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On arc-analytic functions definable by a Weierstrass system

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Abstract. This paper presents certain characterizations through blowing up of arc-analytic functions definable by a convergent Weierstrass system closed under complexification.

Consider a real convergent Weierstrass system $\mathcal{W}$ closed under complexification and the o-minimal expansion $\mathcal{R}$ of the real field by restricted $\mathcal{W}$-analytic functions. Our previous article [6] presented several theorems about the rectilinearization of $\mathcal{W}$-subanalytic functions and their application to quantifier elimination for the structure $\mathcal{R}$. In this paper, we shall apply those results to the theory of arc-$\mathcal{W}$-analytic functions definable in the structure $\mathcal{R}$. In what follows, the word "definable" will mean "definable in the structure $\mathcal{R}$".

The notion of a definable arc-$\mathcal{W}$-analytic function generalizes that of an arc-analytic function, considered by Kurdyka [3] in relation with arc-symmetric semialgebraic sets. Kurdyka also posed the question whether every such function can be, perhaps under additional assumptions, modified by means of blowing up to an analytic function. We should mention that functions of this type (i.e. analytic after composition with certain modifications) were investigated by Kuo [2] as well.

In the classical, real-analytic case, the affirmative answer was first given by Bierstone–Milman [1], and next by Parusiński [7]. Also developed in [1] was a method for rectilinearization of a continuous subanalytic function in such a way that every such function becomes analytic after composing it with a locally finite family of modifications, each of which is a composite of finitely many local blowings-up and local power substitutions. Parusiński improved the above result so that it is enough to substitute powers only at the last step after all local blowings-up.

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The main objective of this paper is to carry over the foregoing results to the real field with restricted $\mathcal{W}$-analytic functions (see [6] for basic definitions). What is crucial for our approach to arc-$\mathcal{W}$-analytic functions is a theorem from [6] on rectilinearization of a continuous definable function (the corollary to Theorem 2*). We first recall some necessary notation from that paper.

By a *quadrant* in $\mathbb{R}^m$ we mean a subset of $\mathbb{R}^m$ of the form

$$\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i = 0,\ x_j > 0,\ x_k < 0 \text{ for } i \in I_0,\ j \in I_+,\ k \in I_-\},$$

where $\{I_0, I_+, I_-\}$ is a disjoint partition of $\{1, \ldots, m\}$; its trace Q on the cube $[-1, 1]^m$ will be called a *bounded quadrant*; put

$$Q_+ := \{x \in [0, 1]^m : x_i = 0, x_j > 0 \text{ for } i \in I_0, j \in I_+ \cup I_-\}.$$

The interior $\mathrm{Int}(Q)$ of the quadrant Q is its trace on the open cube $(-1, 1)^m$. A *bounded closed quadrant* is the closure $\overline{Q}$ of a bounded quadrant Q, i.e. a subset of $\mathbb{R}^m$ of the form

$$\overline{Q} := \{x \in [-1, 1]^m : x_i = 0,\ x_j \geq 0,\ x_k \leq 0 \text{ for } i \in I_0,\ j \in I_+,\ k \in I_-\}.$$

A quadrant of dimension m in $\mathbb{R}^m$ is called an *orthant*.

We say that a function g on a bounded quadrant Q in $\mathbb{R}^m$ is a *fractional normal crossing* on Q if it is the superposition of a normal crossing f in the vicinity of the closure $\overline{Q_+}$ of Q_+ and a rational power substitution ψ given by the equality

$$\psi : \mathbb{R}^m \to \mathbb{R}^m, \qquad \psi(x_1, \ldots, x_m) = (|x_1|^{\alpha_1}, \ldots, |x_m|^{\alpha_m}),$$

where $\alpha_1, \ldots, \alpha_m$ are non-negative rational numbers. In other words, a fractional normal crossing g on Q is a function of the form

$$g(x_1, \ldots, x_m) = |x_1|^{n_1/N} \cdot \ldots \cdot |x_m|^{n_m/N} \cdot u(|x_1|^{1/N}, \ldots, |x_m|^{1/N}),$$

where N is a positive integer, $n_1, \ldots, n_m$ are non-negative integers such that $n_i = 0$ for $i \in I_0$, and u is a $\mathcal{W}$-analytic function near $\overline{Q_+}$ which vanishes nowhere on $\overline{Q_+}$.

Let U be an open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}$ a definable function. Given a bounded orthant Q and a collection of modifications φ_i, we denote by $\mathrm{dom}_i(Q)$ the union of Q and all those bounded quadrants that are adjacent to Q and disjoint from $\varphi_i^{-1}(\partial U)$; it is, of course, an open subset of the closure $\overline{Q}$. Moreover, the open subset $\varphi_i^{-1}(U)$ of the cube $[-1, 1]^m$ coincides with the union of $\mathrm{dom}_i(Q)$, where Q ranges over the bounded orthants that are contained in $\varphi_i^{-1}(U)$, and with the union of those bounded quadrants that are contained in $\varphi_i^{-1}(U)$.

Consequently, the union of the images $\varphi_i(\mathrm{Int}(Q))$, where Q ranges over the bounded quadrants that are contained in $\varphi_i^{-1}(U)$, coincides with the union of the images

$$\varphi_i(\mathrm{dom}_i(Q) \cap (-1,1)^m),$$

where Q range over the bounded orthants Q that are contained in $\varphi_i^{-1}(U)$.

We now recall the following rectilinearization result from our paper [6], which plays a key role in this article.

THEOREM 1 (On rectilinearization of a continuous definable function). *Let U be a bounded open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}$ be a continuous definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \to \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

(1) *each φ_i extends to a $\mathcal{W}$-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centres;*
(2) *each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;*
(3) *each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;*
(4) *U is the union of the images $\varphi_i(\mathrm{dom}_i(Q) \cap (-1,1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1, \ldots, p$;*
(5) *for every bounded orthant Q, the restriction to $\mathrm{dom}_i(Q)$ of each function $f \circ \varphi_i$ either vanishes or is a fractional normal crossing or a reciprocal fractional normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$.*

A function $f : U \to \mathbb{R}$ on an open subset $U \subset \mathbb{R}^m$ will be called *arc-$\mathcal{W}$-analytic* if, for every $\mathcal{W}$-analytic arc $\gamma : (-1,1) \to U$, the superposition $f \circ \gamma$ is $\mathcal{W}$-analytic too.

It is well known that every definable arc-$\mathcal{W}$-analytic function is continuous (cf. [3, 1]). For the reader's convenience, we give here a short proof of this fact.

PROPOSITION 1. *Given an open subset U in $\mathbb{R}^m$, every definable arc-$\mathcal{W}$-analytic function $f : U \to \mathbb{R}$ is continuous.*

Suppose, on the contrary, that the function f is not continuous at a point $a \in U$. Then there are $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, such that

$$a \in \overline{E_1} \cap \overline{E_2} \quad \text{with} \quad E_1 := \{x \in U : f(x) \le \alpha\},\ E_2 := \{x \in U : f(x) \ge \beta\}.$$

The structure $\mathcal{R}$ admits $\mathcal{W}$-analytic cell decomposition, and thus also finite $\mathcal{W}$-analytic stratifications of definable subsets (see e.g. [4, 5, 8]). Therefore, one can partition the set U into finitely many definable $\mathcal{W}$-analytic submanifolds $M_1, \ldots, M_p$ on each of which the function f is $\mathcal{W}$-analytic. Take a definable $\mathcal{W}$-analytic stratification $\Gamma_1, \ldots, \Gamma_s$ of $\mathbb{R}^m$ compatible with the sets E_1, E_2 and $M_1, \ldots, M_p$.

Due to the curve selection lemma and the arc-$\mathcal{W}$-analyticity of f, we get the following implication (considered by Bierstone–Milman [1]):

$$\Gamma_j \subset \overline{\Gamma_i} \;\Rightarrow\; f(\Gamma_j) \subset \overline{f(\Gamma_i)}\,.$$

Consequently, the sets E_1 and E_2 are closed, whence $a \in E_1 \cap E_2$. This contradiction proves Proposition 1.

We now turn to the main purpose of this paper.

THEOREM 2 (Rectilinearization of a definable arc-$\mathcal{W}$-analytic function). *Assume that a definable function $f \not\equiv 0$ is arc-$\mathcal{W}$-analytic on a connected open subset U in $\mathbb{R}^m$. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \to \mathbb{R}^m, \qquad i = 1,\dots,p,$$

such that

(1) *each φ_i extends to a $\mathcal{W}$-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centres;*
(2) *each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;*
(3) *each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;*
(4) *U is the union of the images $\varphi_i(\mathrm{dom}_i(Q) \cap (-1,1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1,\dots,p$;*
(5) *each function $f \circ \varphi_i$ is a $\mathcal{W}$-analytic $\mathcal{W}$-subanalytic function on the union*

$$\bigcup_Q \mathrm{dom}_i(Q) \cap (-1,1)^m$$

with Q ranging over the bounded orthants that are contained in $\varphi_i^{-1}(U)$, which is an open rectangular subset of the open cube $(-1,1)^m$.

Theorem 2 follows directly from Theorem 1 and the proposition below.

PROPOSITION 2. *Let*

$$\Omega_1 := \{x \in \mathbb{R}^m : x_m \ge 0\}, \qquad \Omega_2 := \{x \in \mathbb{R}^m : x_m \le 0\}$$

and F_1 and F_2 be two $\mathcal{W}$-analytic functions in the vicinity of the set Ω_1. Suppose that F_1 and F_2 coincide on $\Omega_1 \cap \Omega_2$. For a positive integer $r \in \mathbb{N}$, consider the functions

$$f_i : \Omega_i \to \mathbb{R}, \quad f_i(x) := F_i(x_1,\dots,x_{m-1},|x_m|^{1/r}), \qquad i = 1,2,$$

and denote by $f : \mathbb{R}^m \to \mathbb{R}$ their gluing. If for all $x_1,\dots,x_{m-1} \in \mathbb{R}$ the functions $f(x_1,\dots,x_{m-1},\cdot)$ of one variable x_m are $\mathcal{W}$-analytic, then so is the function f.

We start with the obvious observation that, under the circumstances, the Taylor series of the functions F_1 and F_2 at each point $(a_1,\dots,a_{m-1}) \in \mathbb{R}^{m-1}$

belong to the ring $\mathbb{R}[[x_1, \dots, x_{m-1}, x_m^r]]$. Further, the assumption about f_1 and f_2 implies that these two functions glue to an analytic function f. Consequently, f is a $\mathcal{W}$-analytic function, because the Weierstrass system $\mathcal{W}$ is convergent, and thus closed under analytic prolongation. This completes the proof.

Corollary 1 (Criterion for arc-$\mathcal{W}$-analyticity). *A definable function $f : U \to \mathbb{R}$ is arc-$\mathcal{W}$-analytic iff there exists a finite collection of definable modifications*

$$\varphi_i : (-1,1)^m \to \mathbb{R}^m, \quad i = 1, \dots, p,$$

such that

(1) $\bigcup_{i=1}^{p} \varphi_i((-1,1)^m) = U$;
(2) *each φ_i is a definable mapping which is a composite of finitely many local blowings-up with smooth centres;*
(3) *each $f \circ \varphi_i$ is a $\mathcal{W}$-analytic function.*

Indeed, whereas the "if" direction is obvious, the "only if" is a special case of Theorem 2.

Remark. The above criterion embraces the classical characterization of arc-analytic functions due to Bierstone–Milman [1].

Corollary 2. *If $f : U \to \mathbb{R}$ is an arc-$\mathcal{W}$-analytic function, then f is a $\mathcal{W}$-analytic function outside a closed definable $\mathcal{W}$-analytic subset $Z \subset U$ of codimension ≥ 2.*

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References

[1] E. Bierstone and P. D. Milman, *Arc-analytic functions*, Invent. Math. 101 (1990), 411–424.
[2] T.-C. Kuo, *On classification of real singularities*, Invent. Math. 82 (1985), 257–262.
[3] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann. 282 (1988), 445–462.
[4] K. J. Nowak, *Decomposition into special cubes and its application to quasi-subanalytic geometry*, Ann. Polon. Math. 96 (2009), 65–74.
[5] —, *Quantifier elimination, valuation property and preparation theorem in quasi-analytic geometry via transformation to normal crossings*, ibid. 96 (2009), 247–282.
[6] —, *Rectilinearization of functions definable by a Weierstrass system and its applications*, ibid. 99 (2010), 129–141.
[7] A. Parusiński, *Subanalytic functions*, Trans. Amer. Math. Soc. 344 (1994), 583–595.

[8] J.-P. Rolin, P. Speissegger and A. J. Wilkie, *Quasianalytic Denjoy–Carleman classes and o-minimality*, J. Amer. Math. Soc. 16 (2003), 751–777.

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