ANNALES
POLONICI MATHEMATICI
100.1 (2011)

On iteration of higher order jets and prolongation of connections

by Miroslav Doupovec (Brno) and
Włodzimierz M. Mikulski (Kraków)

Abstract. We introduce exchange natural equivalences of iterated nonholonomic, holonomic and semiholonomic jet functors, depending on a classical linear connection on the base manifold. We also classify some natural transformations of this type. As an application we introduce prolongation of higher order connections to jet bundles.

In general, the idea of iteration plays an important role in the theory of jets. For example, the rth nonholonomic prolongation $\widetilde{J}^rY$ of a fibered manifold $Y \to M$ is defined by iteration,

$$\widetilde{J}^rY = J^1(\widetilde{J}^{r-1}Y \to M),$$

which yields the natural identification $\widetilde{J}^r(\widetilde{J}^sY) = \widetilde{J}^{r+s}Y$. Denoting by J^rY the classical rth holonomic prolongation of $Y \to M$, we have the canonical inclusion $J^rY \subset \widetilde{J}^rY$ given by $j^r_xs \mapsto j^1_x(u \mapsto j^{r-1}_us)$ for every local section s of Y. One can also recursively define the semiholonomic prolongation $\bar{J}^rY$ (see e.g. [8] and [19]). Then for $r > 1$ we have

$$J^rY \subset \bar{J}^rY \subset \widetilde{J}^rY,$$

while for $r = 1$ all such spaces coincide. For the theory of jets we refer to [14], [13], [17], [19], [26].

Taking into account applications of jet theory in higher order mechanics and mathematical physics, it is also useful to study iteration of higher order jets. This leads to the problem of exchange natural transformations of iterated jet functors, which has direct applications in the prolongation of higher order connections. But in [7] we have proved that for $r \neq s$ there is no natural transformation $J^rJ^s \to J^sJ^r$ and by [3] the only natural transformation $J^rJ^s \to J^rJ^s$ is the identity. On the other hand, M. Modugno [24]

2010 *Mathematics Subject Classification*: 58A05, 58A20.
Key words and phrases: holonomic jets, semiholonomic jets, nonholonomic jets, connections.

DOI: 10.4064/ap100-1-6

has defined an exchange isomorphism $\mathrm{ex}_\Lambda : J^1J^1Y \to J^1J^1Y$ depending on a linear connection Λ on M. In this paper we solve the general problem:

PROBLEM 1. *Introduce exchange natural equivalences $B_\Lambda^{F,G} : FG \to GF$ for any couple F, G of higher order holonomic, semiholonomic or nonholonomic jet functors, depending on some linear connection Λ on the base manifold.*

First, in Section 2 we recall the exchange isomorphism $J^rJ^sY \to J^sJ^rY$ from [6]. In Section 3 we introduce a natural equivalence $\widetilde{B}_\Lambda^{r,s} : \widetilde{J}^r\widetilde{J}^s \to \widetilde{J}^s\widetilde{J}^r$, which has a simple coordinate description. Section 4 is devoted to the solution of Problem 1. Unfortunately, the complete description of all natural transformations $J^rJ^s \to J^sJ^r$ depending on Λ is a difficult problem, which has been solved only for $r = s = 1$, [15]. In Section 5 we classify all natural transformations $J^2J^1 \to J^2J^1$ and $J^2J^1 \to J^1J^2$ depending on a torsion free connection Λ on the base manifold.

In Section 7 we apply natural equivalences $B_\Lambda^{F,G}$ to prolongation of higher order connections. We recall that an rth order nonholonomic connection in the sense of C. Ehresmann [10] is a smooth section

$$\Gamma : Y \to \widetilde{J}^rY.$$

Such a connection is called *holonomic* or *semiholonomic* if it has values in J^rY or in $\bar{J}^rY$, respectively. Clearly, first order connections $Y \to J^1Y$ can also be interpreted as lifting maps $Y \times_M TM \to TY$. Furthermore, a linear smooth section $TM \to J^rTM$ is called a *linear rth order connection* on M. For $r = 1$ we obtain the concept of classical linear connection on M, which can be equivalently interpreted as the covariant derivative $\chi(M) \times \chi(M) \to \chi(M)$. In general, higher order connections have many applications in differential geometry and in the geometric approach to mathematical physics (see e.g. [1], [2], [8], [9], [12], [18], [27], [28]). For example, in [8] we have shown that rth order connections can be used to obtain a geometric description of higher order geometric object fields.

Roughly speaking, by prolongation of connections we understand geometric constructions transforming a connection on $Y \to M$ into a connection on $FY \to M$ or on $FY \to Y$, where F is some bundle functor. We recall that prolongation of first order connections was studied e.g. in [4], [14], [20], [24]. However, prolongation of higher order connections has not been studied systematically up till now. The second author [23] has recently defined prolongation of rth order holonomic connections from $Y \to M$ to $FY \to M$ by means of some classical linear connection on M. In Section 7 we introduce another prolongation of rth order holonomic, semiholonomic and nonholonomic connections to any sth order (holonomic, semiholonomic or nonholonomic) jet bundle by means of some classical linear connection on M.

1. Preliminaries. In what follows we denote by $\mathcal{M}f_m$ the category of m-dimensional manifolds and their local diffeomorphisms, by $\mathcal{FM}$ the category of fibered manifolds and fiber respecting mappings, by $\mathcal{FM}_m$ the subcategory of fibered manifolds with m-dimensional bases with fibered maps over local diffeomorphisms, and by $\mathcal{FM}_{m,n} \subset \mathcal{FM}_m$ the subcategory with n-dimensional fibers and local fibered diffeomorphisms. All manifolds and maps are assumed to be infinitely differentiable.

Obviously, $J^r : \mathcal{FM}_m \to \mathcal{FM}_m \subset \mathcal{FM}$ is a bundle functor transforming a fibered manifold $Y \to M$ into its r-jet prolongation $J^rY \to M$, and any $\mathcal{FM}_m$-morphism $\varphi : Y_1 \to Y_2$ covering $\underline{\varphi} : M_1 \to M_2$ into $J^r\varphi : J^rY_1 \to J^rY_2$, $J^r\varphi(j^r_x\sigma) = j^r_{\underline{\varphi}(x)}(\varphi \circ \sigma \circ \underline{\varphi}^{-1})$. Similarly, $\bar{J}^r$ and $\widetilde{J}^r$ are also bundle functors on $\mathcal{FM}_m$.

Denoting by (x^i, y^p) the canonical coordinates on Y, the induced coordinates on J^1Y are $y^p_i = \partial y^p/\partial x^i$. The canonical coordinates on $\widetilde{J}^rY$ can be introduced by the following induction. First, assume we have the coordinates $(x^i, y^p_{i_1\dots i_{r-1}})$ on $\widetilde{J}^{r-1}Y$, where $i_1, \dots, i_{r-1} \in \{0, 1, \dots, m\}$. Then the induced coordinates on $\widetilde{J}^rY$ are

$$x^i, \quad y^p_{i_1\dots i_{r-1}0} = y^p_{i_1\dots i_{r-1}}, \quad y^p_{i_1\dots i_{r-1}i} = \frac{\partial}{\partial x^i} y^p_{i_1\dots i_{r-1}}.$$

Next, the semiholonomic prolongation $\bar{J}^rY$ can be characterized by the following condition: $y^p_{i_1\dots i_r} = y^p_{j_1\dots j_r}$ provided the sequences obtained from $(i_1 \dots i_r)$ and $(j_1 \dots j_r)$ by deleting all zeros and preserving the order of nonzero indices coincide. So the local coordinates on $\bar{J}^rY$ are $(x^i, y^p_{i_1\dots i_s})$, $s = 0, \dots, r$. Finally, the holonomic prolongation J^rY is characterized by full symmetry in all subscripts.

Let $G_1, G_2 : \mathcal{FM}_m \to \mathcal{FM}$ be bundle functors and Con be the bundle of classical linear connections on the base manifold. By *$\mathcal{FM}_m$-natural transformations $A_\Lambda : G_1 \to G_2$ depending on Λ* we understand $\mathcal{FM}_m$-natural operators $A : Con \rightsquigarrow (G_1, G_2)$ in the sense of [14] transforming classical linear connections $\Lambda \in Con(M)$ into the space $C^\infty_M(G_1Y, G_2Y)$ of all $\mathcal{FM}$-morphisms $G_1Y \to G_2Y$ covering the identity of M. According to [14], such an operator A is a family of invariant regular operators (functions) $A_Y : Con(M) \to C^\infty_M(G_1Y, G_2Y)$ for any $\mathcal{FM}_m$-object $Y \to M$. The invariance of A means that if $\Lambda_1 \in Con(M_1)$ and $\Lambda_2 \in Con(M_2)$ are related by a local diffeomorphism $\varphi : M_1 \to M_2$ and $\Phi : Y_1 \to Y_2$ is an $\mathcal{FM}_m$-map covering φ, then $G_2\Phi \circ A_{Y_1}(\Lambda_1) = A_{Y_2}(\Lambda_2) \circ G_1\Phi$. The regularity means that A transforms smoothly parametrized families of classical linear connections into smoothly parametrized families. Quite analogously one can also define $\mathcal{FM}_{m,n}$-natural transformations. If we want to stress the fibered manifold, we write $(A_\Lambda)_Y$ instead of A_Λ.

2. Exchange natural equivalence of iterated holonomic jets. Write $\mathbb{R}^{m,n}$ for the product fibered manifold $\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$. In what follows we identify sections of $\mathbb{R}^{m,n}$ with maps $\mathbb{R}^m\to\mathbb{R}^n$ and we use the notation

$$j^r_0 j^s(f(x,\underline{x})) = j^r_0(x\to j^s_x(\underline{x}\to f(x,\underline{x})))\in J^r_0J^s(\mathbb{R}^{m,n}). \tag{1}$$

LEMMA 1. *For any $\mathcal{FM}_m$-map $\Phi:\mathbb{R}^{m,n}\to\mathbb{R}^{m,k}$ with $\Phi(x,y)=(\varphi_1(x),\varphi_2(x,y))$ and $\varphi_1(0)=0$ we have*

$$J^rJ^s\Phi(j^r_0j^sf(x,\underline{x})) = j^r_0j^s(\varphi_2(\varphi_1^{-1}(\underline{x}),f(\varphi_1^{-1}(x),\varphi_1^{-1}(\underline{x})))). \tag{2}$$

Proof. Indeed,

$$\begin{aligned} J^rJ^s\Phi(j^r_0j^sf(x,\underline{x})) &= J^rJ^s\Phi(j^r_0(x\to j^s_x(\underline{x}\to f(x,\underline{x}))))\\ &= j^r_0(x\to J^s\Phi(j^s_{\varphi_1^{-1}(x)}(\underline{x}\to f(\varphi_1^{-1}(x),\underline{x}))))\\ &= j^r_0(x\to j^s_x(\underline{x}\to \varphi_2(\varphi_1^{-1}(\underline{x}),f(\varphi_1^{-1}(x),\varphi_1^{-1}(\underline{x})))))\\ &= j^r_0j^s(\varphi_2(\varphi_1^{-1}(\underline{x}),f(\varphi_1^{-1}(x),\varphi_1^{-1}(\underline{x})))). \ \blacksquare \end{aligned}$$

In [6] we correctly defined a linear isomorphism

$$\begin{aligned} &A^{r,s}_{m,n}: J^r_0J^s(\mathbb{R}^{m,n})\to J^s_0J^r(\mathbb{R}^{m,n}),\\ &A^{r,s}_{m,n}(j^r_0j^s(f(x,\underline{x}))) = j^s_0j^r(f(\underline{x}-x,\underline{x})), \end{aligned} \tag{3}$$

and we proved the following invariance condition:

LEMMA 2. *Let $\Phi:\mathbb{R}^{m,n}\to\mathbb{R}^{m,k}$ be an $\mathcal{FM}_m$-map of the form $\Phi(x,y)=(\varphi_1(x),\varphi_2(x,y))$, where $\varphi_1:\mathbb{R}^m\to\mathbb{R}^m$ is a linear isomorphism and $\varphi_2:\mathbb{R}^{m,n}\to\mathbb{R}^k$. Then for any $v\in J^r_0J^s(\mathbb{R}^{m,n})$ we have*

$$A^{r,s}_{m,k}(J^rJ^s\Phi(v)) = J^sJ^r\Phi(A^{r,s}_{m,n}(v)). \tag{4}$$

Now let $Y\to M$ be an $\mathcal{FM}_{m,n}$-object, Λ be a classical linear connection on the base manifold M and $v\in J^r_zJ^sY$, $z\in M$. Choose any fibered coordinate system $\Psi=(\underline{\psi},\psi):Y\to\mathbb{R}^{m,n}$ such that $\underline{\psi}:M\to\mathbb{R}^m$ is a normal coordinate system of Λ with center z, $\underline{\psi}(z)=0$. In [6] we have defined an $\mathcal{FM}_m$-natural equivalence depending on Λ,

$$(A^{r,s}_\Lambda)_Y: J^rJ^sY\to J^sJ^rY,\quad (A^{r,s}_\Lambda)_Y(v):=J^sJ^r\Psi^{-1}(A^{r,s}_{m,n}(J^rJ^s\Psi(v))). \tag{5}$$

Using Lemma 2 we show easily that the definition of $A^{r,s}_\Lambda$ does not depend on the choice of Ψ with the above property. So $A^{r,s}_\Lambda$ is defined correctly and globally.

Denote by $(x^i, y^p=y^p_{00}, y^p_i=y^p_{i0}, Y^p_i=y^p_{0i}, y^p_{ij}=\partial y^p_i/\partial x^j)$ the canonical coordinates on J^1J^1Y and let Λ^k_{ij} be the coordinates of Λ. By [6], the coordinate expression of $A^{1,1}_\Lambda$ is

$$\overline{y}^p_i = Y^p_i,\quad \overline{Y}^p_i = y^p_i,\quad \overline{y}^p_{ij} = y^p_{ji} + (y^p_k - Y^p_k)\Lambda^k_{ji} + \frac{1}{2}(y^p_k-Y^p_k)(\Lambda^k_{ij}-\Lambda^k_{ji}). \tag{6}$$

3. Exchange natural equivalences of iterated nonholonomic jets. Starting from $A^{1,1}_{\Lambda}$, we can introduce an $\mathcal{FM}_m$-natural equivalence depending on Λ,

$$\widetilde{A}^{r,s}_{\Lambda} : \widetilde{J}^r\widetilde{J}^s \to \widetilde{J}^s\widetilde{J}^r, \tag{7}$$

as follows. First, we have $(\widetilde{A}^{1,1}_{\Lambda})_Y := (A^{1,1}_{\Lambda})_Y : \widetilde{J}^1\widetilde{J}^1Y \to \widetilde{J}^1\widetilde{J}^1Y$. Then we can define $(\widetilde{A}^{r,1}_{\Lambda})_Y : \widetilde{J}^r\widetilde{J}^1Y \to \widetilde{J}^1\widetilde{J}^rY$ by

$$(\widetilde{A}^{r,1}_{\Lambda})_Y := (\widetilde{A}^{1,1}_{\Lambda})_{\widetilde{J}^{r-1}Y} \circ J^1(\widetilde{A}^{r-1,1}_{\Lambda})_Y,$$

where $(\widetilde{A}^{r-1,1}_{\Lambda})_Y : \widetilde{J}^{r-1}\widetilde{J}^1Y \to \widetilde{J}^1\widetilde{J}^{r-1}Y$. Finally, we define the map (7) by

$$(\widetilde{A}^{r,s}_{\Lambda})_Y := J^1(\widetilde{A}^{r,s-1}_{\Lambda})_Y \circ (\widetilde{A}^{r,1}_{\Lambda})_{\widetilde{J}^{s-1}Y}.$$

Roughly speaking, $\widetilde{A}^{r,s}_{\Lambda}$ is defined by recursion from $A^{1,1}_{\Lambda}$. Then the coordinate form of $\widetilde{A}^{r,s}_{\Lambda}$ for any r, s can be computed directly from (6) by differentiation.

In general, $(A^{r,s}_{\Lambda})_Y$ is not the restriction of $(\widetilde{A}^{r,s}_{\Lambda})_Y$ to J^rJ^sY. Indeed, we have

Lemma 3. *For some $\Lambda \in Con(M)$ and some $\mathcal{FM}_m$-object $Y \to M$, the isomorphism $(\widetilde{A}^{2,1}_{\Lambda})_Y : \widetilde{J}^2\widetilde{J}^1Y \to \widetilde{J}^1\widetilde{J}^2Y$ does not send J^2J^1Y into J^1J^2Y.*

Proof. For the sake of simplicity we use the following notation of standard coordinates. First, on $\widetilde{J}^1\mathbb{R}^{m,n} = \mathbb{R}^{m,n_1}$ we have the coordinates $(x^i, y^p = y^p_0, y^p_i = \partial y^p/\partial x^i)$ and on $\widetilde{J}^1\widetilde{J}^1\mathbb{R}^{m,n} = \mathbb{R}^{m,n_2}$ we have the induced coordinates $(x^i, y^p = y^p_{00}, y^p_i = y^p_{i0}, Y^p_j = y^p_{0j}, y^p_{ij})$. Then the local coordinates on $\widetilde{J}^1(\widetilde{J}^1\widetilde{J}^1)\mathbb{R}^{m,n} = (\widetilde{J}^1\widetilde{J}^1)\widetilde{J}^1\mathbb{R}^{m,n} = \widetilde{J}^3\mathbb{R}^{m,n} = \mathbb{R}^{m,n_3}$ are

$$(x^i, y^p = y^p_{000}, y^p_i = y^p_{i00}, Y^p_j = y^p_{0j0}, y^p_{ij} = y^p_{ij0}, y^p_{00k}, y^p_{i0k}, y^p_{0jk}, y^p_{ijk}).$$

Obviously,

$$y^p_{ijk}(\sigma) = y^p_{jik}(\sigma) \quad \text{for } \sigma \in J^1J^2\mathbb{R}^{m,n}. \tag{8}$$

Now let ${}^t\Lambda$, $t \in \mathbb{R}$, be a family of connections on $\mathbb{R}^m$ such that

$${}^t\Lambda^1_{12} = {}^t\Lambda^1_{21} = tx^3 \quad \text{and other } {}^t\Lambda^p_{rs} \text{ are zero.}$$

Then ${}^t\Lambda$ is torsion free, so that (6) yields the following coordinate form of $\tilde{A}^{1,1}_{{}^t\Lambda}$:

$$\overline{y}^p_i = Y^p_i, \quad \overline{Y}^p_i = y^p_i, \quad \overline{y}^p_{ij} = y^p_{ji} + (y^p_k - Y^p_k)\,{}^t\Lambda^k_{ji}. \tag{9}$$

Choose a section $\eta : \mathbb{R}^m \to \widetilde{J}^1\widetilde{J}^1\mathbb{R}^{m,1}$ such that

$$y^1_1(\eta(0)) - Y^1_1(\eta(0)) \neq 0 \quad \text{and} \quad j^1_0\eta \in J^2J^1\mathbb{R}^{m,1}. \tag{10}$$

By (9), we have

$$y^1_{ij}(\widetilde{A}^{1,1}_{{}^t\Lambda}(\eta(x))) = y^1_{ji}(\eta(x)) + (y^1_1(\eta(x)) - Y^1_1(\eta(x)))\,{}^t\Lambda^1_{ji}(x), \tag{11}$$

which yields (because ${}^t\Lambda^1_{ji}(0)=0$)

$$\begin{aligned} y^1_{ijk}((J^1\widetilde{A}^{1,1}_{{}^t\Lambda})(j^1_0\eta)) &= y^1_{ijk}(j^1_0(\widetilde{A}^{1,1}_{\Lambda}\circ\eta)) = \frac{\partial}{\partial x^k}_0(y^1_{ij}(\widetilde{A}^{1,1}_{{}^t\Lambda}(\eta(x)))) \\ &= \frac{\partial}{\partial x^k}_0(y^1_{ji}(\eta(x))) + \frac{\partial}{\partial x^k}_0(\ldots){}^t\Lambda^1_{ji}(0) + (y^1_1(\eta(0)) - Y^1_1(\eta(0))){}^t\Lambda^1_{ji;k}(0) \\ &= y^1_{jik}(j^1_0\eta) + (y^1_1(\eta(0)) - Y^1_1(\eta(0))){}^t\Lambda^1_{ji;k}(0). \end{aligned}$$

From ${}^t\Lambda^p_{rs}(0)=0$ and (9) it follows that

$$y^1_{ijk}(\widetilde{A}^{1,1}_{{}^t\Lambda}(\sigma)) = (y^1_i)_{jk}(\widetilde{A}^{1,1}_{{}^t\Lambda}(\sigma)) = (y^1_i)_{kj}(\sigma) = y^1_{ikj}(\sigma)$$

for any $\sigma\in((\widetilde{J}^1\widetilde{J}^1)\widetilde{J}^1\mathbb{R}^{m,1})_0 = (\widetilde{J}^3\mathbb{R}^{m,1})_0$. This implies

$$\begin{aligned} y^1_{ijk}(\widetilde{A}^{2,1}_{{}^t\Lambda}(j^1_0\eta)) &= y^1_{ijk}(\widetilde{A}^{1,1}_{{}^t\Lambda}((J^1\widetilde{A}^{1,1}_{{}^t\Lambda})(j^1_0\eta))) = y^1_{ikj}((J^1\widetilde{A}^{1,1}_{{}^t\Lambda})(j^1_0\eta)) \\ &= y^1_{kij}(j^1_0\eta) + (y^1_1(\eta(0)) - Y^1_1(\eta(0))){}^t\Lambda^1_{ki;j}(0). \end{aligned}$$

From ${}^t\Lambda^1_{12;3}(0)=t$ and the first condition of (10) it follows that $y^1_{231}(\widetilde{A}^{2,1}_{{}^t\Lambda}(j^1_0\eta))$ depends on t in an essential way. On the other hand, $y^1_{321}(\widetilde{A}^{2,1}_{{}^t\Lambda}(j^1_0\eta))$ does not depend on t (as ${}^t\Lambda^1_{13;2}(0)=0$). By (8), $\widetilde{A}^{2,1}_{{}^t\Lambda}(j^1_0\eta)\notin(J^1J^2\mathbb{R}^{m,1})_0$ for some t. ∎

Now we define another $\mathcal{FM}_m$-natural equivalence $\widetilde{B}^{r,s}_{\Lambda}:\widetilde{J}^r\widetilde{J}^s\to\widetilde{J}^s\widetilde{J}^r$ depending on Λ, which is an extension of $A^{r,s}_{\Lambda}$. Let Λ^0 be the canonical flat connection on $\mathbb{R}^m$ and consider natural equivalences (5) and (7) depending on Λ^0. We first prove

LEMMA 4. *The natural isomorphism $(\widetilde{A}^{r,s}_{\Lambda^0})_{\mathbb{R}^{m,n}}:\widetilde{J}^r\widetilde{J}^s\mathbb{R}^{m,n}\to\widetilde{J}^s\widetilde{J}^r\mathbb{R}^{m,n}$ sends $(J^rJ^s\mathbb{R}^{m,n})_0$ into $(J^sJ^r\mathbb{R}^{m,n})_0$. Moreover, the restriction of $\widetilde{A}^{r,s}_{\Lambda^0}$ to $(J^rJ^s\mathbb{R}^{m,n})_0$ coincides with the restriction of $A^{r,s}_{\Lambda^0}$ to $(J^rJ^s\mathbb{R}^{m,n})_0$.*

Proof. We proceed by induction with respect to (r,s). By the definition of $\widetilde{A}^{r,s}_{\Lambda}$, the assertion is true for $(r,s)=(1,1)$. We first compute $(A^{r,s}_{\Lambda^0})_{\mathbb{R}^{m,n}}(j^r_{x_0}j^sf(x,\underline{x}))$, $x_0\in\mathbb{R}^m$. Clearly, the translation $\tau_{-x_0}:\mathbb{R}^m\to\mathbb{R}^m$ by $-x_0$ is a normal coordinate system of Λ^0 with center x_0. By (5) we have

$$\begin{aligned} (12)\quad & A^{r,s}_{\Lambda^0}(j^r_{x_0}j^sf(x,\underline{x})) \\ &= J^sJ^r(\tau_{x_0}\times\mathrm{id}_{\mathbb{R}^n})(A^{r,s}_{m,n}(J^rJ^s(\tau_{-x_0}\times\mathrm{id}_{\mathbb{R}^n})(j^r_{x_0}j^sf(x,\underline{x})))) \\ &= J^sJ^r(\tau_{x_0}\times\mathrm{id}_{\mathbb{R}^n})(A^{r,s}_{m,n}(j^r_0j^s(f(x+x_0,\underline{x}+x_0)))) \\ &= J^sJ^r(\tau_{x_0}\times\mathrm{id}_{\mathbb{R}^n})(j^s_0j^rf(\underline{x}-x+x_0,\underline{x}+x_0)) \\ &= j^s_{x_0}j^rf((\underline{x}-x_0)-(x-x_0)+x_0,(\underline{x}-x_0)+x_0) = j^s_{x_0}j^rf(\underline{x}-x+x_0,\underline{x}). \end{aligned}$$

Now we prove the assertion for $(r,1)$ from $(r-1,1)$. Using (12) for $(r-1,1)$

and the inductive assumption we obtain

$$\begin{aligned}
\widetilde{A}^{r,1}_{\Lambda^0}(j^r_0 j^1 f(x,\underline{x})) &= \widetilde{A}^{1,1}_{\Lambda^0}(J^1(\widetilde{A}^{r-1,1}_{\Lambda^0})(j^1_0(x_0 \to j^{r-1}_{x_0}(x \to j^1_x(\underline{x} \to f(x,\underline{x})))))) \\
&= A^{1,1}_{\Lambda^0}(J^1(A^{r-1,1}_{\Lambda^0})(j^1_0(x_0 \to j^{r-1}_{x_0}(x \to j^1_x(\underline{x} \to f(x,\underline{x})))))) \\
&= A^{1,1}_{\Lambda^0}(j^1_0(x_0 \to A^{r-1,1}_{\Lambda^0}(j^{r-1}_{x_0}(x \to j^1_x(\underline{x} \to f(x,\underline{x})))))) \\
&= A^{1,1}_{\Lambda^0}(j^1_0(x_0 \to j^1_{x_0}(x \to j^{r-1}_x(\underline{x} \to f(\underline{x} - x + x_0,\underline{x}))))) \\
&= j^1_0(x_0 \to j^1_{x_0}(x \to j^{r-1}_x(\underline{x} \to f(\underline{x} - x + (x - x_0),\underline{x})))) \\
&= j^1_0(x_0 \to j^1_{x_0}(x \to j^{r-1}_x(\underline{x} \to f(\underline{x} - x_0,\underline{x})))) \\
&= j^1_0(x_0 \to j^r_{x_0}(x \to f(x - x_0, x))) = j^1_0(x \to j^r_x(\underline{x} \to f(\underline{x} - x,\underline{x}))) \\
&= j^1_0 j^r f(\underline{x} - x,\underline{x}) = A^{r,1}_{\Lambda^0}(j^r_0 j^1 f(x,\underline{x})).
\end{aligned}$$

Finally we prove the assertion for (r,s) from $(r,s-1)$. Applying (12), the above equality and the inductive assumption we obtain

$$\begin{aligned}
\widetilde{A}^{r,s}_{\Lambda^0}(j^r_0 j^s f(x,\underline{x})) &= J^1(\widetilde{A}^{r,s-1}_{\Lambda^0})(\widetilde{A}^{r,1}_{\Lambda^0}(j^r_0(x \to j^1_x(x_0 \to j^{s-1}_{x_0}(\underline{x} \to f(x,\underline{x})))))) \\
&= J^1(A^{r,s-1}_{\Lambda^0})(A^{r,1}_{\Lambda^0}(j^r_0(x \to j^1_x(x_0 \to j^{s-1}_{x_0}(\underline{x} \to f(x,\underline{x})))))) \\
&= J^1(A^{r,s-1}_{\Lambda^0})(j^1_0(x \to j^r_x(x_0 \to j^{s-1}_{x_0}(\underline{x} \to f(x_0 - x,\underline{x}))))) \\
&= j^1_0(x \to A^{r,s-1}_{\Lambda^0}(j^r_x(x_0 \to j^{s-1}_{x_0}(\underline{x} \to f(x_0 - x,\underline{x}))))) \\
&= j^1_0(x \to j^{s-1}_x(x_0 \to j^r_{x_0}(\underline{x} \to f(\underline{x} - x_0 - x + x,\underline{x})))) \\
&= j^s_0(x \to j^r_x(\underline{x} \to f(\underline{x} - x,\underline{x}))) = j^s_0 j^r f(\underline{x} - x,\underline{x}) = A^{r,s}_{\Lambda^0}(j^r_0 j^s f(x,\underline{x})). \ \blacksquare
\end{aligned}$$

LEMMA 5. *Let (x^i, y^p) be the usual fibered coordinate system on $\mathbb{R}^{m,n}$ and Λ_0 be the canonical flat linear connection on $\mathbb{R}^m$. Then $(\widetilde{A}^{r,s}_{\Lambda_0})_{\mathbb{R}^{m,n}} : \widetilde{J}^r\widetilde{J}^s\mathbb{R}^{m,n} \to \widetilde{J}^s\widetilde{J}^r\mathbb{R}^{m,n}$ treated as $(\widetilde{A}^{r,s}_{\Lambda_0})_{\mathbb{R}^{m,n}} : \widetilde{J}^{r+s}\mathbb{R}^{m,n} \to \widetilde{J}^{r+s}\mathbb{R}^{m,n}$ has the coordinate form*

$$x^i \circ (\widetilde{A}^{r,s}_{\Lambda_0})_{\mathbb{R}^{m,n}} = x^i, \qquad y^p_{j_1,\dots,j_r,i_1,\dots,i_s} \circ (\widetilde{A}^{r,s}_{\Lambda_0})_{\mathbb{R}^{m,n}} = y^p_{i_1,\dots,i_s,j_1,\dots,j_r} \tag{13}$$

for all $i = 1,\dots,m$, $p = 1,\dots,n$ and $i_1,\dots,i_s,j_1,\dots,j_r = 0,1,\dots,m$.

Proof. We proceed by induction with respect to (r,s). The case $(r,s) = (1,1)$ follows from (6) and from the fact $(\Lambda_0)^k_{ij} = 0$. Assume that (13) is true for $(r-1,1)$ and we prove it for $(r,1)$. Using the relevant definitions and the inductive assumption we have

$$\begin{aligned}
y^p_{j_1,\dots,j_r,i_1} \circ (\widetilde{A}^{r,1}_{\Lambda_0})_{\mathbb{R}^{m,n}} &= y^p_{j_1,\dots,j_r,i_1} \circ (\widetilde{A}^{1,1}_{\Lambda_0})_{\widetilde{J}^{r-1}\mathbb{R}^{m,n}} \circ J^1(\widetilde{A}^{r-1,1}_{\Lambda_0})_{\mathbb{R}^{m,n}} \\
&= y^p_{j_1,\dots,j_{r-1},i_1,j_r} \circ (J^1\widetilde{A}^{r-1,1}_{\Lambda_0})_{\mathbb{R}^{m,n}} = y^p_{i_1,j_1,\dots,j_r}.
\end{aligned}$$

Assume now that (13) is true for $(r,s-1)$ and we prove it for (r,s). We can write

$$y^p_{j_1,\dots,j_r,i_1,\dots,i_s} \circ (\widetilde{A}^{r,s}_{\Lambda_0})_{\mathbb{R}^{m,n}} = y^p_{j_1,\dots,j_r,i_1,\dots,i_s} \circ J^1(\widetilde{A}^{r,s-1}_{\Lambda_0})_{\mathbb{R}^{m,n}} \circ (\widetilde{A}^{r,1}_{\Lambda_0})_{\widetilde{J}^{s-1}\mathbb{R}^{m,n}}$$
$$= y^p_{i_1,\dots,i_{s-1},j_1,\dots,j_r,i_s} \circ (\widetilde{A}^{r,1}_{\Lambda_0})_{\widetilde{J}^{s-1}\mathbb{R}^{m,n}} = y^p_{i_1,\dots,i_s,j_1,\dots,j_r}. \ \blacksquare$$

Write

$$\widetilde{B}^{r,s}_{m,n} : (\widetilde{J}^r\widetilde{J}^s\mathbb{R}^{m,n})_0 \to (\widetilde{J}^s\widetilde{J}^r\mathbb{R}^{m,n})_0$$

for the restriction of $(\widetilde{A}^{r,s}_{\Lambda^0})_{\mathbb{R}^{m,n}}$ to the fiber over $0 \in \mathbb{R}^m$. Quite analogously to the linear isomorphism (3), the map $\widetilde{B}^{r,s}_{m,n}$ satisfies the invariance condition:

Lemma 6. *Let $\Phi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,k}$ be as in Lemma 2. Then for any $v \in (\widetilde{J}^r\widetilde{J}^s\mathbb{R}^{m,n})_0$ we have*

$$\widetilde{B}^{r,s}_{m,k}(\widetilde{J}^r\widetilde{J}^s\Phi(v)) = \widetilde{J}^s\widetilde{J}^r\Phi(\widetilde{B}^{r,s}_{m,n}(v)). \tag{14}$$

Proof. Clearly, φ_1 preserves Λ^0. Then our assertion follows immediately from the invariance of $\widetilde{A}^{r,s}_{\Lambda^0}$ with respect to Φ. ■

Let $\Psi = (\underline{\psi}, \psi) : Y \to \mathbb{R}^{m,n}$ be any fibered coordinate system such that $\underline{\psi} : M \to \mathbb{R}^m$ is a normal coordinate system of Λ with center z, $\underline{\psi}(z) = 0$.

Definition 1. Let $Y \to M$ be an $\mathcal{FM}_{m,n}$-object and let Λ be a classical linear connection on M. We define an $\mathcal{FM}_m$-natural equivalence depending on Λ,

$$(\widetilde{B}^{r,s}_{\Lambda})_Y : \widetilde{J}^r\widetilde{J}^sY \to \widetilde{J}^s\widetilde{J}^rY, \quad (\widetilde{B}^{r,s}_{\Lambda})_Y(v) := \widetilde{J}^s\widetilde{J}^r\Psi^{-1}(\widetilde{B}^{r,s}_{m,n}(\widetilde{J}^r\widetilde{J}^s\Psi(v))), \tag{15}$$

$v \in \widetilde{J}^r_z\widetilde{J}^sY$, $z \in M$.

From Lemma 6 it follows that this definition does not depend on the choice of a fibered coordinate system Ψ with the above property. Using Lemma 4 we obtain directly

Proposition 1. *$\widetilde{B}^{r,s}_{\Lambda} : \widetilde{J}^r\widetilde{J}^s \to \widetilde{J}^s\widetilde{J}^r$ sends J^rJ^sY into J^sJ^rY and the restriction of $\widetilde{B}^{r,s}_{\Lambda}$ to J^rJ^sY is equal to $A^{r,s}_{\Lambda}$.*

Moreover, we have the following coordinate description of $\widetilde{B}^{r,s}_{\Lambda}$:

Proposition 2. *Let (x^i, y^p) be the usual fibered coordinate system on $\mathbb{R}^{m,n}$ and let Λ be a linear connection on $\mathbb{R}^m$ such that (x^i) is a normal coordinate system of Λ with center 0. Then for all $v \in (\widetilde{J}^{r+s}\mathbb{R}^{m,n})_0$ we have*

$$y^p_{j_1,\dots,j_r,i_1,\dots,i_s} \circ (\widetilde{B}^{r,s}_{\Lambda})_{\mathbb{R}^{m,n}}(v) = y^p_{i_1,\dots,i_s,j_1,\dots,j_r}(v).$$

Proof. This follows from Lemma 5 and from the definition of $\widetilde{B}^{r,s}_{\Lambda}$. ■

4. Solution of Problem 1

PROPOSITION 3. *Let F be either J^r or $\bar{J}^r$ or $\widetilde{J}^r$, and G be either J^s or $\bar{J}^s$ or $\widetilde{J}^s$. Then $\widetilde{B}^{r,s}_\Lambda : \widetilde{J}^r\widetilde{J}^s \to \widetilde{J}^s\widetilde{J}^r$ sends FGY into GFY.*

Proof. Because of the invariance of $\widetilde{B}^{r,s}_\Lambda$ it suffices to show that $(\widetilde{B}^{r,s}_\Lambda)_{\mathbb{R}^{m,n}}$ sends $(FG\mathbb{R}^{m,n})_0$ into $(GF\mathbb{R}^{m,n})_0$ for any linear connection Λ on $\mathbb{R}^m$ such that the usual coordinate system on $\mathbb{R}^m$ is a normal one for Λ with center $0 \in \mathbb{R}^m$. This follows immediately from Proposition 2 and from the coordinate characterization of holonomic and semiholonomic prolongation. ∎

DEFINITION 2. Let F be either J^r or $\bar{J}^r$ or $\widetilde{J}^r$, and G be either J^s or $\bar{J}^s$ or $\widetilde{J}^s$. The restriction from Proposition 3 defines $\mathcal{FM}_m$-natural equivalences depending on Λ,

$$B^{F,G}_\Lambda : FG \to GF. \tag{16}$$

In particular, $B^{J^r,J^s}_\Lambda = A^{r,s}_\Lambda$ and $B^{\widetilde{J}^r,\widetilde{J}^s}_\Lambda = \widetilde{B}^{r,s}_\Lambda$.

PROPOSITION 4. *We have $\widetilde{B}^{s,r}_\Lambda \circ \widetilde{B}^{r,s}_\Lambda = \mathrm{id}$. In particular, $B^{G,F}_\Lambda \circ B^{F,G}_\Lambda = \mathrm{id}$.*

Proof. This is an immediate consequence of Proposition 2 and the definition of $\widetilde{B}^{r,s}_\Lambda$. ∎

An important feature of the canonical involution $TTN \to TTN$ of the iterated tangent bundle is that this map interchanges the two projections of TTN into TN. This concept can be generalized as follows.

DEFINITION 3. Let F and G be two bundle functors on $\mathcal{FM}_m$ and denote by $p^F_Y : FY \to Y$, $p^G_Y : GY \to Y$ the bundle projections. An $\mathcal{FM}_m$-natural equivalence $A_\Lambda : FG \to GF$ depending on Λ is called an *involution* if

$$p^G_{FY} \circ (A_\Lambda)_Y = F(p^G_Y). \tag{17}$$

This means that $(A_\Lambda)_Y$ interchanges the projections $F(p^G_Y) : FGY \to FY$ and $p^G_{FY} : GFY \to FY$. One verifies easily

PROPOSITION 5. *$B^{F,G}_\Lambda : FG \to GF$ is an involution. In particular, $A^{r,s}_\Lambda : J^rJ^s \to J^sJ^r$ and $\widetilde{B}^{r,s}_\Lambda : \widetilde{J}^r\widetilde{J}^s \to \widetilde{J}^s\widetilde{J}^r$ are involutions.*

REMARK 1. It is well known that for every pair F, G of product preserving functors on the category $\mathcal{M}f$ there is an exchange natural equivalence $FG \to GF$, [14]. In particular, denoting by $T^r_m N = J^r_0(\mathbb{R}^m, N)$ the bundle of m-dimensional velocities of order r, we have a natural equivalence $\kappa : T^r_m T^s_m \to T^s_m T^r_m$, which generalizes the classical involution of iterated tangent bundle. In [6] we have shown that the isomorphism (3) corresponds to the canonical isomorphism κ. This means that our natural equivalences $B^{F,G}_\Lambda$ from Problem 1 generalize the exchange map of iterated velocities functors to the case of fibered manifolds. However, to define exchange natural equivalences of iterated jet functors, the use of some auxiliary connection Λ is unavoidable. We also point out that there is an open problem to define an exchange natural equivalence $FG \to GF$ depending on Λ, for any couple F, G of fiber product preserving functors on $\mathcal{FM}_m$. So our natural equivalences $B_\Lambda^{F,G}$ are particular solutions of this general problem for any couple of higher order jet functors.

5. Classification of some natural transformations of iterated jets. Write $J^0Y := Y$ and denote by $\pi^r_s : J^rY \to J^sY$ the jet projection. It is well known that $\pi^r_{r-1} : J^rY \to J^{r-1}Y$ is an affine bundle, the associated vector bundle of which is the pullback of $S^rT^*M \otimes VY$ over $J^{r-1}Y$. If $v \in (J^2J^1Y)_y$, $y \in Y_x$, $x \in M$, then we have two elements $v' = \pi^2_0(v) \in (J^1Y)_y$ and $v'' = J^1\pi^1_0 \circ \pi^2_1(v) \in (J^1Y)_y$. So we can define a fibered map covering the identity of Y,

$$\sigma : J^2J^1Y \to T^*M \otimes VY, \quad \sigma(v) := v'' - v' \in T^*_xM \otimes V_yY. \tag{18}$$

Suppose we have an $\mathcal{M}f_m$-natural operator $C : Con_\tau \rightsquigarrow T \otimes S^2T^* \otimes T^*$ transforming torsion free classical linear connections Λ on M into tensor fields $C(\Lambda)$ of type $T \otimes S^2T^* \otimes T^*$ on M. Given a torsion free classical linear connection Λ on M, we have the contraction

$$\langle C(\Lambda), \sigma\rangle : J^2J^1Y \to S^2T^*M \otimes T^*M \otimes VY$$

covering the identity of Y (we contract T from C with T^* from σ). This can obviously be treated as the fibered map

$$\langle C(\Lambda), \sigma\rangle : J^2J^1Y \to (\pi^1_0 \circ \pi^1_0)^*(S^2T^*M \otimes T^*M \otimes VY)$$

covering the identity of J^1J^1Y, where $(\ )^*$ denotes pullback.

Obviously, $\pi^2_1 : J^2J^1Y \to J^1J^1Y$ is an affine bundle with the associated vector bundle $(\pi^1_0)^*(S^2T^*M \otimes VJ^1Y)$, where $\pi^1_0 : J^1J^1Y \to J^1Y$ and $VJ^1Y \to J^1Y$ is the vertical bundle of $J^1Y \to M$. But $\pi^1_0 : J^1Y \to Y$ is an affine bundle with the associated vector bundle $T^*M \otimes VY$. Then the vertical bundle V^YJ^1Y of $\pi^1_0 : J^1Y \to Y$ is $(\pi^1_0)^*(T^*M \otimes VY)$. The obvious inclusion $V^YJ^1Y \subset VJ^1Y$ yields the induced inclusion (vector bundle monomorphism)

$$(\pi^1_0 \circ \pi^1_0)^*(S^2T^*M \otimes T^*M \otimes VY) \subset (\pi^1_0)^*(S^2T^*M \otimes VJ^1Y)$$

over the identity of J^1J^1Y. So we have an $\mathcal{FM}_m$-natural transformation depending on Λ

$$(A^C_\Lambda)_Y := \mathrm{id}_{J^2J^1Y} + \langle C(\Lambda), \sigma\rangle : J^2J^1Y \to J^2J^1Y$$

covering the identity of M (even the identity of J^1J^1Y). In the next section we prove

THEOREM 1. *Let $n \geq 2$. All $\mathcal{FM}_{m,n}$-natural transformations $A_\Lambda : J^2J^1 \to J^2J^1$ depending on a torsion free classical linear connection Λ on the base manifold are of the form*

$$A_\Lambda := A_\Lambda^C = \mathrm{id}_{J^2J^1} + \langle C(\Lambda), \sigma \rangle$$

for an $\mathcal{M}f_m$-natural operator $C : Con_\tau \rightsquigarrow T \otimes S^2T^* \otimes T^*$. Given an $\mathcal{FM}_{m,n}$-natural transformation A_Λ as above, the corresponding operator C is determined uniquely.

REMARK 2. An example of such an $\mathcal{M}f_m$-natural operator $C : Con_\tau \rightsquigarrow T \otimes S^2T^* \otimes T^*$ is

$$C(\Lambda)(\omega, u, w, v) = (\mathrm{sym}(\mathrm{Ric}_\Lambda))(u, w)\langle \omega, v \rangle, \quad u, v, w \in T_xM, \omega \in T_x^*M,$$

where Ric_Λ is the Ricci tensor field of Λ and sym denotes symmetrization. By Theorem 33.16 from [14], given a linear connection Λ on M, all $C(\Lambda)$ can be obtained from the curvature tensor field R_Λ of Λ by the following procedure: (1) we tensor R_Λ several times with the identity tensor field on M; (2) we apply permutations of arguments of such tensor fields; (3) we apply appropriate contractions of such tensor fields to obtain tensor fields of type $(1, 3)$; (4) we take appropriate symmetrizations of such tensor fields to obtain tensor fields of type $T \otimes S^2T^* \otimes T^*$; (5) we take linear combinations of such tensor fields.

REMARK 3. Using $A_\Lambda^{r,s} : J^rJ^s \to J^sJ^r$ defined by (5), Theorem 1 also gives the classification of all $\mathcal{FM}_{m,n}$-natural transformations $B_\Lambda : J^2J^1 \to J^1J^2$, $D_\Lambda : J^1J^2 \to J^2J^1$ and $E_\Lambda : J^1J^2 \to J^1J^2$ depending on a torsion free connection Λ. For example, all such B_Λ are of the form

$$B_\Lambda := A_\Lambda^{2,1} \circ A_\Lambda^C = A_\Lambda^{2,1} \circ (\mathrm{id}_{J^2J^1} + \langle C(\Lambda), \sigma \rangle) : J^2J^1 \to J^1J^2 \tag{19}$$

for any $\mathcal{M}f_m$-natural operator $C : Con_\tau \rightsquigarrow T \otimes S^2T^* \otimes T^*$.

6. Proof of Theorem 1. We denote the usual coordinates on $\mathbb{R}^{m,n}$ by $x^1, \ldots, x^m$, $y^1, \ldots, y^n$. Taking into account the notation (1), the coordinates on $\mathbb{R}^m \times \mathbb{R}^m$ will be denoted by $x^1, \ldots, x^m$, $\underline{x}^1, \ldots, \underline{x}^m$. In this section we will use the methods from [14] and [16].

LEMMA 7. *Let $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. Let $\mathcal{K} \subset J_0^rJ^s\mathbb{R}^{m,n}$ be a vector subspace such that*

$$j_0^rj^s(x^i, 0, \ldots, 0) \in \mathcal{K} \quad \text{for } i = 1, \ldots, m, \tag{20}$$

$$J_0^rJ^s\varphi(\mathcal{K}) \subset \mathcal{K} \tag{21}$$

for any $\mathcal{FM}_{m,n}$-map $\varphi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ covering $\mathrm{id}_{\mathbb{R}^m}$. If $n \geq 2$, then $\mathcal{K} = J_0^rJ^s\mathbb{R}^{m,n}$.

Proof. In the proof we use several times the formula (2) for special $\mathcal{FM}_m$-maps Φ. First, using the invariance (21) of $\mathcal{K}$ with respect to the $\mathcal{FM}_{m,n}$-map

$(x^1,\dots,x^m,y^1,y^2+x^\alpha,\dots,y^n)$, from $j^r_0 j^s(0,\dots,0)\in\mathcal{K}$ we obtain

$$j^r_0 j^s(0,\underline{x}^\alpha,0,,0)\in\mathcal{K} \tag{22}$$

for any m-tuple α. Applying (21) for the map $(x^1,\dots,x^m,y^1+(y^1)^k, y^2,\dots,y^n)$, from (20) we get

$$j^r_0 j^s((x^i)^k,0,\dots,0)\in\mathcal{K} \tag{23}$$

for any $k=0,1,2,\dots$ and $i=1,\dots,m$. By induction on $s\le m$ we prove

$$j^r_0 j^s((x^1)^{k_1}\dots(x^s)^{k_s},0,\dots,0)\in\mathcal{K}. \tag{24}$$

For $s=1$ we apply (23). Suppose that (24) is true for $s<m$. To prove it for $s+1$, applying (21) for permutations of fibered coordinates, from (23) we get

$$j^r_0 j^s(0,(x^{s+1})^{k_{s+1}},0,\dots,0)\in\mathcal{K}. \tag{25}$$

Taking the sum of (24) and (25) and considering the induction assumption we deduce

$$j^r_0 j^s((x^1)^{k_1}\dots(x^s)^{k_s},(x^{s+1})^{k_{s+1}},0,\dots,0)\in\mathcal{K}. \tag{26}$$

Applying (21) for the $\mathcal{FM}_{m,n}$-map $(x^1,\dots,x^m,y^1+y^1y^2,y^2,\dots,y^n)$, from (26) we deduce (24) for $s+1$ in place of s, so that the proof of (24) is complete. For $s=m$ we have

$$j^r_0 j^s(x^\alpha,0,\dots,0)\in\mathcal{K} \tag{27}$$

for any m-tuple α. From (22) and (27) it follows that

$$j^r_0 j^s(x^\alpha,\underline{x}^\beta,0,\dots,0)\in\mathcal{K}.$$

Then (21) for the map $(x^1,\dots,x^m,y^1+y^1y^2,y^2,\dots,y^n)$ yields

$$j^r_0 j^s(x^\alpha\underline{x}^\beta,0,\dots,0)\in\mathcal{K}$$

for any m-tuples α and β. Finally, applying (21) for permutations of fibered coordinates we obtain

$$j^r_0 j^s(0,\dots,x^\alpha\underline{x}^\beta,\dots,0)\in\mathcal{K} \tag{28}$$

for any m-tuples α and β with $x^\alpha\underline{x}^\beta$ in position $s=1,\dots,n$. Since all elements (28) generate the vector space $J^r_0 J^s\mathbb{R}^{m,n}$, we have proved $\mathcal{K}=J^r_0 J^s\mathbb{R}^{m,n}$. ∎

Lemma 8. *If $n\ge 2$, then $(A_\Lambda)_Y : J^2J^1Y\to J^2J^1Y$ covers the identity of J^1J^1Y for any $\mathcal{FM}_{m,n}$-object $Y\to M$ and any torsion free classical linear connection Λ on M. More precisely, $\pi^2_1\circ A_\Lambda=\pi^2_1$.*

Proof. By the existence of normal coordinates it suffices to show that

$$\pi^2_1\circ A_\Lambda(v)=\pi^2_1(v)$$

for any $v\in J^2_0J^1\mathbb{R}^{m,n}$ and any torsion free classical linear connection Λ on $\mathbb{R}^m$ with vanishing Christoffel symbols at $0\in\mathbb{R}^m$. (We may assume that the

Christoffel symbols of Λ vanish at $0 \in \mathbb{R}^m$ because the Christoffel symbols of torsion free classical linear connections at the center of normal coordinates vanish.) We fix such a Λ and denote

$$\mathcal{K} := \{v \in J_0^2 J^1 \mathbb{R}^{m,n} \mid \pi_1^2 \circ A_\Lambda(v) = \pi_1^2(v)\}.$$

It suffices to show $\mathcal{K} = J_0^2 J^1 \mathbb{R}^{m,n}$. Now we use several times the formula (2) for special $\mathcal{FM}_m$-maps Φ. Because of Lemma 7 we have to verify that $\mathcal{K}$ is a vector subspace and that the conditions (20) and (21) of Lemma 7 for $r = 2$ and $s = 1$ are satisfied. Applying the invariance of $\pi_1^2 \circ A_\Lambda$ and π_1^2 with respect to fiber homotheties we deduce

$$\pi_1^2 \circ A_\Lambda(tv) = t\pi_1^2 \circ A_\Lambda(v) \quad \text{and} \quad \pi_1^2(tv) = t\pi_1^2(v) \quad \text{for } t \neq 0.$$

By the homogeneous function theorem [14, Th. 24.1], $\pi_1^2 \circ A_\Lambda(v)$ and $\pi_1^2(v)$ are linear in v. Hence $\mathcal{K}$ is a vector subspace in $J_0^2 J^1 \mathbb{R}^{m,n}$. Next, the invariance of A yields

$$\pi_1^2 \circ A_\Lambda \circ J^2 J^1 \varphi = J^1 J^1 \varphi \circ \pi_1^2 \circ A_\Lambda, \quad \pi_1^2 \circ J^2 J^1 \varphi = J^1 J^1 \varphi \circ \pi_1^2$$

for $\mathcal{FM}_{m,n}$-maps $\varphi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ covering $\mathrm{id}_{\mathbb{R}^m}$. Therefore $\mathcal{K}$ satisfies (21). So it remains to verify (20). Write

$$\begin{aligned}
&\pi_1^2 \circ A_\Lambda(j_0^2 j^1(x^i, 0, \dots, 0)) \\
&= j_0^1 j^1 \Big(\Big(a^k(\Lambda) + \sum_{j=1}^m b_j^k(\Lambda) x^j + \sum_{j=1}^m c_j^k(\Lambda) \underline{x}^j + \sum_{a,b=1}^m \sum_{k=1}^n d_{ab}^k(\Lambda) x^a \underline{x}^b\Big)\Big)
\end{aligned}$$

for unique real numbers $a^k(\Lambda)$, $b_j^k(\Lambda)$, $c_j^k(\Lambda)$ and $d_{ab}^k(\Lambda)$ depending on the system $(\Lambda_{q,r;\alpha}^p(0))$ of all derivatives $\Lambda_{q,r;\alpha}^p(0)$ at 0 of the Christoffel symbols $\Lambda_{q,r}^p$ of Λ. Below, for the sake of simplicity we usually omit the brackets and write $\Lambda_{q,r;\alpha}^p(0)$ instead of $(\Lambda_{q,r;\alpha}^p(0))$. By the nonlinear Peetre theorem [14, Th. 19.7] we can assume $\Lambda_{q,r:\alpha}^p(0) = 0$ for $|\alpha| \geq K$ for some finite K. Using the invariance of A with respect to the base homotheties $t\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n}$ we obtain

$$\begin{aligned}
a^k(t^{|\alpha|+1}\Lambda_{q,r:\alpha}^p(0)) &= \frac{1}{t} a^k(\Lambda_{q,r:\alpha}^p(0)), \qquad & b_j^k(t^{|\alpha|+1}\Lambda_{q,r;\alpha}^p(0)) &= b_j^k(\Lambda_{q,r;\alpha}^p(0)), \\
c_j^k(t^{|\alpha|+1}\Lambda_{q,r;\alpha}^p(0)) &= c_j^k(\Lambda_{q,r:\alpha}^p(0)), \qquad & d_{ab}^k(t^{|\alpha|+1}\Lambda_{q,r;\alpha}^p(0)) &= t d_{ab}^k(\Lambda_{q,r;\alpha}^p(0))
\end{aligned}$$

for $t \neq 0$. As $\Lambda_{q,r}^p(0) = 0$, by the homogeneous function theorem we deduce

$$a^k(\Lambda) = 0, \quad b_j^k(\Lambda) = b_j^k(\Lambda_0), \quad c_j^k(\Lambda) = c_j^k(\Lambda_0), \quad d_{ab}^k(\Lambda) = 0,$$

where Λ_0 is the connection on $\mathbb{R}^m$ with vanishing Christoffel symbols. Thus

$$\pi_1^2 \circ A_\Lambda(j_0^2 j^1(x^i, 0, \dots, 0)) = \pi_1^2 \circ A_{\Lambda_0}(j_0^2 j^1(x^i, 0, \dots, 0)).$$

So we can assume $\Lambda = \Lambda_0$. Applying the invariance of A_{Λ_0} with respect to fiber homotheties we show that $(A_{\Lambda_0})_0 : J_0^2 J^1 \mathbb{R}^{m,n} \to J_0^2 J^1 \mathbb{R}^{m,n}$ is linear. Next, using the invariance of A_{Λ_0} with respect to $(t^1x^1, \ldots, t^mx^m, y^1, \tau y^2, \ldots, \tau y^n)$ for $t^1, \ldots, t^m, \tau \neq 0$ we prove easily

$$A_{\Lambda_0}(j_0^2 j^1(x^i, 0, \ldots, 0)) = j_0^2 j^1(ax^i + b\underline{x}^i, 0, \ldots, 0) \tag{29}$$

for some $a, b \in \mathbb{R}$. By the invariance of A_{Λ_0} with respect to the $\mathcal{FM}_{m,n}$-map $(x^1, \ldots, x^m, y^1 - x^i, y^2, \ldots, y^n)$ we deduce from (29) that

$$A_{\Lambda_0}(j_0^2 j^1(x^i - \underline{x}^i, 0, \ldots, 0)) = j_0^2 j^1(ax^i + b\underline{x}^i - \underline{x}^i, 0, \ldots, 0). \tag{30}$$

By the invariance of A_{Λ_0} with respect to the $\mathcal{FM}_{m,n}$-map $(x^1, \ldots, x^m, y^1 + (y^1)^2, y^2, \ldots, y^n)$ we get from (30)

$$A_{\Lambda_0}(j_0^2 j^1((x^i - \underline{x}^i)^2, 0, \ldots, 0)) = j_0^2 j^1((ax^i + b\underline{x}^i - \underline{x}^i)^2, 0, \ldots, 0).$$

Since $j_0^2 j^1(\underline{x}^i - x^i)^2 = 0$ (as $j_x^1(\underline{x}^i - x^i)^2 = 0$), we have

$$\begin{aligned} 0 &= j_0^2 j^1((a + b - 1)x^i + (b - 1)(\underline{x}^i - x^i))^2 \\ &= j_0^2 j^1((a + b - 1)^2(x^i)^2 + 2(a + b - 1)(b - 1)x^i(\underline{x}^i - x^i)). \end{aligned}$$

Then $(a + b - 1)^2 = 0$ and $2(a + b - 1)(b - 1) = 0$, so that $a + b = 1$. Similarly, applying the invariance of A_{Λ_0} with respect to the $\mathcal{FM}_{m,n}$-morphism $(x^1, \ldots, x^m,\ y^1 + (y^1)^3, y^2, \ldots, y^n)$, we get from (29)

$$A_{\Lambda_0}(j_0^2 j^1((x^i)^3, 0, \ldots, 0)) = j_0^2 j^1((ax^i + b\underline{x}^i)^3, 0, \ldots, 0).$$

Since $j_0^2 j^1(x^i)^3 = 0$ (as $j_0^2(x^i)^3(\ldots) = 0$), we have

$$\begin{aligned} 0 &= j_0^2 j^1(ax^i + b\underline{x}^i)^3 = j_0^2 j^1((a + b)x^i + b(\underline{x}^i - x^i))^3 = j_0^2 j^1((a + b)^3(x^i)^3 \\ &\quad + 3(a + b)^2(x^i)^2 b(\underline{x}^i - x^i) + 3(a + b)x^i b^2(\underline{x}^i - x^i)^2 + b^3(\underline{x}^i - x^i)^3) \\ &= 3(a + b)^2 b j_0^2 j^1((x^i)^2(\underline{x}^i - x^i)). \end{aligned}$$

Hence $3(a + b)^2 b = 0$. But $a + b = 1$, so that $b = 0$. Then

$$A(\Lambda_0)(j_0^2 j^1(x^i, 0, \ldots, 0)) = j_0^2 j^1(x^i, 0, \ldots, 0),$$

which yields

$$\pi_1^2 \circ A_{\Lambda_0}(j_0^2 j^1(x^i, 0, \ldots, 0)) = \pi_1^2(j_0^2 j^1(x^i, 0, \ldots, 0)).$$

So we have obtained the condition (20) of Lemma 7, which completes the proof. ■

Let A_Λ be the natural transformation from Theorem 1. Using Lemma 8 and the affine bundle structure, for any torsion free classical linear connection Λ on M and any $\mathcal{FM}_{m,n}$-object $Y \to M$ we have the unique fibered map $B(\Lambda) : J^2J^1Y \to (\pi_0^1)^*(S^2T^*M \otimes VJ^1Y)$ covering the identity of J^1J^1Y such that $(A_\Lambda)_Y = \mathrm{id}_{J^2J^1Y} + B(\Lambda)$. Obviously, $B(\Lambda)$ can be treated as the fibered map

$$B(\Lambda) : J^2J^1Y \to S^2T^*M \otimes VJ^1Y$$

covering the identity of J^1Y, where $VJ^1Y \to J^1Y$ is the vertical bundle of $J^1Y \to M$.

Lemma 9. *We have*

$$B(\Lambda): J^2J^1Y \to S^2T^*M \otimes V^Y J^1Y \subset S^2T^*M \otimes VJ^1Y,$$

where $V^Y J^1Y$ *is the vertical bundle of* $\pi^1_0 : J^1Y \to Y$.

Proof. Composing $B(\Lambda)$ with the differential of $\pi^1_0 : J^1Y \to Y$ we obtain the fibered map

$$\tilde{B}(\Lambda): J^2J^1Y \to S^2T^*M \otimes VY$$

covering the identity of Y. Because of the $\mathcal{FM}_{m,n}$-invariance of $B(\Lambda)$, the existence of normal coordinates of Λ and the fact that Λ is torsion free, under standard identifications it suffices to show

$$\tilde{B}(\Lambda)(v) = 0 \in S^2\mathbb{R}^{m*} \otimes \mathbb{R}^n = S^2T^*_0\mathbb{R}^m \otimes T_y\mathbb{R}^n$$

for any $v \in (J^2_0 J^1\mathbb{R}^{m,n})_y$, $y \in (\mathbb{R}^{m,n})_0 = \mathbb{R}^n$ and for a torsion free classical linear connection Λ with vanishing Christoffel symbols at $0 \in \mathbb{R}^m$. We will use similar methods to the proof of Lemma 8. Fix a Λ on $\mathbb{R}^m$ with the above mentioned properties and write

$$\mathcal{K} = \{v \in J^2_0 J^1\mathbb{R}^{m,n} \mid \tilde{B}(\Lambda)(v) = 0\}.$$

It remains to show $\mathcal{K} = J^2_0 J^1\mathbb{R}^{m,n}$. For this, we prove the assumptions of Lemma 7. By the invariance of $\tilde{B}(\Lambda)$ with respect to fiber homotheties we deduce that $\tilde{B}(\Lambda)(v)$ is linear in v. Hence $\mathcal{K}$ is a vector subspace. Applying the invariance of $\tilde{B}(\Lambda)$ with respect to $\mathcal{FM}_{m,n}$-maps covering the identity of $\mathbb{R}^m$ we deduce the condition (21) of Lemma 7. It remains to show

$$\tilde{B}(\Lambda)(j^2_0 j^1(x^i, 0, \dots, 0)) = 0 \in S^2T^*_0\mathbb{R}^m \otimes T_0\mathbb{R}^n. \tag{31}$$

For this we use the invariance of $\tilde{B}$ with respect to $t^{-1}\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n}$ to obtain the homogeneity condition

$$t\tilde{B}(\Lambda^p_{q,r;\alpha}(0))(j^2_0 j^1(x^i, 0, \dots, 0)) = \tilde{B}(t^{|\alpha|+1}\Lambda^p_{q,r;\alpha}(0))(j^2_0 j^1(x^i, 0, \dots, 0)).$$

Taking into account $\Lambda^p_{q,r}(0) = 0$ and using the same arguments as in the proof of Lemma 8 (i.e. the nonlinear Peetre theorem and the homogeneous function theorem) we obtain (31). ∎

But we remarked above that $V^Y J^1Y = (\pi^1_0)^*(T^*M \otimes VY)$. Hence by Lemma 9 we can treat $B(\Lambda)$ as the fibered map

$$B(\Lambda): J^2J^1Y \to S^2T^*M \otimes T^*M \otimes VY$$

covering the identity of Y. Thus we have proved

Proposition 6. *If* $n \geq 2$, *then* A_Λ *can be decomposed in the form*

$$(A_\Lambda)_Y = \mathrm{id}_{J^2J^1Y} + B(\Lambda) \tag{32}$$

for some fibered map $B(\Lambda): J^2J^1Y \to S^2T^*M \otimes T^*M \otimes VY$ *covering the identity of* Y.

PROPOSITION 7. *Let $n \geq 2$ and consider $B(\Lambda)$ from Proposition 6. There is a unique $\mathcal{M}f_m$-natural operator $C : Q_\tau \rightsquigarrow T \otimes S^2T^* \otimes T^*$ such that*

$$B(\Lambda) = \langle C(\Lambda), \sigma\rangle : J^2J^1Y \to S^2T^*M \otimes T^*M \otimes VY \tag{33}$$

for any torsion free classical linear connection Λ on M and any $\mathcal{FM}_{m,n}$-object $Y \to M$, where $\sigma : J^2J^1Y \to T^*M \otimes VY$ is the fibered map (18).

Proof. Let Λ be a torsion free classical linear connection on an m-dimensional manifold M. We have the fibered map

$$B(\Lambda) : J^2J^1(M \times \mathbb{R}^n) \to S^2T^*M \otimes T^*M \otimes V(M \times \mathbb{R}^n)$$

covering the identity of $M \times \mathbb{R}^n$, where $M \times \mathbb{R}^n$ is the trivial bundle over M with fiber $\mathbb{R}^n$. Thus, for $x_0 \in M$ we have

$$B(\Lambda)_{(x_0,0)} : (J^2J^1(M \times \mathbb{R}^n))_{(x_0,0)} \to S^2T^*_{x_0}M \otimes T^*_{x_0}M \otimes T_0\mathbb{R}^n.$$

We define a tensor field $C(\Lambda)$ of type $T \otimes S^2T^* \otimes T^*$ on M by

$$\begin{aligned}\langle C(\Lambda)_{x_0}, \omega \otimes (u \odot v) \otimes w\rangle \\ = dy^1(\langle B(\Lambda)_{(x_0,0)}(j^2_{x_0}j^1(f(x), 0, \ldots, 0)), (u \odot v) \otimes w\rangle) \in \mathbb{R}\end{aligned} \tag{34}$$

for any $x_0 \in M$, any $\omega \in T^*_{x_0}M$ and any $u, v, w \in T_{x_0}M$, where $y^1 : \mathbb{R}^n \to \mathbb{R}$ is the first coordinate and $f : M \to \mathbb{R}$ is such that $f(x_0) = 0$ and $d_{x_0}f = \omega$. By Lemma 10 below, the definition of $C(\Lambda)$ is correct and (33) follows from Lemma 11. It remains to prove the uniqueness. If $C' : Q_\tau \rightsquigarrow T \otimes S^2T^* \otimes T^*$ is another $\mathcal{M}f_m$-natural operator such that $B = \langle C', \sigma\rangle$, then we obtain (38) with C' instead of C, so that by (39) we get (34) with C' instead of C over $0 \in \mathbb{R}^m$. By $\mathcal{M}f_m$-invariance we have $C = C'$, which completes the proof. ∎

LEMMA 10. *The element $C(\Lambda)_{x_0}$ is well-defined and belongs to $T_{x_0}M \otimes S^2T^*_{x_0}M \otimes T^*_{x_0}M$.*

Proof. It suffices to prove that $B(\Lambda)_{(x_0,0)}$ is linear and

$$B(\Lambda)_{(x_0,0)}(j^2_{x_0}j^1(f_1(x)f_2(x), 0, \ldots, 0)) = 0 \tag{35}$$

for any $f_1, f_2 : M \to \mathbb{R}$ with $f_1(x_0) = f_2(x_0) = 0$. Then the definition (34) is independent of the choice of f and $C(\Lambda)_{x_0} \in T_{x_0}M \otimes S^2T^*_{x_0}M \otimes T^*_{x_0}M$. The linearity is a simple consequence of the invariance of $B(\Lambda)$ with respect to the fiber homotheties $\mathrm{id}_M \times t\,\mathrm{id}_{\mathbb{R}^n}$ and the homogeneous function theorem. Applying the $\mathcal{FM}_{m,n}$-map $\mathrm{id}_M \times (y^1 + y^1y^2, y^2, \ldots, y^n)$ to $B(\Lambda)_{(x_0,0)}(j^2_{x_0}j^1(f_1(x), f_2(x), 0, \ldots, 0))$, we obtain (35). More precisely,

$$\begin{aligned}B(\Lambda)_{(x_0,0)}(j^2_{x_0}j^1(f_1(x) + f_1(x)f_2(x), f_2(x), 0, \ldots, 0)) \\ = B(\Lambda)_{(x_0,0)}(j^2_{x_0}j^1(f_1(x), f_2(x), 0, \ldots, 0)),\end{aligned}$$

which implies (because of linearity) the equality (35). ∎

LEMMA 11. *We have $B(\Lambda) = \langle C(\Lambda), \sigma\rangle$.*

Proof. Because of the $\mathcal{FM}_{m,n}$-invariance of B and C it suffices to show

$$
\begin{aligned}
(36)\qquad B(\Lambda)(v) = \langle C(\Lambda)_0, \sigma(v)\rangle &\in S^2\mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^n \\
&= S^2T_0^*\mathbb{R}^m \otimes T_0^*\mathbb{R}^m \otimes T_y\mathbb{R}^n
\end{aligned}
$$

for any torsion free classical linear connection Λ on $\mathbb{R}^m$ with vanishing Christoffel symbols at $0 \in \mathbb{R}^m$ and any $v \in (J_0^2J^1\mathbb{R}^{m,n})_y$, $y \in (\mathbb{R}^{m,n})_0 = \mathbb{R}^n$. Define

$$\mathcal{K} = \{v \in J_0^2J^1\mathbb{R}^{m,n} \mid B(\Lambda)(v) = \langle C(\Lambda)_0, \sigma(v)\rangle\}.$$

We have to verify the assumptions of Lemma 7. Because of the invariance of $B(\Lambda)$ and $\langle C(\Lambda), \sigma\rangle$ with respect to fiber homotheties and the homogeneous function theorem, both sides of (36) are linear in v. So $\mathcal{K}$ is a vector space. The invariance of $B(\Lambda)$ and $\langle C(\Lambda), \sigma\rangle$ with respect to $\mathcal{FM}_{m,n}$-maps covering the identity implies (21). It remains to prove (20), i.e.

$$(37)\qquad B(\Lambda)(j_0^2j^1(x^i, 0, \ldots, 0)) = \langle C(\Lambda)_0, \sigma(j_0^2j^1(x^i, 0, \ldots, 0))\rangle$$

for $i = 1, \ldots, m$. Using the invariance of both sides of (37) (separately) with respect to $(x^1, \ldots, x^m, y^1, ty^2, \ldots, ty^n)$ we deduce that the dy^j coordinates of both sides of (37) are zero for $j = 2, \ldots, n$. Thus it remains to verify

$$
\begin{aligned}
(38)\qquad & dy^1(\langle B(\Lambda)(j_0^2j^1(x^i, 0, \ldots, 0)), (u \odot v) \otimes w\rangle) \\
&\qquad = dy^1(\langle\langle C(\Lambda)_0, (u \odot v) \otimes w\rangle, \sigma(j_0^2j^1(x^i, 0, \ldots, 0))\rangle)
\end{aligned}
$$

for any $u, v, w \in T_0\mathbb{R}^m$. By the definition of σ it is easy to see that

$$(39)\qquad \sigma(j_0^2j^1(x^i, 0, \ldots, 0)) = d_0x^i \otimes \frac{\partial}{\partial y^1}_{|(0,0)}.$$

So the right hand side of (38) is equal to $\langle C(\Lambda)_0, d_0x^i \otimes (u \odot v) \otimes w\rangle$ and (38) follows from the definition (34). ∎

7. Prolongation of higher order connections

A. Prolongation of connections into connections on $FY \to M$. Let F be any of the functors J^r, $\bar{J}^r$, $\widetilde{J}^r$. In [4] (for $r = 1$) and in [5] (for all r) we have proved that there is no $\mathcal{FM}_{m,n}$-natural operator transforming rth order holonomic connections on $Y \to M$ into sth order holonomic connections on $FY \to M$. So the use of a classical linear connection Λ in the following geometric constructions is unavoidable.

Definition 4. Let G be either $\widetilde{J}^r$ or $\bar{J}^r$ or J^r. A *G-connection* on a fibered manifold $Y \to M$ is a section $\Gamma : Y \to GY$.

In particular, for $G = J^r$ we obtain the concept of an rth order holonomic connection.

Proposition 8. *Let F be either J^r or $\bar{J}^r$ or $\widetilde{J}^r$, let G be either J^s or $\bar{J}^s$ or $\widetilde{J}^s$, and let $\Gamma : Y \to GY$ be a G-connection on $Y \to M$. If $B_\Lambda^{F,G}$:*

$FG \to GF$ is an $\mathcal{FM}_m$-natural equivalence (16) *depending on Λ, then*

$$\mathcal{F}(\Gamma,\Lambda) := (B^{F,G}_\Lambda)_Y \circ F\Gamma : FY \to GFY$$

is a G-connection on $FY \to M$.

Proof. By Proposition 5, $B^{F,G}_\Lambda$ is an involution. So $p^G_{FY} \circ (B^{F,G}_\Lambda)_Y \circ F\Gamma = F(p^G_Y) \circ F\Gamma = F(p^G_Y \circ \Gamma) = \mathrm{id}_{FY}$. ∎

Moreover, formula (19) describes the classification of all $\mathcal{FM}_{m,n}$-natural transformations $B_\Lambda : J^2J^1 \to J^1J^2$ depending on a torsion free connection Λ.

COROLLARY 1. *Let $\Gamma : Y \to J^1Y$ be a connection on $Y \to M$ and let $B_\Lambda : J^2J^1 \to J^1J^2$ be any $\mathcal{FM}_{m,n}$ natural transformation depending on a torsion free connection Λ. Then*

$$\mathcal{J}^2(\Gamma,\Lambda) := (B_\Lambda)_Y \circ J^2\Gamma : J^2Y \to J^1J^2Y$$

is a connection on $J^2Y \to M$.

Proof. We know that all $(A^C_\Lambda)_Y$ from Theorem 1 cover the identity of J^1J^1Y. Then the composition $B_\Lambda := A^{2,1}_\Lambda \circ A^C_\Lambda$ is an involution. ∎

COROLLARY 2. *Let $\Gamma : Y \to J^2Y$ be a second order holonomic connection on $Y \to M$ and let $D_\Lambda := A^C_\Lambda \circ A^{1,2}_\Lambda$ be any $\mathcal{FM}_{m,n}$-natural transformation $J^1J^2 \to J^2J^1$ depending on a torsion free connection Λ. Then*

$$\mathcal{J}^1(\Gamma,\Lambda) := (D_\Lambda)_Y \circ J^1\Gamma : J^1Y \to J^2J^1Y$$

is a second order holonomic connection on $J^1Y \to M$.

REMARK 4. In [14] all natural operators transforming a connection Γ on $Y \to M$ and a classical linear connection Λ on M into a connection on $J^1Y \to M$ are classified. However, for $r > 1$ there is an open problem of classifying all natural operators transforming couples (Γ,Λ) into connections on $J^rY \to M$.

B. Prolongation of connections into connections on $FY \to Y$. By item A above, prolongation of higher order connections from $Y \to M$ to $FY \to M$ can be defined by means of an auxiliary classical linear connection Λ on M. However, prolongation of connections from $Y \to M$ to $FY \to Y$ has a quite different character.

PROPOSITION 9 ([20]). *There is no $\mathcal{FM}_{m,n}$-natural operator C transforming connections $\Gamma : Y \to J^1Y$ and classical linear connections ∇ on M into connections $C(\Gamma,\nabla)$ on $J^rY \to Y$.*

Now we show that to construct a connection on $J^rY \to Y$ from a connection on $Y \to M$, it suffices to use some classical linear connection on Y.

Example 1. Let ∇ be a classical linear connection on Y. Then we have a general connection $D(\nabla)$ on $J^rY \to Y$ defined as follows. By Section 12.16 of [14], $J^rY \to Y$ is an open subbundle in the bundle K^r_mY of rth order contact elements on Y of dimension m. More precisely, J^rY is the (open) subset in K^r_mY of all contact elements in K^r_mY transversal to the fibers of $Y \to M$. But $K^r_m : \mathcal{M}f_{m+n} = \mathcal{FM}_{m+n,0} \to \mathcal{FM}$ is a bundle functor. Thus by Section 45.4 of [14], our ∇ on Y together with the trivial general connection Γ^Y on the trivial bundle $\mathrm{id}_Y : Y \to Y$ induces a general connection $\mathcal{K}^r_m(\Gamma^Y, \nabla^{\exp,r})$ on $K^r_mY \to Y$, where $\nabla^{\exp,r}$ is the rth order linear connection on Y (the exponential lift of ∇, [22]), and then by restriction to the open subbundle $J^rY \to Y$ we have a general connection $D(\nabla)$ on $J^rY \to Y$. We remark that this geometric construction is due to I. Kolář.

Remark 5. Using local coordinates, J. Janyška and M. Modugno have also constructed a general connection on $J^1Y \to Y$ from a classical linear connection on Y, [11]. Then the second author extended this construction to $J^rY \to Y$ for all r (unpublished).

Applying Proposition 9 and Example 1 we recover the following result from [21]:

Proposition 10. *There is no $\mathcal{FM}_{m,n}$-natural operator A transforming connections $\Gamma : Y \to J^1Y$ and classical linear connections Λ on M into classical linear connections $A(\Gamma, \Lambda)$ on Y.*

Proof. If such an A exists, then given Γ and Λ as above we have a classical linear connection $A(\Gamma, \Lambda)$ on Y. Then according to Example 1 we have a general connection $C(\Gamma, \Lambda) := D(A(\Gamma, \Lambda))$ on $J^rY \to Y$. This contradicts Proposition 9. ∎

Using different methods than those from [21], we present another proof of Proposition 10. We will use the following well-known facts saying that any affine transformation of a connected manifold is determined by its first jet at a point.

Lemma 12 (see Proposition 2.116 in [25]). *Let ∇ be a classical linear connection on a connected manifold N. Let $f, g : N \to N$ be ∇-affine maps. If $j^1_xf = j^1_xg$ at some point $x \in N$, then $f = g$.*

Second proof of Proposition 10. Suppose that such an A exists. Let Γ^0 be the trivial general connection on $\mathbb{R}^{m,n}$ and Λ^0 be the usual classical linear flat connection on $\mathbb{R}^m$. Then we have a classical linear connection $\nabla := A(\Gamma^0, \Lambda^0)$ on $\mathbb{R}^m \times \mathbb{R}^n$. Consider diffeomorphisms $\varphi_1, \varphi_2 : \mathbb{R}^n \to \mathbb{R}^n$ such that $j^1_0\varphi_1 = j^1_0\varphi_2$ and $\varphi_1 \neq \varphi_2$. The $\mathcal{FM}_{m,n}$-maps $\Phi_a = \mathrm{id}_{\mathbb{R}^m} \times \varphi_a$ for $a = 1, 2$ preserve Λ^0 and Γ^0. By the invariance of A, they also preserve $\nabla = A(\Gamma^0, \Lambda^0)$. Then Φ_1 and Φ_2 are two different ∇-affine maps with $j^1_{(0,0)}\Phi_1 = j^1_{(0,0)}\Phi_2$, contrary to Lemma 12. ■

Remark 6. Let $Y = P \to M$ be a principal bundle. Section 54.7 of [14] yields a construction of a classical linear connection $N_P(\Gamma, \Lambda)$ on P from a principal connection Γ on $P \to M$ and a classical linear connection Λ on M. So Proposition 10 says that the assumption $Y = P \to M$ is a principal bundle in the construction of $N_P(\Gamma, \Lambda)$ is unavoidable.

Replacing Γ^0 in the second proof of Proposition 10 by the trivial sth order connection on $\mathbb{R}^{m,n}$ we obtain the following generalization of Proposition 10.

Proposition 11. *There is no $\mathcal{FM}_{m,n}$-natural operator A transforming sth order connections $\Gamma : Y \to J^sY$ and classical linear connections Λ on M into classical linear connections $A(\Gamma, \Lambda)$ on Y.*

Remark 7. The second author recently classified all bundle functors $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ which admit prolongation of rth order connections Γ on $Y \to M$ into qth order connections $A(\Gamma, \Lambda)$ on $FY \to Y$ by means of torsion free classical linear connections Λ on M, [23].

Acknowledgments. The first author was supported by a grant of the GA ČR No. 201/09/0981.

References

[1] A. Cabras and I. Kolář, *Connections on some functional bundles*, Czechoslovak Math. J. 45 (1995), 529–548.
[2] —, —, *Prolongation of second order connections to vertical Weil bundles*, Arch. Math. (Brno) 37 (2001), 333–347.
[3] M. Doupovec and I. Kolář, *Iteration of fiber product preserving bundle functors*, Monatsh. Math. 134 (2001), 39–50.
[4] M. Doupovec and W. M. Mikulski, *On the existence of prolongation of connections*, Czechoslovak Math. J. 56 (131) (2006), 1323–1334.
[5] —, —, *On involutions of iterated bundle functors*, Colloq. Math. 106 (2006), 135–145.
[6] —, —, *Higher order jet involution*, Czechoslovak Math. J. 57 (132) (2007), 933–945.
[7] —, —, *Non-existence of exchange transformations of iterated jet functors*, in: Banach Center Publ. 76, Inst. Math., Polish Acad. Sci., 2007, 497–504.
[8] —, —, *Holonomic extension of connections and symmetrization of jets*, Rep. Math. Phys. 60 (2007), 299–316.
[9] —, —, *Reduction theorems for principal and classical connections*, Acta Math. Sinica Engl. Ser. 26 (2010), 169–184.
[10] C. Ehresmann, *Sur les connexions d'ordre supérieur*, in: Atti del V. Cong. del'Unione Mat. Italiana, 1955, Cremonese, Roma, 1956, 344–346.
[11] J. Janyška and M. Modugno, *Relations between linear connections on the tangent bundle and the connections on the jet bundle on a fibered manifold*, Arch. Math. (Brno) 32 (1996), 281–288.

[12] I. Kolář, *Higher order absolute differentiation with respect to generalized connections*, in: Banach Center Publ. 12, PWN, Warszawa, 1984, 153–162.

[13] —, *A general point of view to nonholonomic jet bundles*, Cahiers Topol. Géom. Diff. Catég. 44 (2003), 149–160.

[14] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations in Differential Geometry,* Springer, 1993.

[15] I. Kolář and M. Modugno, *Natural maps on the iterated jet prolongation of a fibered manifold*, Ann. Mat. Pura Appl. 158 (1991), 151–165.

[16] D. Krupka and J. Janyška, *Lectures on Differential Invariants*, Univ. J. E. Purkyně, Brno, 1990.

[17] D. Krupka and M. Krupka, *Jets and contact elements*, in: Proc. Seminar on Differential Geometry (Opava, 2000), Math. Publ. 2, Silesian Univ. Opava, 2000, 39–85.

[18] M. Kureš, *Generalizations of jets and reflections in the theory of connections*, Publ. Math. Debrecen 59 (2001), 339–352.

[19] P. Libermann, *Introduction to the theory of semiholonomic jets*, Arch. Math. (Brno) 33 (1977), 173–189.

[20] W. M. Mikulski, *On the existence of some type constructions on connections*, Rep. Math. Phys. 58 (2006), 411–416.

[21] —, *Negative answers to some questions about constructions on connections*, Demonstratio Math. 39 (2006), 685–689.

[22] —, *Higher order connections from first order ones*, Arch. Math. (Brno) 43 (2007), 285–288.

[23] —, *On prolongation of connections*, Ann. Polon. Math. 97 (2010), 101–121.

[24] M. Modugno, *Jet involution and prolongation of connections*, Čas. Pěst. Mat. 114 (1989), 356–365.

[25] W. A. Poor, *Differential Geometric Structures*, McGraw-Hill, 1981.

[26] D. J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lecture Note Ser. 142, Cambridge Univ. Press, 1989.

[27] V. V. Shurygin, *Higher-order connections and lifts of fields of geometric objects*, Izv. Vyssh. Uchebn. Zaved. Mat. 1992, no. 5, 96–104 (in Russian).

[28] J. Virsik, *On the holonomity of higher order connections*, Cahiers Topol. Géom. Diff. 12 (1971), 197–212.

Miroslav Doupovec
Department of Mathematics
Brno University of Technology
FSI VUT, Technická 2
616 69 Brno, Czech Republic
E-mail: doupovec@fme.vutbr.cz

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: mikulski@im.uj.edu.pl

Received 17.12.2009
and in final form 15.3.2010 (2134)