Phi$, then $\phi \circ \psi^{-1} \in H$.

An étale morphism is called an *equivalence* if the collection $\Phi^{-1} = \{\phi^{-1} : \phi \in \Phi\}$ is also an étale morphism of H into G. We say that an étale morphism $\Phi : G \to H$ is *generated* by a subset $\Phi_0 \subset \Phi$ if

$$\Phi = \{h \circ \phi \circ g : g \in G,\ h \in H,\ \phi \in \Phi_0\}.$$

Finally, the pseudogroups G and H are said to be *equivalent* if there exists an equivalence $\Phi : G \to H$; moreover, G and H are *finitely equivalent* if the equivalence $\Phi : G \to H$ is generated by a finite collection Φ_0.

Following the lines of the proof of Lemma 8.8 in [AC] one can establish the following.

PROPOSITION 1. *If two pseudogroups Γ_1 and Γ_2 are finitely equivalent and one of them is distal, then so is the other.*

Proof. Denote by S a symmetric set of generators of Γ_1 that is closed under compositions and satisfies the conditions of our definition of distality. We may also assume (Definition 8.4(ii) in [AC]) that S is closed under restrictions to open sets, thus each $g \in \Gamma_1$ is a combination of maps in S. Assume that Γ_1 is a distal pseudogroup and choose a finite set Φ_0 generating an equivalence Φ of Γ_1 and Γ_2.

Using the same arguments as in the proof of Lemma 8.8 in [AC] we deduce that the set

$$S' := \{\phi \circ g \circ \psi^{-1} : g \in S,\ \phi, \psi \in \Phi_0\}$$

is symmetric, generates Γ_2 and is closed under compositions. Take arbitrary distinct points $x_1, y_1 \in X$. The distality of Γ_1 yields

$$\inf\{d(g(x_1), g(y_1)) : g \in S,\ x_1, y_1 \in D_g\} > 0. \tag{1.3}$$

Fix ψ in Φ_0, put $x_\psi := \psi(x_1)$ and $y_\psi := \psi(y_1)$. Notice that Φ_0 is a finite family of local homeomorphisms, thus

$$\inf\{d(\phi \circ g \circ \psi^{-1}(x_\psi), \phi \circ g \circ \psi^{-1}(y_\psi)) : g \in S,\ \phi \in \Phi_0,\ x_\psi, y_\psi \in D_{\phi \circ g \circ \psi^{-1}}\} > 0 \tag{1.4}$$

and therefore we conclude that for any pair of distinct points $x_2, y_2 \in Y$,

$$\inf\{d(\phi\circ g\circ\psi^{-1}(x_2),\phi\circ g\circ\psi^{-1}(y_2)) : g\in S, \phi,\psi\in\Phi_0, x_2, y_2\in D_{\phi\circ g\circ\psi^{-1}}\}>0,$$

which completes the proof. ∎

Since holonomy pseudogroups (acting on different transversals) of a given foliation (or lamination) $\mathcal{F}$ are equivalent (and finitely equivalent when the foliated space under consideration is compact), one may call a foliation (lamination) of a compact manifold (space) *distal* whenever its holonomy pseudogroup is distal. For example, Riemannian foliations and foliated bundles obtained from suspensions of discrete distal groups are distal. (Several examples of distal groups and a classification of all such groups are provided in [Fu].)

2. Entropy. Let $\mathcal{G}$ be a compactly generated pseudogroup on a compact metric space (X,d) generated by a finite symmetric set $\mathcal{G}_1$. Following [GLW] we shall say that two points x and y of X are (n,ϵ)-*separated* whenever there exists a product $g\in\mathcal{G}$ of $k\le n$ generators such that x and y lie in the domain of g and $d(g(x),g(y))\ge\epsilon$. Let $s(n,\epsilon,\mathcal{G}_1)$ denote the maximal number of (n,ϵ)-separated points of X. Let

$$s(\epsilon,\mathcal{G}_1)=\limsup_{n\to\infty}\frac{1}{n}\log s(n,\epsilon,\mathcal{G}_1)$$

and

$$h(\mathcal{G},\mathcal{G}_1)=\lim_{\epsilon\to 0}s(\epsilon,\mathcal{G}_1). \tag{2.1}$$

Obviously, the limit (either finite or infinite) in (2.1) exists.

The quantity $h(\mathcal{G},\mathcal{G}_1)$ is called the (topological) *entropy* of $\mathcal{G}$ with respect to $\mathcal{G}_1$. It can be defined not only in terms of (n,ϵ)-separated sets but also in terms of (n,ϵ)-spanning sets. Namely, one can just rewrite the formulae defining the entropy replacing the numbers $s(n,\epsilon)$ by $r(n,\epsilon)$, where $r(n,\epsilon)$ is the minimal cardinality of an (n,ϵ)-spanning subset of X; a subset A of X is said to be (n,ϵ)-*spanning* whenever for any $x\in X$ there exists $y\in A$ such that $d(g(x),g(y))<\epsilon$ for all products g of $k\le n$ generators for which x and y lie in the domain of g.

If G is a finitely generated group of homeomorphisms of X, then the entropy of G with respect to a finite generating set G_1 is defined as the entropy of the pseudogroup $\mathcal{G}=\mathcal{G}(G_1)$ generated by G_1 (with respect to the same G_1, of course). Therefore, we may write

$$h(G,G_1)=h(\mathcal{G}(G_1),G_1).$$

Clearly, the entropy of $\mathcal{G}$ depends strongly on the choice of $\mathcal{G}_1$. In fact, if $\mathcal{G}_1'$ is another generating set, then there exists $m\in\mathbb{N}$ such that

$$s(n,\epsilon,\mathcal{G}_1) \le s(mn,\epsilon,\mathcal{G}_1') \quad \text{and} \quad s(n,\epsilon,\mathcal{G}_1') \le s(mn,\epsilon,\mathcal{G}_1)$$

for all n and ϵ. Consequently,

$$\frac{1}{m}h(\mathcal{G},\mathcal{G}_1) \le h(\mathcal{G},\mathcal{G}_1') \le mh(\mathcal{G},\mathcal{G}_1).$$

However, if $h(\mathcal{G},\mathcal{G}_1)=0$ for one generating set $\mathcal{G}_1$, then $h(\mathcal{G},\mathcal{G}_1')=0$ for any other generating set $\mathcal{G}_1'$. Therefore, we can distinguish between pseudogroups (and groups) with vanishing entropy and those with nonvanishing entropy.

Given a compact subset K of X, the restricted entropy $h(\mathcal{G},\mathcal{G}_1,K)$ can be defined by replacing the numbers $s(n,\epsilon)$ (resp., $r(n,\epsilon)$) above by $s(n,\epsilon,K)$ (resp., $r(n,\epsilon,K)$), the maximal cardinalities of (n,ϵ)-separated (resp., the minimal cardinality of (n,ϵ)-spanning) subsets of K. Certainly, $h(\mathcal{G},\mathcal{G}_1,K) \le h(\mathcal{G},\mathcal{G}_1)$ for all K contained in X.

It is easy to see that the entropy of the group generated by a single transformation f of X equals twice the topological entropy of f. For more about entropy of groups and pseudogroups we refer to [GLW], [CC, Chapter 13] and [Wa].

Now, if $\mathcal{F}$ is a foliation (or lamination) of a compact manifold (or space) M and g is a leafwise Riemannian structure on M, then the *geometric entropy* $h(\mathcal{F},g)$ of $\mathcal{F}$ with respect to g can be defined by

$$h(\mathcal{F},g) = \sup_{\mathcal{U}} \frac{1}{\Delta(\mathcal{U})} h(\mathcal{H}_{\mathcal{U}},(\mathcal{H}_{\mathcal{U}})_1) \tag{2.2}$$

where $\mathcal{U}$ ranges over all good coverings of M by foliated charts, $\Delta(\mathcal{U})$ denotes the lower upper bound for the diameters (in the metric induced by g) of plaques of charts in $\mathcal{U}$, $\mathcal{H}_{\mathcal{U}}$ is the holonomy pseudogroup determined by $\mathcal{U}$ on the space of plaques and $\mathcal{H}_{\mathcal{U}_1}$ is the set of elementary holonomy maps corresponding to all pairs of overlapping charts of $\mathcal{U}$.

The original definition in [GLW] was different but it has been shown ([GLW, Thm. 3.4], see also [CC] and [Wa]) that the two approaches provide the same number. The geometric entropy of a foliation (lamination) depends on the choice of the leafwise Riemannian structure but again, as in the case of the entropy for pseudogroups, its vanishing or non-vanishing does not: $h(\mathcal{F},g)=0$ for some g if and only if $h(\mathcal{F},g')=0$ for any other Riemannian structure g'. As before, we refer to [GLW], [CC] and [Wa] for more about the entropy of foliations (laminations). Here, we will just remark that the geometric entropy of a suspension vanishes if and only if the topological entropy of the suspended group does.

3. Quotients. Let X and Y be compact metric spaces and $\pi : X \to Y$ a continuous surjection. Let G be a finitely generated group acting simultaneously on X and Y in such a way that

$$\pi(g(x)) = g(\pi(x))$$

for all $x \in X$ and $g \in G$. Let G_1 be a finite symmetric set generating G.

Recall that the growth of G is *linear* (*polynomial*, *exponential* etc.) whenever there exists a function $f : \mathbb{N} \to \mathbb{N}$ which is linear (polynomial, exponential etc.) such that the cardinalities $g(n)$ of the sets $G_n := \{g_{i_1} \circ \cdots \circ g_{i_n} : g_{i_j} \in G_1\}$ $(n \in \mathbb{N})$ satisfy

$$af(bn) \le g(n) \le Af(Bn)$$

for all $n \in \mathbb{N}$ and some positive numbers a, b, A and B. Certainly this definition is correct: the growth type of G is independent of the choice of the generating set G_1. Already in [GLW], it was observed that the entropy of a group depends *a priori* on two factors: the topological entropies of generators and the growth of the group. For example,

(1) any homeomorphism of a circle has topological entropy zero but the free group generated by two such homeomorphisms h_1 and h_2 which have sinks x_1 and x_2 and sources y_1 and y_2 such that $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$ has exponential growth and positive entropy, while

(2) any homeomorphism of positive topological entropy generates a group of linear growth and positive entropy.

The main result of this section reads as follows.

THEOREM 1. *If G has linear growth, then*

$$h(G, G_1, X) \le h(G, G_1, Y) + C \cdot \sup\{h(G, G_1, \pi^{-1}(y)) : y \in Y\}. \tag{3.1}$$

for some constant $C \ge 1$.

Proof. Most of the arguments follow those of Theorem 17 in [Bo], a modification is needed just in final steps.

So, denote by A the second term in (3.1), assume that A is finite (otherwise there is nothing to prove), and fix $\epsilon, \eta > 0$. For any $y \in Y$ choose $N(y) \in \mathbb{N}$ in such a way that

$$\log r(N(y), \epsilon, \pi^{-1}(y)) \le N(y)(A + \eta)$$

Choose $(N(y), \epsilon)$-spanning subsets E_y of the fibres $\pi^{-1}(y)$, $y \in Y$, of minimal cardinality and set

$$U_y = \bigcup_{z \in E_y} \bigcap_{g \in G_{N(y)}} g^{-1}(B(g(z), 2\epsilon)).$$

Certainly, U_y's are open neighbourhoods of the fibres $\pi^{-1}(y)$, therefore for any $y \in Y$ there exists $\theta(y) > 0$ for which $\pi^{-1}(B(y, \theta(y))) \subset U_y$. Choose a finite cover of Y consisting of sets $B(y_i, \theta(y_i))$, $i = 1, \ldots, p$, and let $\delta > 0$ be a Lebesgue number of this cover.

Let now E_n be an (n,δ)-spanning subset of Y of minimal cardinality. For any $z \in E_n$ and $g \in G_n$ choose an index $\iota(z,g) \in \{1,\dots,p\}$ for which the closure of $B(g(z),\delta/2)$ is contained in $B(y_{\iota(z,g)},\theta(y_{\iota(z,g)}))$. Choose also minimal subsets $A_n(z)$ of G_n for which the balls of centres $g \in A_n(z)$ and radii $N(y_{\iota(z,g)})$ cover G_n in the Cayley graph of G. Define

$$\begin{aligned} &V(z,(\xi_g, g \in A_n(z))) \\ &\qquad = \{x \in X : d(h(g(x)),h(\xi_g)) < 2\epsilon,\ g \in A_n(z),\ h \in G_{N(y_{\iota(g,z)})}\} \end{aligned}$$

for all $z \in E_n$ and $\xi_g \in E_{y_{\iota(z,g)}}$. Notice that

$$V(z,(\xi_g, g \in A_n(z))) = \bigcap_{g \in A_n(z)} g^{-1}\Big(\bigcap_{h \in G_{N(y_{\iota(z,g)})}} h^{-1}B(h(\xi_g),2\epsilon)\Big)$$

Take an arbitrary point $x_0 \in X$ and let $x_0' = \pi(x_0)$. For $x_0' \in Y$ there exists $z \in E_n$ such that $d(g(x_0'),g(z)) < \delta$, for any $g \in G_n$.

Since the ball $B(g(z),\delta/2)$ is contained in $B(y_{\iota(z,g)},\theta(y_{\iota(z,g)}))$ and

$$\pi^{-1}(B(y_{\iota(z,g)},\theta(y_{\iota(z,g)}))) \subset \bigcup_{\xi \in E_{y_0}} \bigcap_{h \in G_{N(y_0)}} h^{-1}(B(h(\xi),2\epsilon)),$$

where $y_0 = y_{\iota(z,g)}$, we conclude that for any $g \in A_n(z)$ there exists $\xi_g' \in E_{y_0}$ such that

$$g(x_0') \in \bigcap_{h \in G_{N(y_{\iota(z,g)})}} h^{-1}B(h(\xi_g'),2\epsilon).$$

The points $g(x_0')$ and $g(x_0)$ are in the same fibre E_{y_0}, so there exists $\xi_g \in E_{y_0}$ such that

$$g(x_0) \in \bigcap_{h \in G_{N(y_{\iota(z,g)})}} h^{-1}B(h(\xi_g),2\epsilon)$$

and

$$x_0 \in \bigcap_{g \in A_n(z)} g^{-1}\Big(\bigcap_{h \in G_{N(y_{\iota(z,g)})}} h^{-1}B(h(\xi_g),2\epsilon)\Big).$$

Thus all such sets $V(z,(\xi_g))$ cover X while any $(n,4\epsilon)$-separated subset of X may have at most one point in common with each of them. Indeed, assume that, on the contrary, two distinct points x_1 and x_2 which are $(n,4\epsilon)$-separated in X, belong to the same set

$$\bigcap_{g \in A_n(z)} g^{-1}\Big(\bigcap_{h \in G_{N(y_{\iota(z,g)})}} h^{-1}B(h(\xi_g),2\epsilon)\Big).$$

Then for any $g \in A_n(z)$ and $h \in G_{N(y_{\iota(z,g)})}$ we have $hg(x_1), hg(x_2) \in B(h(\xi_g),2\epsilon)$. Thus, $d(hg(x_1),hg(x_2)) < 4\epsilon$ while $hg \in G_n$, which contradicts that the points x_1, x_2, are $(n,4\epsilon)$-separated.

For any $z \in E_n$, the number $\nu(z)$ of such sets $V(z, (\xi_g, g \in A_n(z)))$ is bounded from above by

$$\prod_{g \in A_n(z)} r(N(y_{\iota(z,g)}), \epsilon, \pi^{-1}(y_{\iota(z,g)})).$$

So,

$$\begin{aligned}\log \nu(z) &\le \sum_{g \in A_n(z)} \log r(N(y_{\iota(z,g)}), \epsilon, \pi^{-1}(y_{\iota(z,g)})) \\ &\le \# A_n(z) \max\{N(y_1), \ldots, N(y_p)\} \cdot (A + \eta).\end{aligned}$$

Therefore, taking into account that the sequence $A_n(z) \subset G_n$, $n \in \mathbb{N}$, has linear growth

$$\log \nu(z) \le Cn(A + \eta)$$

for some constant C depending on the growth of G. Consequently,

$$s(n, 4\epsilon, X) \le r(n, \delta, Y) \exp(Cn(A + \eta))$$

and

$$\frac{1}{n} \log s(n, 4\epsilon, X) \le \frac{1}{n} \log r(n, \delta, Y) + C(A + \eta).$$

Passing to the limits yields (3.1). ∎

4. Entropy and distality for groups and foliated bundles. Recall after [Fu] that any minimal distal action of a group G on a compact metric space X can be expressed as (X_η, G), where η is an ordinal, (X_0, G) is a trivial action on a singleton, $(X_{\xi+1}, G)$ $(\xi < \eta)$ is an isometric extension of (X_ξ, G), and (X_ξ, G) is the limit of the family (X_ζ, G), $\zeta < \xi$, when ξ is a limit ordinal $\le \eta$. Recall also that an action (Y, G) is an *isometric extension* of an action (Z, G) if there exists a mapping π from Y onto Z which commutes with the corresponding actions of the group G (that is, $\pi(g(y)) = g(\pi(y))$ for all $y \in Y$) and such that all the elements of G act isometrically on the fibres of π (that is, $d(g(y_1), g(y_2)) = d(y_1, y_2)$ whenever $\pi(y_1) = \pi(y_2)$, d being the distance function on Y). Finally, recall that (Y, G) is the *limit* of a family (Z_α, G), $\alpha \in A$, whenever there exist surjective mappings $\pi_\alpha : Y \to Z_\alpha$ commuting with the corresponding actions of the group G and such that for any distinct points y_1 and y_2 of Y there exists $\alpha \in A$ for which $\pi_\alpha(y_1) \ne \pi_\alpha(y_2)$.

LEMMA 1. *If G has linear growth, (Y, G) is an isometric extension of (Z, G) and the entropy of G on Z is equal to zero, then the entropy of G on Y is zero as well.*

Proof. Follows directly from Theorem 1. ∎

LEMMA 2. *If (Y, G) is a limit of the family (Z_α, G), $\alpha \in A$, and G has zero entropy on Z_α for all α, then G has zero entropy on Y.*

Proof. Denote by d_α a metric on Z_α and put

$$d(y_1, y_2) = \sup_\alpha d_\alpha(\pi_\alpha(y_1), \pi_\alpha(y_2))$$

for $y_1, y_2 \in Y$. Certainly, d is a pseudometric on Y. If $d(y_1, y_2) = 0$, then $d_\alpha(\pi_\alpha(y_1), \pi_\alpha(y_2)) = 0$ and $\pi_\alpha(y_1) = \pi_\alpha(y_2)$ for all $\alpha \in A$, consequently, $y_1 = y_2$ and d occurs to be a metric.

Now, if points $y_1, \dots, y_N$ are (n, ϵ)-spanning on (Y, G), then there exists $\alpha \in A$ for which $\pi_\alpha(y_1), \dots, \pi_\alpha(y_N)$ are all distinct. Choose a point $z_\alpha \in Z_\alpha$. Then $z_\alpha = \pi_\alpha(y)$ and $d(g(y), g(y_i)) \leq \epsilon$ for some $y \in Y$, $i \in \{1, \dots, N\}$ and all products g of $k \leq n$ generators of G. Inequalities $d_\alpha(g(z_\alpha), g(\pi_\alpha(y_i)) \leq d(g(y), g(y_i)) < \epsilon$ show that the set $\{\pi_\alpha(y_1), \dots, \pi_\alpha(y_N)\}$ is (n, ϵ)-spanning on Z_α. Therefore, the entropy of G on Y does not exceed that on Z_α, which is equal to zero. ■

THEOREM 2. *Any finitely generated, linear growth, minimal and distal group G of homeomorphisms of a compact metric space X has zero entropy.*

Proof. Let $(X, G) = (X_\eta, G)$ as above. Denote by Σ the set of ordinals $\xi \leq \eta$ for which the entropy of G on X_ξ is zero. Obviously, $0 \in \Sigma$. By Lemma 1, if $\xi \in \Sigma$, then $\xi + 1 \in \Sigma$. By Lemma 2, $\xi \in \Sigma$ if ξ is a limit ordinal for which all the ordinals $\zeta < \xi$ belong to Σ. By transfinite induction, $\eta \in \Sigma$. ■

Directly from the definitions and the above theorem we obtain

COROLLARY 1. *The geometric entropy of a compact minimal distal foliated bundle vanishes whenever its holonomy group has linear growth.* ■

REMARK. It would be interesting to either prove the theorems of this paper without assuming linear growth or provide examples of distal groups (pseudogroups, foliations) of positive entropy. Such examples, if any, would show better the (already mentioned) influence of growth of groups/pseudogroups/foliations on the value of entropy of such systems as well as the significance of the difference between distality and equicontinuity: certainly, equicontinuous systems have zero entropy. Unfortunately, at the moment, the authors are not able to provide such proofs or examples.

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