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Existence and multiplicity of solutions for a class of damped vibration problems with impulsive effects

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Abstract. Some sufficient conditions on the existence and multiplicity of solutions for the damped vibration problems with impulsive effects

$$\begin{cases} u''(t) + g(t)u'(t) + f(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), & j = 1, \dots, p, \end{cases}$$

are established, where $t_0 = 0 < t_1 < \dots < t_p < t_{p+1} = T$, $g \in L^1(0, T; \mathbb{R})$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, p$, are continuous. The solutions are sought by means of the Lax–Milgram theorem and some critical point theorems. Finally, two examples are presented to illustrate the effectiveness of our results.

1. Introduction. Consider the damped vibration problems with impulsive effects

$$\begin{cases} u''(t) + g(t)u'(t) + f(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), & j = 1, \dots, p, \end{cases} \tag{1.1}$$

where $t_0 = 0 < t_1 < \dots < t_p < t_{p+1} = T$, $g \in L^1(0, T; \mathbb{R})$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, p$, are continuous.

Impulsive effects exist widely in many evolution processes whose states change abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians (see [Ni, NR, ZY, LWCH, ZhaL]). In [Ni], the author proved a new existence theorem for a nonlinear periodic boundary value problem for a first-order differential equation with impulses at fixed times. It includes the cases when

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the nonlinearity and the impulsive functions are either bounded or have sublinear growth. In [ZY], the authors studied the existence and multiplicity of solutions for the nonlinear Dirichlet value problem with impulses by variational methods and critical points theory, and gave some new criteria to guarantee that the impulsive problem has at least one nontrivial solution, assuming that the nonlinearity is superquadratic at infinity, subquadratic at the origin, and the impulsive functions have sublinear growth. Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory [GNA, JL], population dynamics [Ne] and medicine [CGR, D, GCNT].

For a second order differential equation $u'' = f(t, u, u')$, one usually considers impulses in the position u and velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position (see [Ca, NO]). The impulse only in the velocity occurs also in impulsive mechanics (see [P]). An impulsive problem with impulses in the derivative only is considered in [TG].

In recent years, impulsive and periodic boundary value problems have been studied extensively. There have been many approaches to periodic solutions of differential equations, including the method of lower and upper solutions, fixed-point theory, and coincidence degree theory. In [LL], the authors used the method of lower and upper solutions together with monotone iterative technique to study impulsive differential equations. In [LJ], the authors applied the Krasnosel'skiĭ fixed point theorem in a cone to impulsive differential equations and obtained the existence of positive solutions. However, the study of solutions for impulsive differential equations using the variational method has received considerably less attention (see, for example, [ZY, ZhaL, NO, TG, ZhoL]). The variational method we use is, to the best of our knowledge, novel and it may open a new approach to nonlinear problems with some type of discontinuities such as impulses.

When $g(t) \equiv 0$, Nieto and O'Regan [NO] showed the variational structure and obtained the existence of solutions for problem (1.1); Zhou and Li [ZhoL] established the corresponding variational structure and obtained the existence, uniqueness and multiplicity of solutions for (1.1). But, when $g(t) \not\equiv 0$, until now, it has been unknown whether problem (1.1) has a variational structure or not.

In this paper, we investigate the existence of a variational construction for problem (1.1) in an appropriate function space. As applications, we study the existence, uniqueness and multiplicity of solutions for (1.1) by using some critical point theorems. All these results are new.

2. Preliminaries and statements. In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we introduce a variational structure. From this variational structure, we can reduce the problem of finding solutions of (1.1) to the one of seeking the critical points of a corresponding functional.

In the Sobolev space $H_0^1(0,T)$, consider the inner product

$$\langle u,v\rangle_{H_0^1(0,T)} = \int_0^T u'(t)v'(t)\,dt, \qquad \forall u,v\in H_0^1(0,T),$$

and the norm

$$\|u\|_{H_0^1(0,T)} = \Big(\int_0^T (u'(t))^2\,dt\Big)^{1/2}, \qquad \forall u\in H_0^1(0,T).$$

Since $g\in L^1(0,T;\mathbb{R})$, define $G(t)=\int_0^t g(s)\,ds$; then $G:[0,T]\to\mathbb{R}$ is continuous. Therefore

$$m=\min_{t\in[0,T]} e^{G(t)}>0, \qquad M=\max_{t\in[0,T]} e^{G(t)}>0.$$

We also consider the inner product

$$\langle u,v\rangle = \int_0^T e^{G(t)}u'(t)v'(t)\,dt, \qquad \forall u,v\in H_0^1(0,T),$$

and the norm

$$\|u\| = \Big(\int_0^T e^{G(t)}(u'(t))^2\,dt\Big)^{1/2}, \quad \forall u\in H_0^1(0,T).$$

Then we have the following lemma:

Lemma 2.1. *The norms $\|\cdot\|$ and $\|\cdot\|_{H_0^1(0,T)}$ are equivalent.*

Using the Poincaré inequality

$$\Big(\int_0^T (u(t))^2\,dt\Big)^{1/2} \le \frac{1}{\sqrt{\lambda_1}}\Big(\int_0^T (u'(t))^2\,dt\Big)^{1/2},$$

where $\lambda_1=\pi^2/T^2$ is the first eigenvalue of the problem

$$\begin{cases} -u''(t)=\lambda u(t), & t\in[0,T],\\ u(0)=u(T)=0, \end{cases}$$

one can easily prove the following lemma.

Lemma 2.2. *There exists $C_1>0$ such that if $u\in H_0^1(0,T)$, then*

$$\|u\|_\infty \le C_1\|u\|.$$

Let λ_k $(k = 1, 2, \ldots)$ denote the eigenvalues of the eigenvalue problem (2.1) and X_k the eigenspace associated to λ_k. Then $H_0^1(0,T) = \overline{\bigoplus_{i\in\mathbb{N}} X_i}$. We denote by $\|\cdot\|_p$ the norm in $L^p(0,T)$.

For $u \in H^2(0,T)$, the functions u and u' are both absolutely continuous, and $u'' \in L^2(0,T)$. Hence, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for any $t \in [0,T]$.

If $u \in H_0^1(0,T)$, then u is absolutely continuous and $u' \in L^2(0,T)$. In this case, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ may not hold for some $t \in (0,T)$ due to the impulsive effects.

Take $v \in H_0^1(0,T)$, multiply both sides of the equality

$$u''(t) + g(t)u'(t) + f(t,u(t)) = 0$$

by $e^{G(t)}v$ and integrate it from 0 to T, to obtain

$$\int_0^T e^{G(t)}u''(t)v(t)\,dt + \int_0^T e^{G(t)}g(t)u'(t)v(t)\,dt = -\int_0^T e^{G(t)}f(t,u(t))v(t)\,dt. \tag{2.1}$$

Moreover,

$$\begin{aligned}
&\int_0^T \big(e^{G(t)}u''(t)v(t) + e^{G(t)}g(t)u'(t)v(t)\big)\,dt\\
&= \sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} \big(e^{G(t)}u''(t)v(t) + e^{G(t)}g(t)u'(t)v(t)\big)\,dt\\
&= \sum_{j=0}^{p}\int_{t_j}^{t_{j+1}} v(t)\,de^{G(t)}u'(t)\\
&= \sum_{j=0}^{p}\Big(e^{G(t_{j+1}^-)}u'(t_{j+1}^-)v(t_{j+1}^-) - e^{G(t_j^+)}u'(t_j^+)v(t_j^+) - \int_{t_j}^{t_j+1} e^{G(t)}u'(t)v'(t)\,dt\Big)\\
&= -\sum_{j=1}^{p} e^{G(t_j)}\Delta u'(t_j)v(t_j) - u'(0)v(0) + e^{G(T)}u'(T)v(T) - \int_0^T e^{G(t)}u'(t)v'(t)\,dt\\
&= -\sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j))v(t_j) - \int_0^T e^{G(t)}u'(t)v'(t)\,dt.
\end{aligned}$$

Combining this with (2.1), we have

$$\int_0^T e^{G(t)}u'(t)v'(t)\,dt + \sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j))v(t_j) - \int_0^T e^{G(t)}f(t,u(t))v(t)\,dt = 0.$$

Considering the above, we introduce the following concept of solution for problem (1.1).

Definition 2.3. We say that a function $u \in H_0^1(0,T)$ is a *weak solution* of problem (1.1) if the identity

$$\int_0^T e^{G(t)} u'(t)v'(t)\,dt + \sum_{j=1}^p e^{G(t_j)} I_j(u(t_j))v(t_j) = \int_0^T e^{G(t)} f(t,u(t))v(t)\,dt$$

holds for any $v \in H_0^1(0,T)$.

Consider the functional $\varphi : H_0^1(0,T) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^p e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\,dt - \int_0^T e^{G(t)} F(t,u(t))\,dt \tag{2.2}$$

where $F(t,s) = \int_0^s f(t,\tau)\,d\tau$. Using the continuity of f and I_j, $j = 1,\ldots,p$, one finds that $\varphi \in C^1(H_0^1(0,T),\mathbb{R})$. For any $v \in H_0^1(0,T)$, we have

$$\varphi'(u)v = \int_0^T e^{G(t)} u'(t)v'(t)\,dt + \sum_{j=1}^p e^{G(t_j)} I_j(u(t_j))v(t_j) - \int_0^T e^{G(t)} f(t,u(t))v(t)\,dt.$$

Thus, the weak solutions of problem (1.1) correspond to the critical points of φ.

For the sake of convenience, we denote

$$l = \min\{e^{G(t_j)} : j = 1,\ldots,p\}, \qquad L = \max\{e^{G(t_j)} : j = 1,\ldots,p\}.$$

To prove our main results, we need the following definition and critical point theorems.

Definition 2.4 ([MW, P_{81}]). Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. Then I is said to satisfy to *P.S. condition* on X if any sequence $\{x_n\} \subset X$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.

Theorem 2.5 (Lax–Milgram, [Ch]). *Let H be a Hilbert space and $a : H \times H \to \mathbb{R}$ be a bounded bilinear form. If a is coercive, i.e., there exists $\alpha > 0$ such that $a(u,u) \geq \alpha\|u\|^2$ for every $u \in H$, then for any $\sigma \in H'$ (the conjugate space of H), there exists a unique $u \in H$ such that*

$$a(u,v) = (\sigma,v), \qquad \forall v \in H.$$

Moreover, if a is also symmetric, then the functional $\psi : H \to \mathbb{R}$ defined by

$$\psi(v) = \frac{1}{2}a(v,v) - (\sigma,v)$$

attains its minimum at u.

Theorem 2.6 ([R, Theorem 9.12]). *Let E be a Banach space. Let $I \in C^1(E,\mathbb{R})$ be an even functional which satisfies the P.S. condition and $I(0) = 0$. If $E = V \oplus W$, where V is finite-dimensional, and I satisfies*

(I_1) *there are constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B_\rho \cap W} \geq \alpha$, *where* $B_\rho = \{x \in E : \|x\| < \rho\}$,

(I_2) *for each finite-dimensional subspace* $\widetilde{E} \subset E$, *there is an* $R = R(\widetilde{E})$ *such that* $I \leq 0$ *on* $\widetilde{E} \setminus B_{R(\widetilde{E})}$,

then I *possesses an unbounded sequence of critical values.*

3. Main results. We are now in a position to state and prove our main results.

MAIN THEOREM 3.1. *Let* d_j $(j = 1, \ldots, p)$ *be fixed constants. If* $f(t, u) = \sigma(t) \in L^2(0, T)$ *and* $I_j(t) = d_j$ $(j = 1, \ldots, p)$, *then problem* (1.1) *has a unique weak solution* u *and* u *minimizes the functional* (2.2).

Proof. Define

$$a : H_0^1(0,T) \times H_0^1(0,T) \to \mathbb{R}, \qquad a(u,v) = \langle u, v\rangle,$$

and

$$l : H_0^1(0,T) \to \mathbb{R}, \qquad l(v) = \int_0^T e^{G(t)}\sigma(t)v(t)\,dt - \sum_{j=1}^{p} e^{G(t_j)} d_j v(t_j).$$

It is evident that l is linear and that a is bilinear, continuous and symmetric. By the Sobolev embedding theorem, there exists $C_2 > 0$ such that

$$\|u\|_2 \leq C_2\|u\|, \qquad \forall u \in H_0^1(0,T). \tag{3.1}$$

Let $d = \max\{|d_1|, \ldots, |d_p|\}$. Combining this with Lemma 2.2, for every $v \in H_0^1(0,T)$, we have

$$|l(v)| \leq M\|\sigma\|_2\|v\|_2 + Ldp\|u\|_\infty \leq MC_2\|\sigma\|_2\|v\| + LdpC_1\|v\|.$$

This implies that l is bounded. Since $a(u,u) = \|u\|^2$, a is coercive. By Theorem 2.5, problem (1.1) has a unique weak solution u and u minimizes the functional (2.2). ∎

MAIN THEOREM 3.2. *Assume that the following conditions are satisfied.*

(i) *There exist* $a, b > 0$ *and* $\gamma \in [0,1)$ *such that*

$$|f(t,u)| \leq a + b|u|^\gamma \quad \textit{for every } (t,u) \in [0,T] \times \mathbb{R}.$$

(ii) *There exist* $a_j, b_j > 0$ *and* $\gamma_j \in [0,1)$ $(j = 1, \ldots, p)$ *such that*

$$|I_j(u)| \leq a_j + b_j|u|^{\gamma_j} \quad \textit{for every } u \in \mathbb{R}\ (j = 1, \ldots, p).$$

Then problem (1.1) *has at least one weak solution.*

Proof. Let $M_1 = \max\{a_1, \ldots, a_p\}, M_2 = \max\{b_1, \ldots, b_p\}$. In view of (i), (ii) and Lemma 2.2, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\,dt - \int_0^T e^{G(t)} F(t,u(t))\,dt \\
&\ge \frac{1}{2}\|u\|^2 - \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} (a_j + b_j|t|^{\gamma_j})\,dt - \int_0^T e^{G(t)}(a|u| + b|u|^{\gamma+1})\,dt \\
&\ge \frac{1}{2}\|u\|^2 - pLM_1\|u\|_\infty - LM_2 \sum_{j=1}^{p} \|u\|_\infty^{\gamma_j+1} - aMT\|u\|_\infty - bMT\|u\|_\infty^{\gamma+1} \\
&\ge \frac{1}{2}\|u\|^2 - pLM_1C_1\|u\| - LM_2 \sum_{j=1}^{p} C_1^{\gamma_j+1}\|u\|^{\gamma_j+1} \\
&\quad - aMTC_1\|u\| - bMTC_1^{\gamma+1}\|u\|^{\gamma+1}
\end{aligned}$$

for all $u \in H_0^1(0,T)$. This implies that $\lim_{\|u\|\to\infty} \varphi(u) = \infty$, and φ is coercive.

On the other hand, we show that φ is weakly lower semi-continuous. If $\{u_k\}_{k\in\mathbb{N}} \subset H_0^1(0,T)$ and $u_k \rightharpoonup u$, then $\{u_k\}_{k\in\mathbb{N}}$ converges uniformly to u on $[0,T]$ and $\liminf_{k\to\infty} \|u_k\| \ge \|u\|$. Thus

$$\begin{aligned}
&\liminf_{k\to\infty} \varphi(u_k) \\
&\quad = \liminf_{k\to\infty} \left(\frac{1}{2}\|u_k\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u_k(t_j)} I_j(t)\,dt - \int_0^T e^{G(t)} F(t,u_k(t))\,dt \right) \\
&\quad \ge \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\,dt - \int_0^T e^{G(t)} F(t,u(t))\,dt = \varphi(u).
\end{aligned}$$

By Theorem 1.1 of [MW], φ has a minimum point on $H_0^1(0,T)$, which is a critical point of φ. Hence, problem (1.1) has at least one weak solution. ■

We readily have the following corollary.

COROLLARY 3.3. *Assume that f is bounded and that the impulsive functions I_j $(j = 1, \ldots, p)$ are bounded. Then problem* (1.1) *has at least one weak solution.*

EXAMPLE 3.4. Let $T = \pi/2$ and $t_1 = 1/2$. Consider the Dirichlet problem with impulse

$$\begin{cases}
u''(t) + (\sin t + \cos t)u'(t) + (t + \sqrt[4]{u(t)}) = 0, & \text{a.e. } t \in [0, \pi/2], \\
u(0) = u(\pi/2) = 0, \\
\Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = 1 + \sqrt[3]{u(t_1)}.
\end{cases} \tag{3.2}$$

It is easy to see that conditions (i) and (ii) of Theorem 3.2 hold. According to Theorem 3.2, problem (3.2) has at least one weak solution.

Main Theorem 3.5. *Assume that the condition* (ii) *of Theorem* 3.2 *and the following conditions are satisfied.*

(f_1) *$f(t,u)$ is odd in u.*
(f_2) *There exist $r_1, r_2 > 0$ and $\mu \in (1,\infty)$ such that*

$$|f(t,u)| \le r_1 + r_2|u|^\mu \qquad \text{for every } (t,u) \in [0,T] \times \mathbb{R}.$$

(f_3) *There exist $R > 0$ and $\beta > 2$ such that for every $t \in [0,T]$ and every $u \in \mathbb{R}$ with $|u| \ge R$,*

$$0 < \beta F(t,u) \le u f(t,u).$$

Moreover, $f(t,u) = o(u)$ as $u \to 0$ uniformly in t.
(f_4) *I_j ($j = 1, \dots, p$) are odd and nondecreasing.*

Then problem (1.1) *has infinitely many nontrivial weak solutions.*

Proof. We have $\varphi \in C^1(H_0^1(0,T),\mathbb{R})$, by ($f_1$) and ($f_4$), φ is an even functional and $\varphi(0) = 0$.

We divide our proof into three parts.

Firstly, we show that φ satisfies the P.S. condition.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $H_0^1(0,T)$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$.

By (ii), (f_3) and Lemma 2.2, we have

$$\begin{aligned}
&\beta\varphi(u_n) - \langle \varphi'(u_n), u_n\rangle \\
&= \left(\frac{\beta}{2} - 1\right)\|u_n\|^2 + \beta \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u_n(t_j)} I_j(t)\,dt - \sum_{j=1}^{p} e^{G(t_j)} I_j(u_n(t_j))u_n(t_j) \\
&\quad + \int_0^T e^{G(t)}(u_n f(t,u_n) - \beta F(t,u_n))\,dt \\
&\ge \left(\frac{\beta}{2} - 1\right)\|u_n\|^2 - L(\beta+1)\Big(pM_1C_1\|u_n\| + M_2\sum_{j=1}^{p} C_1^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\Big) \\
&\quad - M\int_0^T \max_{t\in[0,T],\,|u_n|\le R} |u_n f(t,u_n) - \beta F(t,u_n)|\,dt \\
&\ge \left(\frac{\beta}{2} - 1\right)\|u_n\|^2 - L(\beta+1)\Big(pM_1C_1\|u_n\| + M_2\sum_{j=1}^{p} C_1^{\gamma_j+1}\|u_n\|^{\gamma_j+1}\Big) - C_3,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(0,T)$. Hence there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that

$$u_n \rightharpoonup u \quad \text{in } H_0^1(0,T), \tag{3.3}$$

$$u_n \to u \quad \text{uniformly in } C([0,T]). \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle &= \int_0^T e^{G(t)} |u_n'(t) - u'(t)|^2\, dt \\ &\quad + \sum_{j=1}^{p} e^{G(t_j)} \big(I_j(u_n(t_j)) - I_j(u(t_j))\big)(u_n(t_j) - u(t_j)) \\ &\quad + \int_0^T e^{G(t)} \big(f(t,u_n(t)) - f(t,u(t))\big)(u_n(t) - u(t))\, dt.\end{aligned} \tag{3.5}$$

From (f_2), (3.3), (3.4) and (3.5), it follows that $u_n \to u$ in $H_0^1(0,T)$. Thus, φ satisfies the P.S. condition.

Secondly, we verify the condition (I_2) of Theorem 2.6.

By (f_3), we have

$$\frac{\beta}{u} \le \frac{f(x,u)}{F(x,u)}, \qquad u \ge R, \tag{3.6}$$

$$\frac{\beta}{u} \ge \frac{f(x,u)}{F(x,u)}, \qquad u \le -R. \tag{3.7}$$

Integrating (3.6) and (3.7) for u from $[R,u]$ and $[u,-R]$, respectively, we have

$$\beta \ln \frac{u}{R} \le \ln \frac{F(x,u)}{F(x,R)}, \qquad u \ge R,$$

$$\beta \ln \frac{R}{-u} \ge \ln \frac{F(x,u)}{F(x,-R)}, \qquad u \le -R.$$

That is,

$$F(x,u) \ge F(x,R)(u/R)^\beta, \qquad u \ge R, \tag{3.8}$$

$$F(x,u) \ge F(x,-R)(-u/R)^\beta, \qquad u \le -R. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$F(x,u) \ge \alpha_1 |u|^\beta, \qquad |u| \ge R, \tag{3.10}$$

where

$$\alpha_1 = R^{-\beta} \min\{\min_{x\in[0,T]} F(x,R), \min_{x\in[0,T]} F(x,-R)\} > 0.$$

On the other hand, by the continuity of $F(x,u)$, $F(x,u)$ is bounded on $[0,T]\times[-R,R]$, there exists $K>0$ such that

$$F(x,u)\geq -K\geq \alpha_1|u|^\beta-\alpha_1-K,\qquad |u|\leq R, \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$F(t,u)\geq \alpha_1|u|^\beta-\alpha_2,\qquad \forall (t,u)\in[0,T]\times\mathbb{R}, \tag{3.12}$$

where $\alpha_2=\alpha_1+K$.

For an arbitrary finite-dimensional subspace $E_1\subset H_0^1(0,T)$, there exists $C_4=C_4(E_1)>0$ such that for any $u\in E_1$,

$$\|u\|_\beta\geq C_4\|u\|. \tag{3.13}$$

By (ii), (3.12), (3.13) and Lemma 2.2, we have

$$\begin{aligned}
\varphi(u)&\leq \frac{1}{2}\|u\|^2+\sum_{j=1}^{p}e^{G(t_j)}\int_0^{u(t_j)}(a_j+b_j|t|^{\gamma_j})\,dt-\int_0^T e^{G(t)}(\alpha_1|u|^\beta-\alpha_2)\,dt\\
&\leq \frac{1}{2}\|u\|^2+pLM_1\|u\|_\infty+LM_2\sum_{j=1}^{p}\|u\|_\infty^{\gamma_j+1}-m\alpha_1\|u\|_\beta^\beta+M\alpha_2T\\
&\leq \frac{1}{2}\|u\|^2+pLM_1C_1\|u\|+LM_2\sum_{j=1}^{p}C_1^{\gamma_j+1}\|u\|^{\gamma_j+1}\\
&\quad -m\alpha_1C_4^\beta\|u\|^\beta+M\alpha_2T
\end{aligned}$$

for every $u\in E_1$. This implies that $\varphi(u)\to-\infty$ as $u\in E_1$ and $\|u\|\to\infty$. So there exists $R(E_1)>0$ such that $\varphi\leq 0$ on $E_1\setminus B_{R(E_1)}$.

Finally, we verify the condition (I_1) of Theorem 2.6.

Let $V=X_1\oplus X_2, W=\overline{\bigoplus_{i=3}^\infty X_i}$, then $H_0^1(0,T)=V+W$ and V is finite-dimensional. Using (f_4), we have

$$\sum_{j=1}^{p}\int_0^{u(t_j)}I_j(t)\,dt\geq 0. \tag{3.14}$$

By (f_3), we find

$$\lim_{u\to 0}\frac{F(t,u)}{u^2}=0.$$

Hence, for $\epsilon=1/(4MC_2^2)$, there exists $\delta>0$ such that for every u with $|u|\leq\delta$,

$$|F(t,u)|\leq \frac{1}{4MC_2^2}u^2. \tag{3.15}$$

Hence, for any $u\in W$ with $\|u\|\leq \delta/C_1$ and $\|u\|_\infty\leq\delta$, by (3.1), (3.14) and

(3.15), we have

$$\begin{aligned}\varphi(u) &\geq \frac{1}{2}\|u\|^2 - \int_0^T e^{G(t)} F(t,u(t))\,dt \geq \frac{1}{2}\|u\|^2 - \frac{1}{4MC_2^2} M\|u\|_2^2 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4MC_2^2} MC_2^2\|u\|^2 = \frac{1}{4}\|u\|^2.\end{aligned}$$

Taking $\alpha = \frac{1}{4}\frac{\delta^2}{C_1^2}$ and $\rho = \frac{\delta}{C_1}$, we obtain

$$\varphi(u) \geq \alpha, \qquad \forall u \in W \cap \partial B_\rho.$$

By Theorem 2.6, φ possesses infinitely many critical points, that is, problem (1.1) has infinitely many nontrivial weak solutions. ∎

EXAMPLE 3.6. Let $T = 1$ and $t_1 = 1/3$. Consider the Dirichlet impulsive problem

$$\begin{cases} u''(t) + (\sin t + t^2)u'(t) + (u(t))^5 = 0, & \text{a.e. } t \in [0,1], \\ u(0) = u(1) = 0, \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \sqrt[5]{u(t_1)}. \end{cases} \tag{3.16}$$

Then, according to Theorem 3.5, problem (3.16) has infinitely many nontrivial weak solutions.

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