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Application of spaces of subspheres to conformal invariants of curves and canal surfaces

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Abstract. We review some techniques from the Möbius geometry of curves and surfaces in the 3-sphere, consider canal surfaces using their characteristic circles, and express the conformal curvature, and conformal torsion, of a vertex-free space curve in terms of its corresponding curve of osculating circles, and osculating spheres, respectively. We accomplish all of this strictly within the framework of Möbius geometry, and compare our results with the literature. Finally, we show how our formulation allows for the re-expression of the conformal invariants in terms of standard Euclidean invariants.

1. Introduction. In Möbius geometry, angles, rather than distances, are considered, and it is a well-known fact that the action of the Möbius group, the group of transformations preserving angles, transforms spheres into spheres and circles into circles. This allows for the study of the Möbius geometry of curves and surfaces in a 3-sphere by examining their associated curves or surfaces in the set of spheres and in the set of circles, simultaneously.

We begin this article by recalling how Euclidean space (or a sphere) of dimension 3 can be isometrically embedded in the Minkowski space $\mathbb{R}^5_1$ as the intersection of the light cone and an isotropic (or respectively, a space-like) affine hyperplane. We then describe the identification of the space of oriented 2-spheres with de Sitter space, which is the "unit sphere" of $\mathbb{R}^5_1$, and the space of oriented circles with a Grassmann manifold of oriented time-like 3-planes, as well as with a Grassmann manifold of oriented space-like 2-planes by taking the orthogonal complement. In doing so, we see how circles and tangent vectors to a curve in the space of circles can be seen as vectors in $\Lambda^2(\mathbb{R}^5_1)$, and moreover, the 2-vector representing a circle has to be pure—that is, of the form $u \wedge v$, $u, v \in \mathbb{R}^5_1$. The identifications of

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the above spaces will be applied in our examination of curves and canal surfaces.

Recall that a canal surface is the envelope of a one-parameter family of spheres which corresponds to a space-like curve σ in de Sitter space. By abuse of terminology, we also call the curve σ a canal surface. A canal surface is tangent to the spheres along the so-called characteristic circles: in other words, a characteristic circle of a canal surface σ is the intersection of two infinitesimally close spheres $\sigma(t)$ and $\sigma(t + dt)$. The canal surface is generated by characteristic circles: for example, the set of osculating spheres of a vertex-free spatial curve C is nothing but a canal surface with non-vanishing light-like geodesic curvature vector. (Recall that a vertex of a curve is a point where the curve has third-order contact with the osculating circle.) In this case, the characteristic circles are the osculating circles of C, and form a null curve in the space of circles. We give a necessary and sufficient condition for a curve in the space of circles to be a set of characteristic circles of a canal surface with non-vanishing geodesic curvature vectors.

It was shown in [F] that a space curve is determined up to Möbius transformations by three conformal invariants: the conformal arc-length, the conformal curvature, and the conformal torsion. The conformal arc-length was found in [L] for space curves and in [P] for plane curves, and the conformal curvature and torsion were given in [V]. These invariants have been studied using conformal derivations [F], Cartan's group-theoretical method of moving frames [Su], and using normal forms [CSW]; conformal torsion was also studied in [MRS] using conformal invariants for pairs of spheres. In [LO2], it was shown that the conformal arc-length can be considered a "1/2-dimensional measure" of the curve of osculating circles, which is a null curve in the space of circles.

In this paper, we use the curve of osculating circles and that of osculating spheres to construct a Möbius invariant moving frame. We also show that the conformal curvature and torsion appear naturally in the Frenet formula with respect to conformal arc-length (this part is an adaptation of [Su] to our context). We then give a Möbius-geometric formula which expands the coordinates of a curve in a series in the conformal arc-length. By eliminating the conformal arc-length parameters, we obtain the normal form of [CSW], which expresses the curve in $\mathbb{R}^3$ as a power series in x, the first coordinate of $\mathbb{R}^3$. We then give a new formula for the conformal curvature and the conformal torsion in terms of a curve of osculating circles and that of osculating spheres.

The above results are obtained within the framework of Möbius geometry. On the other hand, the conformal curvature and the conformal torsion can be expressed by Euclidean invariants: curvature, torsion, and their derivatives with respect to the arc-length of the curve in $\mathbb{R}^3$, as introduced in [CSW]. We show that these expressions can also be derived from our new formula.

1.1. Notations, assumptions and remarks on parameters. We denote by γ a curve in the space of oriented circles, and by σ a curve in the space of spheres. We always assume that σ is a space-like curve, i.e. the velocity vector is space-like. The derivative with respect to the arc-length parameter of σ is expressed by putting $^{\bullet}$ above the functions σ and γ. We remark that this parameter is suitable for expressing characteristic circles and the geodesic curvature vector of σ.

In Section 4, we restrict ourselves to the case when σ is the curve of osculating spheres of a vertex-free curve C in $\mathbb{R}^3$, and γ is a curve of osculating circles of C. We consider three parameters: the arc-length s of C; the conformal arc-length ρ of C; and the arc-length l of σ. The latter two fit our framework, as they are invariant under Möbius transformations, and it turns out that the Frenet formula can be expressed in a more aesthetically pleasing form if we use the conformal arc-length ρ. In order to avoid confusion, we denote derivatives with respect to s, ρ, and l by $'$, $^{\circ}$, and $^{\bullet}$, respectively, in §4. For simplicity, we make the assumption that $dl/d\rho$ never vanishes when C is a space curve; then γ is the set of characteristic circles of σ. Finally, we use the letter m for a point in $C \subset \mathbb{E}^3 \subset \mathbb{R}^5_1$.

2. Preliminaries on Möbius geometry of spaces of spheres. In this section, we introduce a pseudo-Riemannian structure on the space of spheres (or circles) in $\boldsymbol{S}^3$ by identifying it with the 4-dimensional de Sitter space (or, respectively, with the Grassmannian manifold of space-like 2-planes in 5-dimensional Minkowski space $\mathbb{R}^5_1$).

2.1. Realization of $\mathbb{R}^3$ and $\boldsymbol{S}^3$ in Minkowski space $\mathbb{R}^5_1$. We start by recalling commonly used models of the sphere and of Euclidean space in Möbius geometry (cf. [Be, Ce, HJ]).

Minkowski space $\mathbb{R}^5_1$ is $\mathbb{R}^5$ endowed with an indefinite inner product (the *Minkowski product* or the *Lorentz quadratic form*) given by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

Owing to the indefiniteness of the Minkowski product, we organize nonzero vectors $x \in \mathbb{R}^5_1$ into three categories: *space-like* whenever $\langle x, x\rangle > 0$; *time-like* whenever $\langle x, x\rangle < 0$; and *light-like* whenever $\langle x, x\rangle = 0$. Furthermore, by defining the *light cone* $\mathcal{C} = \{x \in \mathbb{R}^5_1 \mid \langle x, x\rangle = 0\}$, we extend this nomenclature to proper subspaces $W \subsetneq \mathbb{R}^5_1$: W is said to be

- *time-like* if it contains a time-like vector;
- *isotropic* if it is tangent to the light cone;

- *space-like* if it is none of the above; i.e. if every non-zero vector in W is space-like.

When $\dim W \geq 2$, it turns out that W is time-like if and only if it intersects the light cone transversally.

The 3-dimensional sphere $\boldsymbol{S}^3$ or $\mathbb{R}^3 \cup \{\infty\}$ can be considered as a projectivization of the light cone. In fact, $\boldsymbol{S}^3$ (or $\mathbb{R}^3$) can be *isometrically* embedded in $\mathbb{R}^5_1$ as the intersection of the light cone and a codimension 1 space-like (or respectively, isotropic) affine subspace H, as follows:

(i) Suppose $H = \{x \mid \langle x, n\rangle = -1\}$ for some unit time-like vector n. In this case, H is tangent to the hyperboloid $\{y \mid \langle y, y\rangle = -1\}$ at n, so $\mathbb{S}^3 := H \cap \mathcal{C}$ is a unit sphere.

(ii) Suppose $H = \{x \mid \langle x, n\rangle = -1\}$ for some light-like vector n. In this case, H is parallel to a hyperplane which is tangent to the light cone along $\mathrm{span}(n)$, so $\mathbb{E}^3 := H \cap \mathcal{C}$ is a paraboloid, and the restriction of the metric induced from $\langle \cdot, \cdot \rangle$ to $\mathbb{E}^3$ is Euclidean. For example, if we take $n = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0)$, the correspondence is given by

$$\mathbb{R}^3 \ni \boldsymbol{x} \mapsto \left(\frac{1}{\sqrt{2}} + \frac{\boldsymbol{x}\cdot\boldsymbol{x}}{2\sqrt{2}}, -\frac{1}{\sqrt{2}} + \frac{\boldsymbol{x}\cdot\boldsymbol{x}}{2\sqrt{2}}, \boldsymbol{x} \right) \in \mathbb{E}^3 = \mathcal{C} \cap \{x \mid \langle x, n\rangle = -1\}.$$

We use the notation $\mathbb{S}^3$ and $\mathbb{E}^3$ to emphasize that they are embedded in $\mathbb{R}^5_1$. The action of the Möbius group is now obtained from the action of the group preserving the Lorentz quadratic form on the light cone (see [HJ]).

2.2. De Sitter space as the set of codimension 1 spheres. An oriented sphere $\Sigma \subset \boldsymbol{S}^3$ is given by the intersection of $\boldsymbol{S}^3$ and an oriented affine hyperplane $P \subset \mathbb{R}^4$. Considering $\mathbb{R}^4$ as the affine hyperplane $\{x_0 = 1\}$ of $\mathbb{R}^5_1$, the cone $\mathrm{span}(\Sigma, O)$ is the intersection of the light cone with the hyperplane $\tilde{P} = \mathrm{span}(P, O) \subset \mathbb{R}^5_1$, which is time-like. Therefore, the set $\mathcal{S}(2,3)$ of oriented spheres in the 3-sphere can be identified with the Grassmann manifold $\widetilde{G}^-_{4,5}$ of oriented time-like 4-dimensional subspaces of $\mathbb{R}^5_1$. By taking the positive unit normal vector, we obtain a bijection between $\widetilde{G}^-_{4,5}$ and the quadric $\Lambda^4 = \{x \in \mathbb{R}^5_1 \mid \langle x, x\rangle = 1\}$, called *de Sitter space* (Figure 1). The bijection from Λ^4 to $\mathcal{S}(2,3)$ is given by

$$\Lambda^4 \ni \sigma \mapsto \Sigma = \mathbb{S}^3 \cap (\mathrm{span}(\sigma))^\perp \in \mathcal{S}(2,3),$$

where Σ is endowed with the orientation of the boundary of the ball $\mathbb{S}^3 \cap \{\langle \sigma, \cdot \rangle \leq 0\} \subset \mathbb{R}^5_1$, and its inverse is given by

$$\mathcal{S}(2,3) \ni \mathbb{S}^3 \cap \mathrm{span}(u^1, u^2, u^3, u^4) \mapsto \frac{u^1 \times u^2 \times u^3 \times u^4}{\|u^1 \times u^2 \times u^3 \times u^4\|} \in \Lambda^4,$$

where $w = u^1 \times u^2 \times u^3 \times u^4$ is the *Lorentz vector product* that is characterized by the following three properties: (i) $\langle w, u_i\rangle = 0$; (ii) the norm $\|w\| =$

$\sqrt{|\langle w, w\rangle|}$ is equal to the volume of the parallelepiped spanned by u_i, which is given by $\sqrt{|\det(\langle u_i, u_j\rangle)|}$; and (iii) $\det(w, u^1, u^2, u^3, u^4) > 0$ ([LO1]; the sign convention here is opposite to that in [LO1]). The coordinates of w are given by $w_0 = -\det(e^0, u^1, u^2, u^3, u^4)$ and $w_i = \det(e^i, u^1, u^2, u^3, u^4)$ $(i \neq 0)$. Direct computation shows that

$$\langle u^1 \times u^2 \times u^3 \times u^4, v^1 \times v^2 \times v^3 \times v^4\rangle = -\det(\langle u^i, v^j\rangle). \tag{2.1}$$

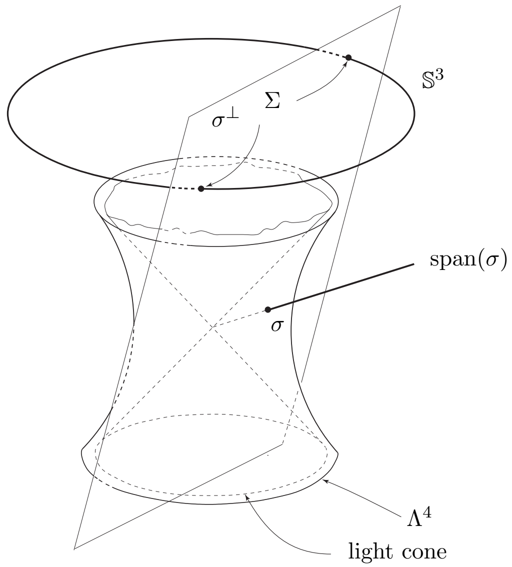

Fig. 1. The correspondence between de Sitter space and the set of oriented 2-spheres

Notice that the restriction of the Lorentz quadratic form of $\mathbb{R}^5_1$ to a tangent hyperplane $T_\sigma \Lambda^4$ is always of signature 1. Therefore, non-zero tangent vectors to Λ^4 can be either space-, time- or light-like.

Given a point $\sigma_0 \in \Lambda^4$ corresponding to a sphere Σ_0, let us construct four pencils of spheres containing Σ_0. First, choose three mutually orthogonal circles $\gamma_1, \gamma_2, \gamma_3$ on Σ_0; then each circle γ_i is contained in a sphere Σ_i orthogonal to Σ_0 along γ_i, and the three spheres Σ_i intersect in a pair of points $\Omega = \{\omega_1, \omega_2\}$. If, as in Figure 2, the three spheres Σ_i are three orthogonal planes, then one of the points ω_i is "at infinity", and the other is the intersection of the three planes.

Each circle γ_i defines a pencil of spheres with base circle γ_i. The fourth pencil has limit points Ω. It is not difficult to check (see [LO1]) that these four pencils correspond to four curves of Λ^4 intersecting orthogonally at σ_0, and any unit vector tangent to a curve corresponding to the pencil with base circle γ_i (or the pencil with limit points) is space-like (or, respectively, time-like). A pseudo-orthogonal basis of $T_{\sigma_0}\Lambda^4$ can be given by the unit tangent vectors to the four pencils of spheres thus defined.

Similarly, the set of oriented circles in the 2-sphere can be identified with de Sitter space Λ^3 in $\mathbb{R}^4_1$.

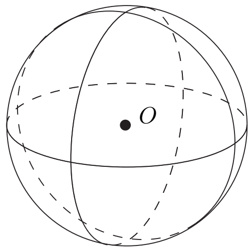

Fig. 2. Three mutually orthogonal great circles γ_i as the intersection of the sphere Σ_0 and three mutually orthogonal planes; they intersect in a pair of points, 0 and ∞, and produce four mutually orthogonal pencils.

2.3. Pseudo-Riemannian structure of indefinite Grassmann manifolds. In general, the set of oriented k-dimensional spheres in an n-sphere can be identified with the Grassmann manifold $\widetilde{G}^-_{k+2,n+2}$ of oriented time-like $(k+2)$-dimensional subspaces in $\mathbb{R}^{n+2}_1$. By taking the orthogonal complement, we obtain a bijection from $\widetilde{G}^-_{k+2,n+2}$ to the Grassmann manifold $\widetilde{G}^+_{n-k,n+2}$ of oriented space-like $(n-k)$-dimensional subspaces in $\mathbb{R}^{n+2}_1$.

We define the indefinite inner product on a Grassmann algebra by

$$\langle u^1 \wedge \cdots \wedge u^q, v^1 \wedge \cdots \wedge v^q \rangle = \det(\langle u^i, v^j \rangle), \tag{2.2}$$

and say that a vector $\boldsymbol{v} \in \bigwedge^q \mathbb{R}^5_1$ is space-like, null, or time-like according to whether $\langle \boldsymbol{v}, \boldsymbol{v} \rangle$ is positive, 0, or negative, respectively. If the dimension of $\Pi = \operatorname{span}(v^1, \ldots, v^q)$ is equal to q, then according to whether Π is space-like, isotropic, or time-like, $v^1 \wedge \cdots \wedge v^q$ is space-like, null, or time-like ([HJ, p. 280]).

2.4. Two Grassmannians as the set of oriented circles in the 3-sphere. In particular, putting $n = 3$ and $k = 1$, we see that the set $\mathcal{S}(1,3)$ of oriented circles in the 3-sphere can be identified with both $\widetilde{G}^-_{3,5}$ and $\widetilde{G}^+_{2,5}$. The wedge product $u \wedge v$ of two vectors $u = (u_0, u_1, \ldots, u_4)$ and $v = (v_0, v_1, \ldots, v_4)$ in $\mathbb{R}^5_1$ is given by the *Plücker coordinates* p_{ij} as

$$u \wedge v = (p_{ij})_{0 \le i < j \le 4} \in \textstyle\bigwedge^2 \mathbb{R}^5_1 \cong \mathbb{R}^{10}, \qquad p_{ij} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}. \tag{2.3}$$

A vector $\boldsymbol{p} = (p_{ij})_{0 \le i < j \le 4}$ in $\bigwedge^2 \mathbb{R}^5_1 \cong \mathbb{R}^{10}$ is a *pure 2-vector* (i.e. it can be expressed as the wedge product of two vectors in $\mathbb{R}^5_1$) if and only if the p_{ij}

satisfy the *Plücker relations*

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0, \tag{2.4}$$

$$p_{01}p_{24} - p_{02}p_{14} + p_{04}p_{12} = 0, \tag{2.5}$$

$$p_{01}p_{34} - p_{03}p_{14} + p_{04}p_{13} = 0, \tag{2.6}$$

$$p_{02}p_{34} - p_{03}p_{24} + p_{04}p_{23} = 0, \tag{2.7}$$

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{2.8}$$

(These are not linearly independent: for example, the relations (2.7) and (2.8) can be derived from the rest if $p_{01} \neq 0$.)

This allows us to identify $\widetilde{G}^+_{2,5}$ with the set of unit, space-like, pure 2-vectors in $\bigwedge^2 \mathbb{R}^5_1 = \mathbb{R}^{10}_4$:

$$\widetilde{G}^+_{2,5} \cong \Big\{ \boldsymbol{p} = (p_{ij})_{0\le i<j\le 4} \;\Big|\; \langle \boldsymbol{p}, \boldsymbol{p}\rangle = -\sum_{k=1}^{4} {p_{0k}}^2 + \sum_{1\le i<j\le 4} {p_{ij}}^2 = 1, \\ p_{ij} \text{ satisfy (2.4)–(2.8)} \Big\}.$$

Now the identification between $\mathcal{S}(1,3)$ and $\widetilde{G}^+_{2,5}$ can be explicitly given by

$$\mathcal{S}(1,3) \ni \mathbb{S}^3 \cap (\operatorname{span}(u,v))^\perp \mapsto \frac{u \wedge v}{\|u \wedge v\|} \in \widetilde{G}^+_{2,5}. \tag{2.9}$$

We will illustrate the pseudo-Riemannian structure of $\mathcal{S}(1,3)$ by constructing a set of six curves in $\mathcal{S}(1,3)$ through a given point γ_0 corresponding to a circle Γ_0, so that the tangent vectors form a pseudo-orthonormal basis of $T_{\gamma_0}\mathcal{S}(1,3)$.

We may assume without loss of generality that Γ_0 is the "horizontal" unit circle of $\mathbb{R}^3$ given by

$$\Gamma_0 = \{(x,y,0) \mid x^2 + y^2 = 1\},$$

i.e. the intersection of the unit sphere $\Sigma_0 \subset \mathbb{R}^3$ with the horizontal plane $P_{xy} = \{z = 0\}$. Using two mutually orthogonal planes $P_{yz} = \{x = 0\}$ and $P_{zx} = \{y = 0\}$ which are orthogonal to both Σ_0 and P_{xy}, let

$$\begin{aligned} A &= \Sigma_0 \cap P_{xy} \cap P_{zx} = \Gamma_0 \cap P_{zx}, \\ B &= \Sigma_0 \cap P_{xy} \cap P_{yz} = \Gamma_0 \cap P_{yz}, \\ \Omega &= \Sigma_0 \cap P_{yz} \cap P_{zx}, \\ \Xi &= (P_{xy} \cap P_{yz} \cap P_{zx}) \cup \{\infty\}, \end{aligned}$$

as illustrated in Figure 3. This construction defines four pencils of circles with base points and two pencils of circles with limit points:

- two pencils of circles in Σ_0 with base points A and B, resp.,
- two pencils of circles in P_0 with base points A and B, resp.,

- the pencil of circles in Σ_0 with limit points Ω,
- the pencil of circles in P_0 with limit points Ξ.

These pencils are geodesics in $\mathcal{S}(1,3)$ through γ_0, and unit tangent vectors to these six curves (four are space-like and two time-like) form a pseudo-orthonormal basis of $T\mathcal{S}(1,3)$ at Γ_0.

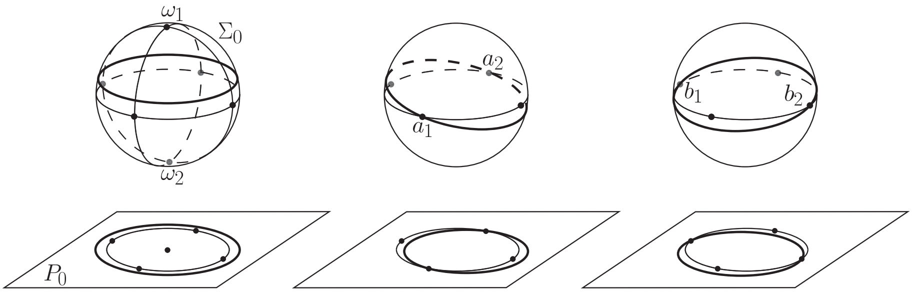

Fig. 3. Six pencils as curves in $\mathcal{S}(1,3)$; $A = \{a_1, a_2\}, B = \{b_1, b_2\}$, and $\Omega = \{\omega_1, \omega_2\}$

3. Canal surfaces. In this section, let σ be a space-like curve in Λ^4.

3.1. Canal surfaces in terms of a curve in $\mathcal{S}(1,3)$

Definition 3.1. A *canal surface* in the 3-sphere $\mathbb{S}^3$ is the envelope of a space-like curve σ in Λ^4. By abuse of terminology, we also call σ a canal surface. A circle $\operatorname{span}(\sigma, \dot{\sigma})^\perp \cap \mathbb{S}^3$ is called a *characteristic circle* of the canal surface σ.

The point in $\mathcal{S}(1,3) \subset \bigwedge^2 \mathbb{R}^5_1$ which corresponds to the characteristic circle of a canal surface σ is given by $\gamma = \sigma \wedge \dot{\sigma}$.

Lemma 3.2. *Let γ be a curve in $\mathcal{S}(1,3) \cong \widetilde{G}^+_{2,5}$. If the derivative $\gamma'(t)$ is a non-zero pure vector, then there is a unique space-like direction* $\operatorname{span}(\Pi(t) \cap P(t))$*, where $\Pi(t)$ and $P(t)$ are 2-dimensional vector subspaces of $\mathbb{R}^5_1$ which correspond to $\gamma(t)$ and $\gamma'(t)$, respectively.*

Proof. Suppose γ can be expressed as $\gamma(t) = \alpha(t) \wedge \beta(t)$ for some $\alpha(t)$ and $\beta(t)$ in $\mathbb{R}^5_1$, with $\langle \alpha(t), \alpha(t) \rangle = \langle \beta(t), \beta(t) \rangle = 1$ and $\langle \alpha(t), \beta(t) \rangle = 0$. We may assume without loss of generality that, at $t = t_0$,

$$\alpha = \alpha(t_0) = (0,1,0,0,0) \quad \text{and} \quad \beta = \beta(t_0) = (0,0,1,0,0).$$

Suppose $\alpha' = \alpha'(t_0)$ and $\beta' = \beta'(t_0)$ are given by

$$\alpha' = (a,b,c,d,e) \quad \text{and} \quad \beta' = (\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}).$$

Then $b = \bar{c} = 0$, as $\langle \alpha, \alpha' \rangle = \langle \beta, \beta' \rangle = 0$, and $\bar{b} + c = 0$, as $\langle \alpha, \beta' \rangle + \langle \alpha', \beta \rangle = 0$. Using formula (2.3), the Plücker coordinates of $\gamma' = \alpha \wedge \beta' + \alpha' \wedge \beta$ are given by

$$(p_{01},\dots,p_{04};\,p_{12},\dots,p_{34})(\alpha\wedge\beta'+\alpha'\wedge\beta) = (-\bar{a},-a,0,0;\,0,\bar{d},\bar{e},-d,-e,0).$$

As they satisfy the Plücker relations (2.4)–(2.8) by assumption, we have $\bar{a}d = a\bar{d}$, $\bar{a}e = a\bar{e}$, and $\bar{d}e = d\bar{e}$, which implies that the two vectors $u = \alpha' - c\beta = (a, 0, 0, d, e)$ and $\bar{u} = \beta' + c\alpha = (\bar{a}, 0, 0, \bar{d}, \bar{e})$ are linearly dependent. Neither is equal to $\mathbf{0}$, since $\gamma' = \alpha\wedge\beta' + \alpha'\wedge\beta = \alpha\wedge\bar{u} - \beta\wedge u \neq \mathbf{0}$ by assumption. Therefore, there is a non-zero vector $v \in (\mathrm{span}(\alpha,\beta))^{\perp}$ and a $\theta\in\mathbb{R}$ such that $\bar{u} = (\cos\theta)v$ and $u = -(\sin\theta)v$, so $\gamma' = ((\cos\theta)\alpha + (\sin\theta)\beta)\wedge v$. It follows that γ and γ' correspond to $\Pi = \mathrm{span}(\alpha,\beta)$ and $P = \mathrm{span}((\cos\theta)\alpha + (\sin\theta)\beta, v)$, respectively, which implies $\mathrm{span}(\Pi\cap P) = \mathrm{span}((\cos\theta)\alpha + (\sin\theta)\beta)$, which is a space-like direction. ∎

Definition 3.3. Using the notation of Lemma 3.2, let $\sigma(t)$ be a unit vector in $\mathrm{span}(\Pi(t)\cap P(t))$. Then σ is called a *supporting sphere* of γ at t.

The following theorem characterizes a canal surface σ with non-vanishing geodesic curvature vector $\vec{k}_g = \sigma + \ddot{\sigma}$ in terms of a curve in the space of circles.

Theorem 3.4. *Let $\mathcal{S}$ be the set of space-like curves σ in Λ^4 with non-vanishing geodesic curvature vector $\vec{k}_g = \sigma + \ddot{\sigma}$, and $\mathcal{C}$ the set of curves γ in $\mathcal{S}(1,3)$ such that $\gamma'(t)$ is a non-zero pure vector for any t and the corresponding supporting spheres σ of γ satisfy $\sigma'(t)\neq\mathbf{0}$ for any t. Then there is a bijection $\varphi:\mathcal{S}\to\mathcal{C}$ such that $\varphi(\sigma)$ is the characteristic circle of σ and $\varphi^{-1}(\gamma)$ is the supporting sphere of γ.*

Proof. Let $\psi(\gamma)$ be a curve of supporting spheres of $\gamma\in\mathcal{C}$.

(1) Suppose σ is a space-like curve in Λ^4 so that $\sigma+\ddot{\sigma}$ never vanishes. If we let $\gamma = \sigma\wedge\dot{\sigma}$, then $\dot{\gamma} = \sigma\wedge\ddot{\sigma}$ is a non-zero pure vector. Moreover, $\mathrm{span}(\sigma,\dot{\sigma})\cap\mathrm{span}(\sigma,\ddot{\sigma}) = \mathrm{span}(\sigma)$, so we see that $\varphi(\mathcal{S})\subset\mathcal{C}$ and $\psi\circ\varphi = \mathrm{id}$.

(2) Let σ be a curve of supporting spheres of γ. If we use the same notation as in the proof of Lemma 3.2, then

$$\begin{aligned}\sigma &= (\cos\theta)\alpha + (\sin\theta)\beta,\\ \sigma' &= (\cos\theta)(c\beta - (\sin\theta)v) + (\sin\theta)(-c\alpha + (\cos\theta)v) - \theta'(\sin\theta)\alpha + \theta'(\cos\theta)\beta\\ &= (\theta'+c)(-(\sin\theta)\alpha + (\cos\theta)\beta).\end{aligned}$$

Since $\sigma'\neq\mathbf{0}$ by assumption, we have $\theta'+c\neq 0$, which implies that σ' is space-like. Computing the wedge product of σ and σ' yields

$$\sigma\wedge\sigma' = (\theta'+c)\alpha\wedge\beta = (\theta'+c)\gamma,$$

so, if we use the arc-length parameter of σ, we then have $\sigma\wedge\dot{\sigma} = \gamma$, and hence $\dot{\gamma} = \sigma\wedge\ddot{\sigma}$. Finally, as $\dot{\gamma}\neq\mathbf{0}$ by assumption, we have $\ddot{\sigma}\neq\pm\sigma$, which means that $\sigma\in\mathcal{S}$, and thus we see that $\psi(\mathcal{C})\subset\mathcal{S}$ and $\varphi\circ\psi = \mathrm{id}$. ∎

As an element of Λ^4, the geodesic curvature vector $\vec{k}_g$ may be either time-like, space-like, light-like, or equal to $\mathbf{0}$. The type of $\vec{k}_g$ coincides with that of the velocity vector of the curve of characteristic circles; namely, we have:

Proposition 3.5. *Let σ be a canal surface, i.e. a space-like curve in Λ^4, $\vec{k}_g = \sigma + \ddot{\sigma}$ be the geodesic curvature vector, and $\gamma = \sigma \wedge \dot{\sigma}$ be the characteristic circle. Then $\langle \vec{k}_g, \vec{k}_g \rangle = \langle \dot{\gamma}, \dot{\gamma} \rangle$, while $\vec{k}_g = \mathbf{0}$ if and only if $\dot{\gamma} = \mathbf{0}$.*

Proof. Recall that $\vec{k}_g = \sigma + \ddot{\sigma}$ and $\dot{\gamma} = \sigma \wedge \ddot{\sigma}$. Since $\langle \sigma, \sigma \rangle = 1$ and $\langle \dot{\sigma}, \dot{\sigma} \rangle = 1$, we see that $\langle \sigma, \dot{\sigma} \rangle = 0$ and $\langle \sigma, \ddot{\sigma} \rangle = -1$, and therefore

$$
\begin{aligned}
(3.1) \qquad \langle \vec{k}_g, \vec{k}_g \rangle = \langle \sigma + \ddot{\sigma}, \sigma + \ddot{\sigma} \rangle = \langle \ddot{\sigma}, \ddot{\sigma} \rangle - 1 &= \begin{vmatrix} \langle \sigma, \sigma \rangle & \langle \sigma, \ddot{\sigma} \rangle \\ \langle \sigma, \ddot{\sigma} \rangle & \langle \ddot{\sigma}, \ddot{\sigma} \rangle \end{vmatrix} \\
&= \langle \sigma \wedge \ddot{\sigma}, \sigma \wedge \ddot{\sigma} \rangle = \langle \dot{\gamma}, \dot{\gamma} \rangle,
\end{aligned}
$$

which implies the equation, where the fourth equality follows from formula (2.2).

If $\vec{k}_g = \mathbf{0}$, then $\ddot{\sigma} = -\sigma$, which implies that $\dot{\gamma} = \sigma \wedge \ddot{\sigma} = \mathbf{0}$. On the other hand, if $\dot{\gamma} = \sigma \wedge \ddot{\sigma} = \mathbf{0}$, then $\ddot{\sigma} = a\sigma$ for some $a \in \mathbb{R}$, which, together with $\langle \sigma, \sigma \rangle = 1$, $\langle \sigma, \ddot{\sigma} \rangle = -1$, and $\langle \ddot{\sigma}, \ddot{\sigma} \rangle = 1$ (which follows from (3.1)), implies that $\ddot{\sigma} = -\sigma$, so $\vec{k}_g = \mathbf{0}$. ■

The properties of a canal surface depend on the type of its geodesic curvature vector $\vec{k}_g$ (the reader is referred to [LS] for details). If $\vec{k}_g$ is always time-like, then the canal surface is locally an immersed cylinder (Figure 4). If $\vec{k}_g$ is always space-like, then the canal surface has two singular curves in general and the characteristic circles are tangent to these two curves. If $\vec{k}_g$ is always light-like, then in general the canal surface is formed by the osculating circles of a curve, except for degenerate cases like families of spheres tangent to a given direction at a point. Lastly, if $\vec{k}_g \equiv \mathbf{0}$, then the canal surface is a space-like geodesic of Λ^4, which is a pencil of spheres that contain a fixed circle.

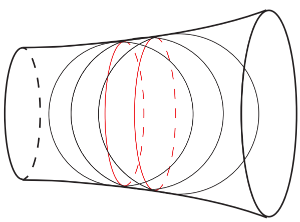

Fig. 4. A canal surface and characteristic circles; a case when $\vec{k}_g$ is always time-like

In Section 4.4.2, we will study a canal surface consisting of osculating spheres of a vertex-free space curve, which is an example of the third type mentioned above, i.e. a canal surface whose geodesic curvature vector is always light-like.

4. Möbius geometry of curves using osculating circles and spheres. Let C be a curve in $\mathbb{E}^3 \subset \mathbb{R}^5_1$, and suppose $m = m(s) \in C$, where s is the arc-length parameter. In what follows, we assume that $'$ means the derivative with respect to s.

4.1. Osculating circles and osculating spheres. Let Π be a time-like vector subspace of $\mathbb{R}^5_1$ of dimension 3 (or 4). Then the curve C has contact of order $\geq k$ with the circle (or the sphere, respectively) $\Pi \cap \mathbb{E}^3$ if and only if $m(s), m'(s), \ldots, m^{(k)}(s)$ belong to Π. Recall that the curve is not in the 3-dimensional space $\mathbb{R}^3$, but in the paraboloid $\mathbb{E}^3$ in the light cone $\mathcal{C}$. Therefore, an osculating circle $\Gamma(s)$ to C at $m(s)$ is given by $\mathbb{E}^3 \cap \operatorname{span}(m(s), m'(s), m''(s))$, and an osculating sphere is given by $\mathbb{E}^3 \cap \operatorname{span}(m(s), m'(s), m''(s), m'''(s))$.

Recall that the Lorentz vector product of four vectors $a \times b \times c \times d$ satisfies $\langle v, a \times b \times c \times d\rangle = \det(v, a, b, c, d)$ for any v. Assume that the curve C is *vertex-free*, i.e. any osculating circle has contact order exactly equal to 2. Then the point $\sigma(s) \in \Lambda^4$ that corresponds to the osculating sphere at a point $m(s)$ is given by

$$\sigma(s) = \frac{m(s) \times m'(s) \times m''(s) \times m'''(s)}{\|m(s) \times m'(s) \times m''(s) \times m'''(s)\|}. \tag{4.1}$$

Note that $\sigma'(s)$ can be expressed as

$$\sigma'(s) = m(s) \times m'(s) \times m''(s) \times (am'''(s) + bm^{(4)}(s)) \tag{4.2}$$

for some $a, b \in \mathbb{R}$. Let us further assume that $\sigma'(s) \neq \mathbf{0}$ for all s. Since $\dim \operatorname{span}(\sigma, \sigma') = 2$, as $\langle \sigma, \sigma'\rangle = 0$, we have

$$\operatorname{span}(\sigma(s), \sigma'(s)) = \big(\operatorname{span}(m(s), m'(s), m''(s))\big)^{\perp} = (\operatorname{span}(\Gamma(s))^{\perp},$$

which implies that σ' is space-like. This means that an osculating sphere intersects another infinitesimally close osculating sphere in an osculating circle. The osculating circle at $m(s)$ corresponds to a point $\gamma(s) \in \widetilde{G}^{+}_{2,5}$ given by

$$\gamma(s) = \pm \frac{\sigma(s) \wedge \sigma'(s)}{\|\sigma(s) \wedge \sigma'(s)\|}.$$

For simplicity, we assume that the sign is + in what follows.

Let l denote the arc-length parameter of the curve of osculating spheres σ in Λ^4, and let us denote the derivative with respect to l by putting $^{\bullet}$ above a symbol. Then we have

$$\gamma = \sigma \wedge \dot{\sigma}. \tag{4.3}$$

The osculating circle to a curve $C \subset \mathbb{E}^2$ at a point $m(s)$ (in this case, the osculating sphere is constantly equal to $\mathbb{E}^2$) can be expressed in a similar way to (4.1). In this case, as $\|m(s) \times m'(s) \times m''(s)\| = 1$ ([LO2]), a point $\gamma(s)$ in Λ^3 which corresponds to the osculating circle at $m(s)$ is given by $\gamma = m(s) \times m'(s) \times m''(s)$.

Similarly, the osculating circle to a curve in $\mathbb{E}^3$ at a point $m(s)$ is given by

$$\gamma(s) = m(s) \wedge m'(s) \wedge m''(s) \in \widetilde{G}^-_{3,5}. \tag{4.4}$$

4.2. Null curve of osculating circles and conformal arc-length. Let us first consider a smooth one-parameter family of circles in $\mathbb{R}^2$ given by their centers $\omega(t)$ and radii $r(t)$. Locally speaking, these circles may admit an envelope which is formed of two curves, or exceptionally one curve. The latter is the case we will consider here, as we consider the family of osculating circles to a curve, so then $\|\omega'(t)\| = |r'(t)|$ everywhere. The fact that the

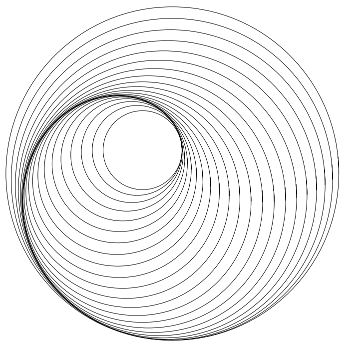

Fig. 5. Osculating circles of a plane curve with monotone curvature

osculating circles of a planar arc with monotone curvature are nested along the arc as illustrated in Figure 5 was observed at the beginning of the 20th century in [K]; in our language:

Lemma 4.1 ([LO2]). *A curve $\gamma \subset \Lambda^3$ of osculating circles to a curve C in $\mathbb{E}^2$ is light-like, and a point $m(s)$ on C is given by $m(s) = \mathbb{E}^2 \cap \mathrm{span}(\gamma'(s))$.*

Similarly, when C is a curve in $\mathbb{E}^3$, a curve γ of osculating circles is a null curve in $\widetilde{G}^+_{2,5}$, with γ' being a pure vector. If $\Pi(s)$ is a plane in $\mathbb{R}^5_1$ which corresponds to $\gamma'(s)$, it is tangent to the light cone along the line $\mathrm{span}(m(s))$.

It is known that $\gamma' = 0$ at a point if and only if the point is a vertex.

PROPOSITION 4.2 ([LO2]). *Let γ denote a curve of osculating circles to a curve C in $\mathbb{E}^2$ or $\mathbb{E}^3$. Then the 1-form $\sqrt[4]{\langle d^2\gamma/dt^2, d^2\gamma/dt^2\rangle}\,dt$ is independent of the parameter t. Let ρ be a parameter so that $d\rho = \sqrt[4]{\langle d^2\gamma/dt^2, d^2\gamma/dt^2\rangle}\,dt$; in other words, the parameter ρ can be characterized by*

$$\left\langle \frac{d^2\gamma}{d\rho^2}, \frac{d^2\gamma}{d\rho^2} \right\rangle = 1.$$

Then it can be uniquely determined up to $\rho \mapsto \pm\rho + c$ for some constant c.

The parameter ρ is called the *conformal arc-length* of C, and serves as a non-singular parameter for a vertex-free curve.

4.3. The moving frame and Frenet formula for a curve in $\mathbb{E}^2$ or $\mathbb{S}^2$. Let γ be a curve in de Sitter space Λ^3 which consists of the osculating circles to a curve C in $\mathbb{E}^2$. We will give a moving frame in terms of γ, and will see how the conformal curvature appears in the Frenet formula using the conformal arc-length.

First, observe that by Lemma 4.1 and Proposition 4.2, the quantities $\langle d^i\gamma/d\rho^i, d^j\gamma/d\rho^j\rangle$ for small i, j can be calculated as in Table 1, where we define (1) $Q_2 = -\frac{1}{2}\langle \overset{\circ\circ\circ}{\gamma}, \overset{\circ\circ\circ}{\gamma}\rangle$. We then see that γ, $\overset{\circ}{\gamma}$, $\overset{\circ\circ}{\gamma}$, and $\overset{\circ\circ\circ}{\gamma}$ are linearly independent.

Table 1. A table of $\langle d^i\gamma/d\rho^i, d^j\gamma/d\rho^j\rangle$

$\langle\cdot,\cdot\rangle$	γ	$\overset{\circ}{\gamma}$	$\overset{\circ\circ}{\gamma}$	$\overset{\circ\circ\circ}{\gamma}$	$\gamma^{(4)}$	$\gamma^{(5)}$	$\gamma^{(6)}$
γ	1	0	0	0	1	0	$2Q_2$
$\overset{\circ}{\gamma}$	0	0	0	-1	0	$-2Q_2$	
$\overset{\circ\circ}{\gamma}$	0	0	1	0	$2Q_2$		
$\overset{\circ\circ\circ}{\gamma}$	0	-1	0	$-2Q_2$			

As in [Br], our moving frame, which is called an *isotropic orthonormal frame*, consists of two space-like vectors v_1, v_2 and two light-like vectors n, n^*, instead of three space-like vectors and a time-like vector. Let us give the frame at $\rho = 0$: first, choose n in span(m), and then assume the following:

- $v_1 = \overset{\circ}{n}$;
- the x- and y-coordinates of the normal form at $\rho = 0$ are given by $\langle m(\rho) - m(0), v_1\rangle = \langle m(\rho), v_1\rangle$ and $\langle m(\rho) - m(0), v_2\rangle = \langle m(\rho), v_2\rangle$, respectively; and
- the x-axis of the normal form corresponds to the osculating circle.

(1) We write Q_2 to indicate that it is a quantity for a plane curve. Later on, we will see that it can be generalized to the conformal curvature Q of a space curve.

Then we have $n = \mathring{\gamma}$, $v_1 = \mathring{n} = \overset{\circ\circ}{\gamma}$, and $v_2 = \gamma$, up to sign. Our last vector n^* is a light-like vector in $(\mathrm{span}(v_1, v_2))^\perp$ that satisfies $\langle n, n^* \rangle = -1$, so it is given by $n^* = -Q_2\mathring{\gamma} + \overset{\circ\circ\circ}{\gamma} = -Q_2\, n + \overset{\circ\circ}{n}$. Hence, in summary, our moving frame consists of the four vectors:

$$n = \mathring{\gamma}, \quad v_1 = \mathring{n} = \overset{\circ\circ}{\gamma}, \quad v_2 = \gamma, \quad n^* = -Q_2 n + \overset{\circ\circ}{n} = -Q_2\mathring{\gamma} + \overset{\circ\circ\circ}{\gamma}. \tag{4.5}$$

We remark that the last vector n^* defines a point $m^* = \mathbb{E}^2 \cap \mathrm{span}(n^*)$, and the osculating circle intersects a circle which corresponds to $\overset{\circ\circ}{\gamma}$ in the pair of points m and m^*.

The Frenet formula with respect to our moving frame is then

$$\frac{d}{d\rho}\begin{pmatrix} n \\ v_1 \\ v_2 \\ n^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ Q_2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & Q_2 & 1 & 0 \end{pmatrix} \begin{pmatrix} n \\ v_1 \\ v_2 \\ n^* \end{pmatrix}. \tag{4.6}$$

4.4. Curves in $\mathbb{E}^3$ or $\mathbb{S}^3$. We now introduce a moving frame using a curve σ of osculating spheres, and hence we start our discussion with the arc-length parameter of σ.

4.4.1. *Osculating spheres and conformal torsion.* Let m be a point on a curve C in $\mathbb{E}^3$, $\gamma \subset \widetilde{G}^+_{2,5}$ be a curve of osculating circles, and $\sigma \subset \Lambda^4$ be a curve of osculating spheres; then, as we saw in Section 4.1, σ is a space-like curve in Λ^4. Let ρ be the conformal arc-length and l the arc-length parameter of the curve σ; here, we will denote differentiation with respect to l and ρ with a $^\bullet$ and $^\circ$ on top, respectively. Let

$$T = \frac{dl}{d\rho}. \tag{4.7}$$

As dl measures the infinitesimal angular variation of the spheres, T measures how an osculating sphere rotates around an osculating circle with respect to the conformal arc-length. For simplicity, let $T > 0$ in what follows; it was proven in [RS] that T coincides with the *conformal torsion* up to sign, and as was pointed out in [Sh], the conformal torsion can be uniquely determined up to sign. We note that our T is denoted by $\pm\lambda_2$ in [Su]. We remark that the conformal torsion is identically 0 if and only if C is a planar or spherical curve.

LEMMA 4.3. $\langle \ddot{\sigma}, \ddot{\sigma} \rangle = 1.$

Proof. As $\langle \dot{\gamma}, \dot{\gamma} \rangle = 0$ by Lemma 4.1, (3.1) implies $\langle \ddot{\sigma}, \ddot{\sigma} \rangle = 1$. ∎

PROPOSITION 4.4. *The conformal torsion T satisfies*

$$T = \frac{1}{\sqrt[4]{\langle \ddot{\gamma}, \ddot{\gamma} \rangle}} = \frac{1}{\sqrt[4]{\langle \dddot{\sigma}, \dddot{\sigma} \rangle - 1}}.$$

Proof. The first equality comes from Proposition 4.2.

Proceeding to the next equality, since $\langle\sigma,\sigma\rangle = \langle\dot\sigma,\dot\sigma\rangle = \langle\ddot\sigma,\ddot\sigma\rangle = 1$, the last equality holding by Lemma 4.3, we have $\langle\sigma,\dot\sigma\rangle = \langle\dot\sigma,\ddot\sigma\rangle = \langle\ddot\sigma,\dddot\sigma\rangle = 0$, and hence $\langle\sigma,\ddot\sigma\rangle = \langle\dot\sigma,\dddot\sigma\rangle = -1$ and $\langle\sigma,\dddot\sigma\rangle = 0$. Therefore, by (2.2),

$$\langle\dot\sigma\wedge\ddot\sigma,\dot\sigma\wedge\ddot\sigma\rangle = 1, \quad \langle\dot\sigma\wedge\ddot\sigma,\sigma\wedge\dddot\sigma\rangle = -1, \quad \langle\sigma\wedge\dddot\sigma,\sigma\wedge\dddot\sigma\rangle = \langle\dddot\sigma,\dddot\sigma\rangle.$$

As $\gamma = \sigma\wedge\dot\sigma$, we have $\ddot\gamma = \sigma\wedge\dddot\sigma + \dot\sigma\wedge\ddot\sigma$, and hence $\langle\ddot\gamma,\ddot\gamma\rangle = \langle\dddot\sigma,\dddot\sigma\rangle - 1$. ∎

It follows that $\langle d^i\sigma/dl^i, d^j\sigma/dl^j\rangle$ for small i,j are as in Table 2.

Table 2. A table of $\langle d^i\sigma/dl^i, d^j\sigma/dl^j\rangle$

$\langle\cdot,\cdot\rangle$	σ	$\dot\sigma$	$\ddot\sigma$	$\dddot\sigma$	$\sigma^{(4)}$
σ	1	0	-1	0	1
$\dot\sigma$	0	1	0	-1	0
$\ddot\sigma$	-1	0	1	0	$-(1+T^{-4})$
$\dddot\sigma$	0	-1	0	$1+T^{-4}$	$-2T^{-5}\dot T$

Lemma 4.5. *If the curve C is vertex-free and $dl/d\rho$ never vanishes, then σ, $\dot\sigma$, $\ddot\sigma$, $\dddot\sigma$, and $\sigma^{(4)}$ are linearly independent.*

Proof. Suppose $a_0\sigma + a_1\dot\sigma + a_2\ddot\sigma + a_3\dddot\sigma + a_4\sigma^{(4)} = \mathbf{0}$ for some $a_0,\dots,a_4 \in \mathbb{R}$. By defining $G_3 = \langle\dddot\sigma,\dddot\sigma\rangle = 1+T^{-4}$ and $G_4 = \langle\sigma^{(4)},\sigma^{(4)}\rangle$, and then taking the pseudo-inner product with $\sigma,\dots,\sigma^{(4)}$, we have:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -G_3 \\ 0 & -1 & 0 & G_3 & \frac{1}{2}\dot G_3 \\ 1 & 0 & -G_3 & \frac{1}{2}\dot G_3 & G_4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \mathbf{0},$$

and the determinant of the coefficient matrix is equal to $(1-G_3)^3 = -T^{-12} \neq 0$. ∎

4.4.2. *Osculating canals*

Proposition 4.6 ([Y], [LS]). *A point m on a curve C is given by the light-ray generated by the geodesic curvature vector $\vec{k}_g$ of the curve σ of osculating spheres of C.*

Proof. Since σ is given by $\sigma = \varphi m \times \dot m \times \ddot m \times \dddot m$ for some function φ, $\ddot\sigma$ can be expressed as a linear combination of vectors of the form $m \times \dot m \times m^{(i)} \times m^{(j)}$, which implies that $\langle m, \sigma + \ddot\sigma\rangle = 0$. On the other hand, Table 2 shows that $\sigma + \ddot\sigma$ is a non-zero light-like vector. As $(\mathrm{span}(m))^\perp$ is tangent to the light cone along $\mathrm{span}(m)$, this means $\mathrm{span}(m) = \mathrm{span}(\sigma + \ddot\sigma)$. ∎

Let σ be a curve in Λ^4 consisting of the osculating spheres of a vertex-free curve C in $\mathbb{E}^3$. The velocity vector σ' is space-like as long as it is non-zero (Section 4.1), so the restriction of σ to a subarc where σ' never vanishes becomes a canal surface, which is called an osculating canal surface. The characteristic circles are the osculating circles of C (Section 4.1), and hence the osculating canal surface coincides with the so-called curvature tube of [Ba-Wh], which is the union of osculating circles. The set γ of characteristic circles is a curve in the space of circles such that $\dot{\gamma}$ is a null pure vector everywhere (Section 4.2). By the assumption that σ' never vanishes, the conformal torsion never vanishes as well (Section 4.4.1). The geodesic curvature vector $\vec{k}_g = \sigma + \ddot{\sigma}$ is always light-like (Prop. 3.5), and a point m on C can be given by $m = \mathbb{E}^3 \cap \operatorname{span}(\vec{k}_g)$ (Prop. 4.6). The converse is also true, in general: if σ is a canal surface with light-like geodesic curvature vector, it is the curve of osculating spheres of a vertex-free curve given by $m = \mathbb{E}^3 \cap \operatorname{span}(\vec{k}_g)$, except in some degenerate cases. (The reader is referred to [Tho] and [LS] for these facts.)

On the other hand, a characterization of the curve of osculating circles of a space curve is given as follows: a curve γ in $\mathcal{S}(1,3)$, with non-vanishing velocity vector, is the set of osculating circles of a vertex-free curve in $\mathbb{E}^3$, also with non-vanishing velocity vector, if and only if ([LO2]):

(4.8a) $\gamma'(t)$ is a null pure vector for all t; and

(4.8b) $\dim\big(\operatorname{span}(\gamma'(t), \gamma''(t))\big) = 2.$

The conditions (4.8) do not depend on the parameter of γ as long as $\gamma'(t)$ never vanishes. One can see that, if γ is the set of osculating circles of an osculating canal, then conditions (4.8) are satisfied (by the discussion in Section 4.2 and Lemma 4.5), but the converse is not necessarily true, as conditions (4.8) do not exclude degenerate cases of osculating spheres. For example, the set of osculating circles of a plane curve (or a curve on a sphere) can also satisfy conditions (4.8), but the set of osculating spheres of the same curve degenerates to a set with exactly one element.

4.4.3. *Moving frames in $\mathbb{R}^5_1$ and the Frenet formula.* We now describe an isotropic orthonormal frame at $\rho = 0$ consisting of two light-like vectors n and n^*, and three space-like vectors v_1, v_2, and v_3, as follows: first, by choosing n in $\operatorname{span}(m) = \operatorname{span}(\sigma + \ddot{\sigma}) = \operatorname{span}(\vec{k}_g)$, we observe that $\operatorname{span}(v_1, v_2, v_3) \subset (\operatorname{span}(\sigma + \ddot{\sigma}))^\perp = \operatorname{span}(\sigma, \dot{\sigma}, \ddot{\sigma}, \dddot{\sigma})$. We then define

$$n = T(\sigma + \ddot{\sigma}), \quad v_1 = \mathring{n} = T\dot{T}(\sigma + \ddot{\sigma}) + T^2(\dot{\sigma} + \dddot{\sigma}), \quad v_2 = \dot{\sigma}, \quad v_3 = -\sigma,$$

where we have chosen $v_3 = -\sigma$, instead of $v_3 = \sigma$, so that the Frenet matrix and the normal form match those in [Su] and [CSW], respectively. Our last vector n^* is a light-like vector in $(\operatorname{span}(v_1, v_2, v_3))^\perp$ that is uniquely determined by the condition that $\langle n, n^* \rangle = -1$; this then defines a point $m^* = \mathbb{E}^3 \cap \operatorname{span}(n^*)$, and this point is on the osculating circle to C, since $\langle \sigma, n^* \rangle = \langle \dot{\sigma}, n^* \rangle = 0$.

In the Frenet formula which we derive, the parameter is now the conformal arc-length ρ, and not the arc-length l of the curve σ. This choice is made to allow for easy comparison with previous work on the topic.

Proposition 4.7 ([Su]). *Define*

$$Q = \langle \mathring{n}, \mathring{n}^* \rangle = -\langle \overset{\circ\circ}{n}, n^* \rangle. \tag{4.9}$$

Then the Frenet formula with respect to the above moving frame is given by

$$\frac{d}{d\rho}\begin{pmatrix} n \\ v_1 \\ v_2 \\ v_3 \\ n^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ Q & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & T & 0 \\ 0 & 0 & -T & 0 & 0 \\ 0 & Q & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} n \\ v_1 \\ v_2 \\ v_3 \\ n^* \end{pmatrix}. \tag{4.10}$$

Proof. First, notice that $\mathring{u} = T\dot{u}$ for any function u, by (4.7). The first, third, and fourth rows of the matrix follow from the definition of the frame. The other two rows can be obtained by differentiating the scalar products of the vectors defining the frame, and then using the fact that the n- (or n^*-) coordinate of a vector u is given by $-\langle u, n^* \rangle$ (or $-\langle u, n \rangle$, respectively). ∎

Since $\overset{\circ\circ}{n} = \mathring{v}_1 = Qn + n^*$, we have $\big\langle \overset{\circ\circ}{n}, \overset{\circ\circ}{n} \big\rangle = \big\langle Qn + n^*, Qn + n^* \big\rangle = -2Q$, and therefore

$$Q = -\tfrac{1}{2}\langle \overset{\circ\circ}{n}, \overset{\circ\circ}{n} \rangle. \tag{4.11}$$

This quantity Q is called the *conformal curvature* (denoted by λ_1 in [Su]).

As $\overset{\circ\circ}{n} = Qn + n^*$, our last vector in the frame is given by $n^* = -Qn + \overset{\circ\circ}{n}$, as in the case of planar curves. Thus, our isotropic orthonormal moving frame is

$$n \in \operatorname{span}(m), \quad v_1 = \mathring{n}, \quad v_2 = \mathring{\sigma}/\|\mathring{\sigma}\|, \quad v_3 = -\sigma, \quad n^* = -Qn + \overset{\circ\circ}{n}.$$

We observe that the above frame also serves as a moving frame for the curve $\sigma \subset \Lambda^4$.

Remark 4.8. Let us consider the asymptotic behavior of the frame as the curve C degenerates to a planar curve, which corresponds to $T \to 0$. We have

$$v_2 \wedge v_3 = \gamma,\ v_1 \wedge v_3 = \overset{\circ\circ}{\gamma} + T^2(\gamma - \dot{\sigma} \wedge \ddot{\sigma}),\ v_1 \wedge v_2 = T(\dot{T}\gamma - \dot{T}\dot{\sigma} \wedge \ddot{\sigma} - T\dot{\sigma} \wedge \dddot{\sigma}).$$

As T goes to 0, $v_2 \wedge v_3$ and $v_1 \wedge v_3$ approach γ and $\overset{\circ\circ}{\gamma}$, which are two space-like vectors in the frame (4.5) for a planar curve, whereas $v_1 \wedge v_2$ approaches $\mathbf{0}$.

Remark 4.9. Our moving frame n, v_1, v_2, v_3, n^* produces a moving frame of γ in $\bigwedge^2 \mathbb{R}^5_1 \cong \mathbb{R}^{10}_4$, which consists of four space-like vectors

$$n \wedge n^*, \quad v_1 \wedge v_2, \quad v_1 \wedge v_3, \quad v_2 \wedge v_3 = \gamma,$$

and six null vectors

$$n \wedge v_1, \quad n \wedge v_2, \quad n \wedge v_3 = \overset{\circ}{\gamma}, \quad v_1 \wedge n^*, \quad v_2 \wedge n^*, \quad v_3 \wedge n^*.$$

The Frenet formula for them can be obtained using (2.2) and (4.10).

Remark 4.10. Let us comment on the correspondence between our expression and some others in the literature on Möbius geometry. For the 5-dimensional Lorentz space equipped with the indefinite form given by $\langle \xi, \eta \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 - \xi_0\eta_4 - \xi_4\eta_0$, Euclidean space can be realized as $\mathbb{E}^3_L = \{(0, x, y, z, 0) \mid (x, y, z) \in \mathbb{R}^3\}$. Let $\mathcal{M}\ddot{o}b_3$ be the Möbius group; then the moving frame along a curve C defines a map $g = [n\, v_1\, v_2\, v_3\, n^*]$ from C to $\mathcal{M}\ddot{o}b_3$, where the vectors are column vectors, and our matrix in the Frenet formula is the transpose of $g^{-1}dg/d\rho$. Notice that we did not choose the same signature as in [CSW]: their indefinite form is given by $\langle \xi, \eta \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + \xi_0\eta_4 + \xi_4\eta_0$, whereas our matrix reverses the signs of the last row and column.

4.4.4. *Power series in ρ and the normal form.* In this section, we expand the x-, y-, and z-coordinates of a curve in power series in the conformal arc-length ρ. By then eliminating ρ, we obtain the normal form, which is a local expression of a curve as a power series in x.

The point $n = n(\rho)$ does not necessarily stay on some constant Euclidean model. We may assume, after a radial projection from the origin if necessary, that the Euclidean model where the curve C lives is given by

$$\mathbb{E}^3_0 = \{v \in \mathbb{R}^5_1 \mid \langle v, n^*(0) \rangle = -1\} = n(0) + \operatorname{span}(v_1(0), v_2(0), v_3(0)).$$

Then the x-, y-, and z-coordinates of the curve can be obtained by taking the inner product of $m(\rho)$ with $v_1(0), v_2(0)$, and $v_3(0)$, respectively. Let $f(\rho)$ be the function defined by $n(\rho) = m(\rho)/f(\rho)$. By using the Frenet formula (4.10) to express $n^{(i)}(\rho)$ in terms of the frame, we then obtain

$$\begin{aligned} -1 = \langle f(\rho) n(\rho), n^*(0) \rangle &= f(\rho) \left\langle n(0) + \overset{\circ}{n}(0)\, \rho + \frac{\overset{\circ\circ}{n}(0)}{2!}\, \rho^2 + O(\rho^3), n^*(0) \right\rangle \\ &= f(\rho) \left(-1 - \frac{Q(0)}{2} \rho^2 + O(\rho^3) \right), \end{aligned}$$

which implies that

$$f(\rho) = 1 - \frac{Q(0)}{2} \rho^2 + O(\rho^3).$$

Therefore, the curve can be expressed in terms of the conformal arc-length as

$$
\begin{aligned}
x &= \langle m(\rho), v_1(0)\rangle = f(\rho)\langle n(\rho), v_1(0)\rangle \\
&= \left(1 - \frac{Q(0)}{2}\rho^2 + O(\rho^3)\right)\left(\rho + \frac{Q(0)}{3}\rho^3 + O(\rho^4)\right) \\
&= \rho - \frac{Q(0)}{6}\,\rho^3 + O(\rho^4), \\
y &= \langle m(\rho), v_2(0)\rangle = f(\rho)\langle n(\rho), v_2(0)\rangle \\
&= \left(1 - \frac{Q(0)}{2}\,\rho^2 + O(\rho^3)\right)\left(\frac{1}{3!}\,\rho^3 + \frac{2Q(0) - T(0)^2}{5!}\,\rho^5 + O(\rho^6)\right) \\
&= \frac{1}{3!}\,\rho^3 + \left(\frac{2Q(0) - T(0)^2}{5!} - \frac{Q(0)}{2\cdot 3!}\right)\rho^5 + O(\rho^6), \\
z &= \langle m(\rho), v_3(0)\rangle = f(\rho)\langle n(\rho), v_3(0)\rangle \\
&= \left(1 - \frac{Q(0)}{2}\,\rho^2 + O(\rho^3)\right)\left(\frac{T}{4!}\rho^4 + \frac{\mathring{T}}{5!}\rho^5 + O(\rho^6)\right) \\
&= \frac{T}{4!}\rho^4 + \frac{\mathring{T}}{5!}\rho^5 + O(\rho^6).
\end{aligned}
$$

Since

$$
\frac{1}{3!}x^3 = \frac{1}{3!}\rho^3 - \frac{Q(0)}{2\cdot 3!}\rho^5 + O(\rho^6),
$$

the normal form of the curve is given by

$$
\begin{aligned}
y &= \frac{1}{3!}x^3 + \frac{(2Q - T^2)}{5!}\,x^5 + O(x^6), \\
z &= \frac{T}{4!}x^4 + \frac{\mathring{T}}{5!}\,x^5 + O(x^6),
\end{aligned}
\tag{4.12}
$$

as in [CSW].

REMARK 4.11. The normal form of a planar curve can be obtained by setting $T = 0$ and $Q = Q_2$, and forgetting the z-coordinate. The direction vectors of the x- and y-axes at $\rho = 0$ are then given by $\overset{\circ\circ}{\gamma}(0)$ and $\gamma(0)$, respectively.

4.4.5. *Conformal curvature and torsion in terms of osculating circles and spheres.* We shall present a new formula for the conformal curvature and conformal torsion in terms of a curve of osculating circles and a curve of osculating spheres. As osculating circles and osculating spheres can be expressed in terms of m and its derivatives (Section 4.1), they are easier to handle than other vectors in our moving frame, and so our new formula can be applied in computing T and Q in terms of Euclidean invariants such as curvature, torsion and their derivatives with respect to the arc-length of the curve C (Prop. 4.13 below).

THEOREM 4.12. *Let $\gamma \subset \widetilde{G}^{+}_{2,5}$ be a curve of osculating circles and $\sigma \subset \Lambda^4$ a curve of osculating spheres. Then the conformal curvature Q and the conformal torsion T satisfy*

$$T = \sqrt{\langle \overset{\circ}{\sigma}, \overset{\circ}{\sigma} \rangle}, \tag{4.13}$$

$$Q = -\tfrac{1}{2}\langle \overset{\circ\circ\circ}{\gamma}, \overset{\circ\circ\circ}{\gamma} \rangle + 3\langle \overset{\circ}{\sigma}, \overset{\circ}{\sigma} \rangle. \tag{4.14}$$

Proof. The first equation is trivial from the definition (4.7) of T, as $\langle \overset{\circ}{\sigma}, \overset{\circ}{\sigma} \rangle = \langle T\overset{\bullet}{\sigma}, T\overset{\bullet}{\sigma} \rangle = T^2$.

Furthermore, $\gamma = \sigma \wedge \overset{\bullet}{\sigma} = -v_3 \wedge v_2$, so we see that

$$\begin{aligned}
\overset{\circ}{\gamma} &= n \wedge v_3, \\
\overset{\circ\circ\circ}{\gamma} &= -2Tv_1 \wedge v_2 - \overset{\circ}{T} n \wedge v_2 + (Q - T^2) n \wedge v_3 - v_3 \wedge n^*.
\end{aligned}$$

It follows that $\langle \overset{\circ\circ\circ}{\gamma}, \overset{\circ\circ\circ}{\gamma} \rangle = 4T^2 - 2(Q - T^2) = -2Q + 6T^2$, which implies the second equation. ∎

This theorem shows that the conformal curvature Q for a plane curve coincides with the quantity Q_2 introduced in Section 4.3.

In the case of planar curves, the last vector of the frame was given by $n^* = \frac{1}{2}\langle \overset{\circ\circ\circ}{\gamma}, \overset{\circ\circ\circ}{\gamma} \rangle \overset{\circ}{\gamma} + \overset{\circ\circ\circ}{\gamma}$. But the same formula does not hold in the case of spatial curves: this is because the proof above shows that $\frac{1}{2}\langle \overset{\circ\circ\circ}{\gamma}, \overset{\circ\circ\circ}{\gamma} \rangle \overset{\circ}{\gamma} + \overset{\circ\circ\circ}{\gamma}$, or in general, any vector of the form $c\overset{\circ}{\gamma} + \overset{\circ\circ\circ}{\gamma}$, is not a pure vector; in other words, although it is a null vector, it does not define a light-like vector in $\mathbb{R}^5_1$.

4.5. Euclidean expression of conformal invariants. The results so far have been obtained within the framework of Möbius geometry. In this last section, we shall derive expressions for conformal invariants in terms of Euclidean invariants. Let κ and τ be the curvature and the torsion of a curve $C \subset \mathbb{E}^3$ respectively. Recall that $'$ means d/ds, where s is the arc-length of C.

Let $\nu = \sqrt{(\kappa')^2 + \kappa^2\tau^2}$. Then $\nu = 0$ at a point of C if and only if the point is a vertex. The conformal arc-length ρ satisfies ([Ta])

$$d\rho = \sqrt{\nu}\, ds = \sqrt[4]{(\kappa')^2 + \kappa^2\tau^2}\, ds. \tag{4.15}$$

PROPOSITION 4.13 ([CSW]). *The conformal curvature Q and conformal torsion T satisfy*

$$\begin{aligned}
Q &= \frac{4(\nu'' - \kappa^2\nu)\nu - 5(\nu')^2}{8\nu^3}, \\
T &= \pm\frac{2(\kappa')^2\tau + \kappa^2\tau^3 + \kappa\kappa'\tau' - \kappa\kappa''\tau}{\nu^{5/2}}.
\end{aligned}$$

Proof. We reprove this using the material from this article, as an application. In particular, we apply Theorem 4.12, and formulas (4.1) for σ, (4.4) for γ, (4.15) for $d\rho/ds$, and (2.1) and (2.2) for our pseudo-Riemannian structure. Using the results listed in Table 3, and letting $F_i = \|d^i m/ds^i\|^2$, we obtain

$$
\begin{aligned}
F_2 &= \kappa^2,\\
F_3 &= \kappa^4 + (\kappa')^2 + \kappa^2\tau^2,\\
F_4 &= 9\kappa^2(\kappa')^2 + (\kappa^3 + \kappa\tau^2 - \kappa'')^2 + (2\kappa'\tau + \kappa\tau')^2.
\end{aligned}
$$

We observe that $\nu = (d\rho/ds)^2$ satisfies $\nu^2 = F_3 - F_2^2$ ([L]).

Table 3. A table of $\langle d^i m/ds^i, d^j m/ds^j\rangle$

$\langle\cdot,\cdot\rangle$	m	m'	m''	m'''	$m^{(4)}$	$m^{(5)}$
m	0	0	-1	0	F_2	$\frac{5}{2}F_2'$
m'		1	0	$-F_2$	$-\frac{3}{2}F_2{}'$	$-2F_2'' + F_3$
m''			F_2	$\frac{1}{2}F_2{}'$	$\frac{1}{2}F_2{}'' - F_3$	$\frac{1}{2}F_2''' - \frac{3}{2}F_3'$
m'''				F_3	$\frac{1}{2}F_3{}'$	$\frac{1}{2}F_3'' - F_4$
$m^{(4)}$					F_4	$\frac{1}{2}F_4'$

We then see that

$$
\begin{aligned}
\overset{\circ\circ\circ}{\gamma} &= \frac{2(\nu')^2 - \nu\nu''}{2\nu^{7/2}}\gamma' - \frac{3\nu'}{2\nu^{5/2}}\gamma'' + \frac{1}{\nu^{3/2}}\gamma''',\\
\langle\overset{\circ\circ\circ}{\gamma}, \overset{\circ\circ\circ}{\gamma}\rangle &= \frac{(2(\nu')^2 - \nu\nu'')^2}{4\nu^7}\langle\gamma',\gamma'\rangle - \frac{3\nu'(2(\nu')^2 - \nu\nu'')}{2\nu^6}\langle\gamma',\gamma''\rangle\\
&\quad + \frac{2(\nu')^2 - \nu\nu''}{\nu^5}\langle\gamma',\gamma'''\rangle + \frac{9(\nu')^2}{4\nu^5}\langle\gamma'',\gamma''\rangle\\
&\quad - \frac{3\nu'}{\nu^4}\langle\gamma'',\gamma'''\rangle + \frac{1}{\nu^3}\langle\gamma''',\gamma'''\rangle\\
&= \frac{-11(\nu')^2 + 4\nu\nu'' + 4\langle\gamma''',\gamma'''\rangle}{4\nu^3},\\
\langle\gamma''',\gamma'''\rangle &= 5F_2^3 - 11F_2F_3 + 8F_2F_2'' - \frac{23}{2}F_2' - F_3'' + 6F_4,\\
\sigma' &= \frac{1}{\nu}m\times m'\times m''\times m^{(4)} - \frac{\nu'}{\nu^2}m\times m\times m'\times m''\times m''',\\
\langle\mathring{\sigma},\mathring{\sigma}\rangle &= \frac{1}{\nu}\langle\sigma',\sigma'\rangle\\
&= \frac{F_2^3 + F_2F_2'' - 2F_2F_3 + F_4 - \frac{9}{4}(F_2')^2 - (\nu')^2}{\nu^3}.
\end{aligned}
$$

The conclusion now follows by direct computation. ∎

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