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On some subspaces of Morrey–Sobolev spaces and boundedness of Riesz integrals

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Abstract. For $1 \le q \le \alpha \le p \le \infty$, $(L^q, l^p)^\alpha$ is a complex Banach space which is continuously included in the Wiener amalgam space (L^q, l^p) and contains the Lebesgue space L^α.

We study the closure $(L^q, l^p)^\alpha_{c,0}$ in $(L^q, l^p)^\alpha$ of the space $\mathcal{D}$ of test functions (infinitely differentiable and with compact support in $\mathbb{R}^d$) and obtain norm inequalities for Riesz potential operators and Riesz transforms in these spaces. We also introduce the Sobolev type space $W^1((L^q, l^p)^\alpha)$ (a subspace of a Morrey–Sobolev space, but a superspace of the classical Sobolev space $W^{1,\alpha}$) and obtain in it Sobolev inequalities and a Kondrashov–Rellich compactness theorem.

1. Introduction. Let d be a fixed positive integer. The space $\mathbb{R}^d$ is endowed with its usual scalar product $(x, \xi) \mapsto x \cdot \xi$, Euclidean norm $|\cdot|$ and Lebesgue measure.

For $1 \le p \le \infty$ we denote by $\| \ \|_p$ the usual norm on the classical Lebesgue space $L^p = L^p(\mathbb{R}^d)$ and by p' the conjugate of p $(1/p + 1/p' = 1)$.

Let I_γ $(0 < \gamma < 1)$ be the *Riesz potential* operator defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^d} |x - y|^{d(\gamma-1)} f(y)\, dy.$$

N. C. Phuc and M. Torres [P-T] have obtained a result which contains the following assertion:

Proposition 1.1. *Let $d/(d-1) < \alpha^* < \infty$ and f be a non-negative locally integrable function on $\mathbb{R}^d$. The following assertions are equivalent:*

(i) *The equation* $\operatorname{div} F = f$ *has a solution F in $(L^{\alpha^*})^d$.*
(ii) $I_{1/d} f \in L^{\alpha^*}$.

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In the proposition below we recall the classical Hardy–Littlewood–Sobolev inequality (see [St]) and a result contained in [D-F-K] [see Section 2 for definition of $(L^q, l^p)^\alpha$].

Proposition 1.2. *Let* $0 < \gamma < 1$, $1/(1-\gamma) < \alpha^* < \infty$ *and* $1/\alpha = 1/\alpha^* + \gamma$. *Then*

$$L^\alpha \subset \{f \in L^1_{\mathrm{loc}} \mid I_\gamma f \in L^{\alpha^*}\} \subset \textit{closure of } L^\alpha \textit{ in } (L^1, l^{\alpha^*})^\alpha.$$

The classical Sobolev spaces $W^{m,\alpha} = W^{m,\alpha}(\mathbb{R}^d)$ ($m \in \mathbb{N}^*$, $\alpha \in [1,\infty]$) have offered a fruitful framework for the study of partial differential equations (see [Br]). The density of smooth functions in L^α (for $\alpha < \infty$), Sobolev–Poincaré inequalities and the Kondrashov–Rellich compactness theorem are among the most important tools in this field.

In view of Propositions 1.1 and 1.2 it is worth:

- introducing Sobolev type spaces $W^1((L^q, l^p)^\alpha)$ for which the spaces $(L^q, l^p)^\alpha$ will take the place of the Lebesgue spaces L^α in the definition of $W^{1,\alpha}$;
- examining the existence in these new spaces of analogues for classical tools useful in the study of partial differential equations.

The paper deals with these questions. Section 2 contains notations, definitions and some known results. In Section 3 we introduce the space $W^1((L^q, l^p)^\alpha)$ and study the closure in $(L^q, l^p)^\alpha$ of the space $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^d)$ of infinitely differentiable real functions on $\mathbb{R}^d$. Section 4 is devoted to the boundedness of Riesz potential operators and Riesz transforms on $(L^q, l^p)^\alpha$, and analogues of the Sobolev inequality and of the Kondrashov–Rellich compactness theorem in the set up of $W^1((L^q, l^p)^\alpha)$. In Section 5 we prove an existence theorem for the equation $\operatorname{div} F = f$ with data $f \in (L^q, l^p)^\alpha$.

2. Preliminaries

Notations 2.1. For any subset E of $\mathbb{R}^d$, χ_E denotes its characteristic function and $|E|$ its Lebesgue measure.

Let r be a positive real number. We set

$$I^r_k = \prod_{j=1}^d [k_j r, (k_j+1)r), \qquad k = (k_1, \ldots, k_d) \in \mathbb{Z}^d,$$

$$J^r_x = \prod_{j=1}^d (x_j - r/2, x_j + r/2), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Definition 2.1. Let $1 \le q, p \le \infty$. For any f in $L^1_{\mathrm{loc}} = L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we set

$$ {}_r\|f\|_{q,p} = \begin{cases} \Big[\sum_{k\in\mathbb{Z}^d} (\|f\chi_{I_k^r}\|_q)^p \Big]^{1/p} & \text{if } p < \infty, \\ \sup_{x\in\mathbb{R}^d} \|f\chi_{J_x^r}\|_q & \text{if } p = \infty, \end{cases} $$

and we define

$$ (L^q, l^p) = \{ f \in L^1_{\mathrm{loc}} \mid {}_1\|f\|_{q,p} < \infty \}. $$

The Wiener amalgam spaces (L^q, l^p) $(1 \le q, p \le \infty)$ were introduced in 1926 by Norbert Wiener who considered the special cases (L^1, l^2), (L^2, l^∞), (L^∞, l^1) and (L^1, l^∞) (see [Wi1] and [Wi2]). In 1975 Finbar Holland undertook the first systematic study of these spaces (see [Ho]). Since then, much work has been dedicated to them (see the survey paper [F-S] and the references therein) and to their generalizations introduced by Hans Feichtinger in 1980 (see [Fe1], [Fe2]).

Let us recall the following results (see [Ho] and [Fo3]).

Proposition 2.1. *Let* $1 \le q, p \le \infty$.

(a) $((L^q, l^p), {}_1\|\ \|_{q,p})$ *is a Banach space and* $(L^q, l^q) = L^q$.
(b) *If* $q, p < \infty$ *then there exist real numbers* A *and* B *such that*

$$ A\, {}_r\|f\|_{q,p} \le r^{-d/p} \Big\{ \int_{\mathbb{R}^d} \Big[\int_{J_x^r} |f(y)|^q\, dy \Big]^{p/q} dx \Big\}^{1/p} \le B\, {}_r\|f\|_{q,p} $$

for all $f \in L^1_{\mathrm{loc}}$, $r > 0$.

Definition 2.2. Let $1 \le q \le \alpha \le p \le \infty$. For any f in L^1_{loc} we set

$$ \|f\|_{q,p,\alpha} = \sup_{r>0} r^{d(1/\alpha - 1/q)}\, {}_r\|f\|_{q,p}, $$

$$ |||f|||_{q,p,\alpha} = \sup_{r>0} r^{d(1/\alpha - 1/q - 1/p)} \Big\{ \int_{\mathbb{R}^d} \Big[\int_{J_x^r} |f(y)|^q\, dy \Big]^{p/q} dx \Big\}^{1/p} \quad \text{if } p < \infty, $$

and we define

$$ (L^q, l^p)^\alpha = \{ f \in L^1_{\mathrm{loc}} \mid \|f\|_{q,p,\alpha} < \infty \}. $$

The spaces $(L^q, l^p)^\alpha$ were introduced in 1988 by Ibrahim Fofana (see [Fo1]–[Fo3]). Results about multipliers and Fourier multipliers between Lebesgue spaces and continuity properties of fractional maximal operators and Riesz potential operators were obtained in this framework (see [Fo3], [Fo4], [F-F-K], [D-F]). We recall some of their properties below (see [Fo3]).

Proposition 2.2. *Let* $1 \le q \le \alpha \le p \le \infty$.

(a) $((L^q, l^p)^\alpha, \|\ \|_{q,p,\alpha})$ *is a Banach space.*
(b) $|||\ |||_{q,p,\alpha}$ *is a norm equivalent to* $\|\ \|_{q,p,\alpha}$ *on* $(L^q, l^p)^\alpha$ *if* $p < \infty$.
(c) $\|f\|_{q,p,\alpha} \le \|f\|_\alpha$ *for* $f \in L^1_{\mathrm{loc}}$, *and therefore* $L^\alpha \subset (L^q, l^p)^\alpha$.

(d) $(L^q, l^p)^\alpha = L^\alpha$ *when* $\alpha \in \{q, p\}$.

(e) *If* $q < \alpha < p$ *then there exists a real number* C *such that*

$$\|f\|_{q,p,\alpha} \le C\|f\|^*_{\alpha,+\infty}, \qquad f \in L^1_{\mathrm{loc}},$$

where

$$\|f\|^*_{\alpha,+\infty} = \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^d \mid |f(x)| > \lambda\}|^{1/\alpha}$$

and therefore the weak-Lebesgue space $L^{\alpha,+\infty} = \{f \in L^1_{\mathrm{loc}} \mid \|f\|^*_{\alpha,\infty} < \infty\}$ *is contained in* $(L^q, l^p)^\alpha$.

(f) $(L^q, l^p)^\alpha \subset (L^{q_1}, l^p)^\alpha$ *if* $1 \le q_1 < q$, *and* $(L^q, l^p)^\alpha \subset (L^q, l^{p_1})^\alpha$ *if* $p < p_1 \le \infty$.

Let us recall that the convolution product $f * g$ of f, g in L^1_{loc} is given by the formula

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$$

at all points $x \in \mathbb{R}^d$ where this integral is defined. It satisfies the following Young inequality (see [Fo3]).

Proposition 2.3.

(a) *Let* $1 \le q_1 \le \alpha_1 \le p_1 \le \infty$, $1 \le q_2 \le \alpha_2 \le p_2 \le \infty$, $1/p_1 + 1/p_2 - 1 = 1/p \ge 0$, $1/\alpha_1 + 1/\alpha_2 - 1 = 1/\alpha$ *and* $1/q_1 + 1/q_2 - 1 = 1/q$. *Then for any* f_1 *in* $(L^{q_1}, l^{p_1})^{\alpha_1}$ *and* f_2 *in* $(L^{q_2}, l^{p_2})^{\alpha_2}$,

$$\|f_1 * f_2\|_{q,p,\alpha} \le C\|f_1\|_{q_1,p_1,\alpha_1}\|f_2\|_{q_2,p_2,\alpha_2}$$

where C *is a real number not depending on* f_1 *and* f_2.

(b) *In particular if* $1 \le q \le \alpha \le p \le \infty$ *then for any* (φ, f) *in* $L^1 \times (L^q, l^p)^\alpha$,

$$\|\varphi * f\|_{q,p,\alpha} \le C\|\varphi\|_1\|f\|_{q,p,\alpha}$$

where C *is a real number not depending on* f *and* φ.

We recall that in the theory of Sobolev spaces, approximation of an element of a Lebesgue space by elements of $\mathcal{C}^\infty$ is an important device based on the continuity of the convolution product (Young inequality) and of the translation operator τ_u with translation vector $u \in \mathbb{R}^d$, defined by

$$(\tau_u f)(x) = f(x-u), \qquad x \in \mathbb{R}^d,\ f \in L^1_{\mathrm{loc}}.$$

It is easy to verify the following assertion.

Proposition 2.4. *Let* $1 \le q \le \alpha \le p \le \infty$. *Then* $(L^q, l^p)^\alpha$ *is translation invariant and there is a real number* C *such that*

$$\|\tau_u f\|_{q,p,\alpha} \le C\|f\|_{q,p,\alpha}, \qquad u \in \mathbb{R}^d,\ f \in L^1_{\mathrm{loc}}.$$

However an analogue of the following property of Lebesgue spaces:

$$\lim_{u\to 0}\|\tau_u f - f\|_\alpha = 0, \qquad f\in L^\alpha,\ 1\le\alpha<\infty,$$

is not true in $(L^q,l^p)^\alpha$ when $1\le q<\alpha<p\le\infty$. So, I. Fofana [Fo3] has considered some special subspaces of $(L^q,l^p)^\alpha$ defined below.

Definition 2.3. For $1\le q\le\alpha\le p\le\infty$ we set

$$(L^q,l^p)^\alpha_c = \Big\{ f\in (L^q,l^p)^\alpha \;\Big|\; \lim_{u\to 0}\|\tau_u f - f\|_{q,p,\alpha}=0\Big\},$$
$$(L^q,l^p)^\alpha_0 = \Big\{ f\in (L^q,l^p)^\alpha \;\Big|\; \lim_{R\to\infty}\|f\chi_{\mathbb{R}^d\setminus J_0^R}\|_{q,p,\alpha}=0\Big\},$$
$$(L^q,l^p)^\alpha_{c,0} = (L^q,l^p)^\alpha_c\cap (L^q,l^p)^\alpha_0.$$

Let us fix some notations.

Notations 2.2.

- ρ is a fixed element of $\mathcal{C}^\infty$, non-negative, with support included in the unit ball $\overline{B}(0;1)=\{x\in\mathbb{R}^d \mid |x|\le 1\}$ and satisfying $\int_{\mathbb{R}^d}\rho(x)\,dx=1$.
- $\rho_m(x)=m^d\rho(mx)$, $x\in\mathbb{R}^d$, $m\in\mathbb{N}^*$.
- ω is a fixed element of $\mathcal{C}^\infty$ satisfying $\chi_{J_0^1}\le\omega\le\chi_{J_0^2}$.
- $\omega_m(x)=\omega(x/m)$, $x\in\mathbb{R}^d$, $m\in\mathbb{N}^*$.

The following results are contained in [Fo3].

Proposition 2.5. *Let* $1\le q\le\alpha\le p\le\infty$.

(a) $(L^q,l^p)^\alpha_c$ *is a closed subspace of* $(L^q,l^p)^\alpha$.
(b) *If* $\alpha<\infty$ *then* $L^\alpha\subset(L^q,l^p)^\alpha_c$.
(c) $(L^q,l^p)^\alpha_c = L^1*(L^q,l^p)^\alpha_c = L^1*(L^q,l^p)^\alpha$.
(d) $\lim_{m\to\infty}\|\rho_m*f-f\|_{q,p,\alpha}=0$ *for* f *in* $(L^q,l^p)^\alpha_c$, *where* ρ_m *is defined as in Notations* 2.2.

We list below some useful properties of $(L^q,l^p)^\alpha_0$ and $(L^q,l^p)^\alpha_{c,0}$.

Proposition 2.6. *Let* $1\le q\le\alpha\le p\le\infty$.

(a) $(L^q,l^p)^\alpha_0$ *and* $(L^q,l^p)^\alpha_{c,0}$ *are closed subspaces of* $(L^q,l^p)^\alpha$.
(b) $\lim_{m\to\infty}\|(f\omega_m)*\rho_m-f\|_{q,p,\alpha}=0$ *for* $f\in(L^q,l^p)^\alpha_{c,0}$, *where* ω_m *and* ρ_m *are defined as in Notations* 2.2.

Proof. (a) It is clear that $(L^q,l^p)^\alpha_0$ and $(L^q,l^p)^\alpha_{c,0}$ are subspaces of $(L^q,l^p)^\alpha$.

Suppose that $(f_n)_{n\ge 1}$ is a sequence of elements of $(L^q,l^p)^\alpha_0$ converging in $(L^q,l^p)^\alpha$ to some f. Let $\varepsilon>0$. For any real $R>0$ we have

$$\begin{aligned}|f-f\chi_{J_0^R}| &\le |f-f_n|+|f_n-f_n\chi_{J_0^R}|+|(f-f_n)\chi_{J_0^R}|\\ &\le 2|f-f_n|+|f_n-f_n\chi_{J_0^R}|, \qquad n\ge 1.\end{aligned}$$

There are $n_\varepsilon \geq 1$ and $R_\varepsilon > 0$ such that

$$\|f - f_{n_\varepsilon}\|_{q,p,\alpha} \leq \varepsilon/3 \quad \text{and} \quad \|f_{n_\varepsilon} - f_{n_\varepsilon}\chi_{J_0^R}\|_{q,p,\alpha} \leq \varepsilon/3, \qquad R \geq R_\varepsilon,$$

and therefore

$$\|f - f\chi_{J_0^R}\|_{q,p,\alpha} < \varepsilon, \qquad R \geq R_\varepsilon.$$

Thus $f \in (L^q, l^p)_0^\alpha$. This means that $(L^q, l^p)_0^\alpha$ is closed in $(L^q, l^p)^\alpha$. Furthermore, by Proposition 2.5(a), $(L^q, l^p)_c^\alpha$ is also closed in $(L^q, l^p)^\alpha$. Thus $(L^q, l^p)_{c,0}$ is also closed.

(b) Let f be in $(L^q, l^p)_{c,0}^\alpha$. We have

$$\|f - (f\omega_m) * \rho_m\|_{q,p,\alpha} \leq \|f - f * \rho_m\|_{q,p,\alpha} + \|(f - f\omega_m) * \rho_m\|_{q,p,\alpha}, \quad m \geq 1,$$

and therefore, by Proposition 2.3(b),

$$\|f - (f\omega_m) * \rho_m\|_{q,p,\alpha} \leq \|f - f * \rho_m\|_{q,p,\alpha} + \|f - f\omega_m\|_{q,p,\alpha}, \quad m \geq 1.$$

It is clear that

$$|f - f\omega_m| \leq |f - f\chi_{J_0^m}|, \qquad m \geq 1,$$

and so

$$\|f - (f\omega_m) * \rho_m\|_{q,p,\alpha} \leq \|f - f * \rho_m\|_{q,p,\alpha} + \|f - f\chi_{J_0^m}\|_{q,p,\alpha}, \qquad m \geq 1,$$

which implies that $\lim_{m\to\infty} \|f - (f\omega_m) * \rho_m\|_{q,p,\alpha} = 0$. ∎

Notice that Propositions 2.5(d) and 2.6(b) together with Proposition 2.2(c) imply that in $(L^q, l^p)^\alpha$:

- $(L^q, l^p)_c^\alpha$ is the closure of $(L^q, l^p)_c^\alpha \cap \mathcal{C}^\infty$,
- $(L^q, l^p)_{c,0}^\alpha$ is the closure of $\mathcal{D}$ (and also of L^α if $\alpha < \infty$).

It is worth recalling the following extension of the well known Kolmogorov–Riesz–Tamarkin compactness theorem (see [S-F]):

PROPOSITION 2.7. *Let $1 \leq q \leq \alpha \leq p \leq \infty$ with $\alpha < \infty$. Any closed subset H of $(L^q, l^p)^\alpha$ satisfying the following conditions:*

(i) $\sup_{f\in H} \|f\|_{q,p,\alpha} < \infty$,
(ii) $\lim_{u\to 0} \sup_{f\in H} \|f - \tau_u f\|_{q,p,\alpha} = 0$,
(iii) $\lim_{R\to\infty} \sup_{f\in H} \|f - f\chi_{J_0^R}\|_{q,p,\alpha} = 0$,

is compact in $(L^q, l^p)^\alpha$.

3. Sobolev spaces. We fix $q, \alpha, p \in [1, \infty]$ such that $q \leq \alpha \leq p$ and $q < \infty$.

DEFINITION 3.1. Let E be one of the spaces $(L^q, l^p)^\alpha$, $(L^q, l^p)_c^\alpha$ or $(L^q, l^p)_{c,0}^\alpha$. We define

$$W^1(E) = \{f \in E \mid \partial f/\partial x_j \in E \text{ for } j \in \{1, \ldots, d\}\}$$

where $\partial f/\partial x_j = D_j f$ stands for the distributional partial derivative.

For any f in $W^1((L^q, l^p)^\alpha)$ we set

$$\|f\|_{W^1((L^q,l^p)^\alpha)} = \|f\|_{q,p,\alpha} + \sum_{j=1}^{d} \left\| \frac{\partial f}{\partial x_j} \right\|_{q,p,\alpha}.$$

We point out that:

- $W^1((L^q, l^p)^\alpha)$ is a subspace of a more general Sobolev type space introduced by Domion Douyon in his thesis ([Do]).
- $W^1((L^q, l^\infty)^\alpha)$ is the Morrey–Sobolev space $W^{1,(q,d(1-q/\alpha))}(\mathbb{R}^d)$ considered by G. Cupini and R. Petti and used in the study of the regularity of minimizers for functionals ([C-P]) and solutions of elliptic equations ([F-L-Y]).

It is easy to verify

Proposition 3.1.

(a) $W^1((L^q, l^p)^\alpha)$ *is a subspace of* $W^{1,q}_{\mathrm{loc}} = \{f \in L^q_{\mathrm{loc}} \mid f \in W^{1,q}(\Omega)$ *for any bounded open subset* Ω *of* $\mathbb{R}^d\}$.

(b) $(W^1(E), \| \ \|_{W^1((L^q,l^p)^\alpha)})$ *is a Banach space if* E *is any of the spaces* $(L^q, l^p)^\alpha$, $(L^q, l^p)^\alpha_c$ *and* $(L^q, l^p)^\alpha_{c,0}$.

Let us recall the following well known result (see [K-J-F]).

Lemma 3.2. *Suppose that* $f \in L^q_{\mathrm{loc}}$ *and* $D_j f \in L^q_{\mathrm{loc}}$ *for some* $j \in \{1, \ldots, d\}$. *Then*

$$\rho_m * f \in \mathcal{C}^\infty,\ D_j(\rho_m * f) = (D_j\rho_m) * f = \rho_m * (D_j f), \qquad m \in \mathbb{N},$$

$$D^\beta(\rho_m * f) = (D^\beta\rho_m) * f, \qquad (\beta, m) \in \mathbb{N}^d \times \mathbb{N}^*,$$

$$\lim_{m\to\infty} \|(\rho_m * f - f)\chi_{J^r_x}\|_q = 0 = \lim_{m\to\infty} \|[D_j(\rho_m * f) - D_j f]\chi_{J^r_x}\|_q, \qquad (x, r) \in \mathbb{R}^d \times (0, \infty),$$

where ρ_m *is as in Notations* 2.2.

From the lemma above and the proof of Proposition IX in [Br] we readily obtain the following result.

Lemma 3.3. *Suppose that* $f \in W^{1,q}_{\mathrm{loc}}$. *Then*

$$\int_{J^r_x} |\tau_u f(y) - f(y)|^q \, dy \leq |u|^q \int_0^1 \int_{J^r_x} |\nabla f(y - tu)|^q \, dy \, dt, \qquad (u, x, r) \in \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty).$$

The lemma above leads to the following property of our Sobolev type space.

Proposition 3.2. *There exists a real number C such that*

$$\|\tau_u f - f\|_{q,p,\alpha} \leq C|u| \, \| \, |\nabla f| \, \|_{q,p,\alpha}, \qquad u \in \mathbb{R}^d, \ f \in W^1((L^q, l^p)^\alpha),$$

and therefore $W^1((L^q, l^p)^\alpha) \subset (L^q, l^p)_c^\alpha$.

Proof. Suppose that $p < \infty$, $f \in W^1((L^q, l^p)^\alpha)$, $u \in \mathbb{R}^d$ and $r \in (0, \infty)$. From Lemma 3.3 we get

$$\begin{aligned} I &:= \bigg\{ \int_{\mathbb{R}^d} \Big[\int_{J_x^r} |\tau_u f(y) - f(y)|^q \, dy \Big]^{p/q} dx \bigg\}^{1/p} \\ &\leq |u| \bigg\{ \int_{\mathbb{R}^d} \Big[\int_0^1 \int_{J_x^r} |\nabla f(y - tu)|^q \, dy \, dt \Big]^{p/q} dx \bigg\}^{1/p}. \end{aligned}$$

Therefore, by the Minkowski inequality for integrals (see [St, p. 271])

$$I \leq |u| \bigg\{ \int_0^1 \Big[\int_{\mathbb{R}^d} \Big(\int_{J_x^r} |\nabla f(y - tu)|^q \, dy \Big)^{p/q} dx \Big]^{q/p} dt \bigg\}^{1/q}.$$

From the inequality above and Proposition 2.4, we obtain

$$\begin{aligned} I &\leq C_1 |u| \bigg\{ \int_0^1 [\| \, |\nabla f| \, \|_{q,p,\alpha} r^{d(1/q + 1/p - 1/\alpha)}]^q \, dt \bigg\}^{1/q} \\ &= C_1 |u| \, \| \, |\nabla f| \, \|_{q,p,\alpha} r^{d(1/q + 1/p - 1/\alpha)} \end{aligned}$$

where C_1 is a real number not depending on f, u and r. Therefore, by Proposition 2.2(b) we have

$$\|\tau_u f - f\|_{q,p,\alpha} \leq C|u| \, \| \, |\nabla f| \, \|_{q,p,\alpha}$$

where C is a real number not depending on f and u.

In the case $p = \infty$ a similar proof works. ■

From Propositions 2.5 and 3.2 we deduce the following result.

Proposition 3.3. *Suppose that $q < \infty$. Then the following assertions are equivalent:*

(i) $f \in (L^q, l^p)_c^\alpha$.
(ii) $f = \lim_{m \to \infty} \rho_m * f$ *in* $(L^q, l^p)^\alpha$ *where* ρ_m *is as in Notations* 2.2.
(iii) f *belongs to the closure in* $(L^q, l^p)^\alpha$ *of*

$$\mathcal{C}^\infty_{(L^q, l^p)^\alpha} = \{ g \in \mathcal{C}^\infty \mid D^\beta g \in (L^q, l^p)^\alpha \text{ for all } \beta \in \mathbb{N}^d \}.$$

Proof. (i)⇒(ii) by Proposition 2.5(d).

Suppose that (ii) is true. Fix a positive integer m and $\beta \in \mathbb{N}^d$. By Lemma 3.2, $\rho_m * f \in \mathcal{C}^\infty$ and $D^\beta(\rho_m * f) = (D^\beta \rho_m) * f$. As $D^\beta \rho_m \in L^1$, Proposition 2.3(b) shows that $D^\beta(\rho_m * f) \in (L^q, l^p)^\alpha$. Therefore $\rho_m * f \in \mathcal{C}^\infty_{(L^q, l^p)^\alpha}$. Furthermore $\lim_{m \to \infty} \|\rho_m * f - f\|_{q,p,\alpha} = 0$ (Proposition 2.5(d)). Thus (ii)⇒(iii).

Suppose that (iii) is true: there exists a sequence $(g_m)_{m\geq 1} \subset \mathcal{C}^{\infty}_{(L^q,l^p)^\alpha}$ converging to f in $(L^q, l^p)^\alpha$. It is clear that any g_m ($m \in \mathbb{N}^*$) belongs to $W^1((L^q, l^p)^\alpha)$. Therefore, from Proposition 3.2 we have

$$g_m \in (L^q, l^p)^\alpha_c, \qquad m \in \mathbb{N}^*.$$

$(L^q, l^p)^\alpha_c$ being closed in $(L^q, l^p)^\alpha$ (Proposition 2.5(a)), f clearly belongs to $(L^q, l^p)^\alpha_c$. Thus (iii)⇒(i). ∎

Proposition 3.2 leads to the following characterization of $W^1((L^q, l^p)^\alpha)$.

PROPOSITION 3.4. *Suppose that $f \in (L^q, l^p)^\alpha$. Then the following assertions are equivalent:*

(i) $f \in W^1((L^q, l^p)^\alpha)$.
(ii) *There exists a real number C such that*

$$\|\tau_u f - f\|_{q,p,\alpha} \leq C|u|, \qquad u \in \mathbb{R}^d.$$

Proof. The implication (i)⇒(ii) follows readily from Proposition 3.2.

Conversely, suppose that (ii) is true. Denote by $\{e_j \mid 1 \leq j \leq d\}$ the canonical basis of $\mathbb{R}^d$.

(a) Let Ω be any bounded open subset of $\mathbb{R}^d$ and Q a closed and bounded cube in $\mathbb{R}^d$ such that $\Omega \subset Q$. We have

$$\begin{aligned}\|(\tau_u f - f)\chi_\Omega\|_q \leq \|(\tau_u f - f)\chi_Q\|_q &\leq 2^{d/p'}|Q|^{1/q-1/\alpha}\|\tau_u f - f\|_{q,p,\alpha} \\ &\leq 2^{d/p'}C|Q|^{1/q-1/\alpha}|u|, \qquad u \in \mathbb{R}^d,\end{aligned}$$

and

$$\|s^{-1}(\tau_{se_j} f - f)\chi_\Omega\|_q \leq 2^{d/p'}|Q|^{1/q-1/\alpha}C, \qquad j \in \{1,\ldots,d\},\ s \in (0,\infty).$$

Hence $\{s^{-1}(\tau_{se_j} f - f)\chi_\Omega \mid s \in (0,\infty),\ j \in \{1,\ldots,d\}\}$ is a bounded subset of L^q. Therefore there exists a sequence $(s_m)_{m\geq 1}$ in $(0,\infty)$ such that

$$\begin{cases} \lim\limits_{m\to\infty} s_m = 0, \\ \text{for any } j \in \{1,\ldots,d\},\ (s_m^{-1}(\tau_{s_m e_j} f - f)\chi_\Omega)_{m\geq 1} \text{ weakly converges} \\ \text{in } L^q \text{ to some } g_j. \end{cases}$$

Notice that, for any $j \in \{1,\ldots,d\}$ and any $\varphi \in \mathcal{C}^\infty$ with support in Ω,

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \varphi(x) g_j(x)\,dx &= \lim_{m\to\infty} \int\limits_{\mathbb{R}^d} \varphi(x) s_m^{-1}[f(x - s_m e_j) - f(x)]\,dx \\ &= -\lim_{m\to\infty} \int\limits_{\mathbb{R}^d} s_m^{-1}[\varphi(x + s_m e_j) - \varphi(x)] f(x)\,dx \\ &= -\int\limits_{\mathbb{R}^d} \frac{\partial\varphi}{\partial x_j} f(x)\,dx.\end{aligned}$$

That is, $g_j = \partial f/\partial x_j$ in Ω for $j \in \{1,\ldots,d\}$. Therefore $f \in W^{1,q}_{\text{loc}}$.

(b) Suppose that $p=\infty$. Let R be a bounded and closed cube, $0<\epsilon<1$, $Q=(1+\epsilon)R$ the cube with side length $(1+\epsilon)|R|^{1/d}$ and the same center as R, and $\Omega=\dot{Q}$ the interior of Q.

Using the notations in (a) we have, for any $j\in\{1,\dots,d\}$ and any $\varphi\in\mathcal{C}^\infty$ with support in Ω,

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}\frac{\partial f}{\partial x_j}(x)\varphi(x)\,dx\Big| &= \lim_{m\to\infty}\Big|\int_{\mathbb{R}^d} s_m^{-1}[f(x-s_me_j)-f(x)]\varphi(x)\,dx\Big|\\ &\le \limsup_{m\to\infty}\|(\tau_{s_me_j}f-f)\chi_Q\|_q\|\varphi\|_{q'}s_m^{-1}\\ &\le \limsup_{m\to\infty}\|\tau_{s_me_j}f-f\|_{q,\infty,\alpha}|Q|^{1/q-1/\alpha}\|\varphi\|_{q'}s_m^{-1}\\ &\le C|Q|^{1/q-1/\alpha}\|\varphi\|_{q'},\end{aligned}$$

and so

$$\Big\|\frac{\partial f}{\partial x_j}\chi_R\Big\|_q \le \Big\|\frac{\partial f}{\partial x_j}\chi_\Omega\Big\|_q \le C|Q|^{1/q-1/\alpha}=(1+\epsilon)^{d(1/q-1/\alpha)}C|R|^{1/q-1/\alpha}.$$

Letting ϵ go to zero, we get, for any $j\in\{1,\dots,d\}$,

$$\Big\|\frac{\partial f}{\partial x_j}\chi_R\Big\|_q\le C|R|^{1/q-1/\alpha}.$$

Thus $\|\partial f/\partial x_j\|_{q,\infty,\alpha}\le C$ for $j\in\{1,\dots,d\}$.

(c) Suppose that $p<\infty$. Let (r,m) be any element of $(0,\infty)\times\mathbb{N}^*$. Set $K_n=\{k\in\mathbb{Z}^d \mid |k|\le n\}$,

$$Q=\{x=(x_j)_{1\le j\le d}\mid -(n+1)r\le x_j\le (n+2)r \text{ for } 1\le j\le d\},$$

and $\Omega=\dot{Q}$. For any $k\in K_n$, let $\varphi_k\in\mathcal{C}_0^\infty$ with support in I_k^r and $\|\varphi_k\|_{q'}\le 1$. Using the notations in (a) we have, for $|s_n|<\min_{k\in K_n} d(\operatorname{supp}\varphi_k,\partial I_k^r)$ and $j\in\{1,\dots,d\}$,

$$\begin{aligned}&\Big[\sum_{k\in K_n}\Big|\int_{\mathbb{R}^d} s_m^{-1}(\tau_{s_me_j}f-f)(x)\varphi_k(x)\,dx\Big|^p\Big]^{1/p}\\ &\quad\le\Big[\sum_{k\in K_n}(s_m^{-1}\|(\tau_{s_me_j}f-f)\chi_{I_k^r}\|_q\|\varphi_k\|_{q'})^p\Big]^{1/p}\\ &\quad\le s_m^{-1}\Big[\sum_{k\in K_n}\|(\tau_{s_me_j}f-f)\chi_{I_k^r}\|_q^p\Big]^{1/p}\\ &\quad\le s_m^{-1}\|\tau_{s_me_j}f-f\|_{q,p,\alpha}r^{d(1/q-1/\alpha)}\le Cr^{d(1/q-1/\alpha)},\end{aligned}$$

and therefore

$$\Big[\sum_{k\in K_n}\Big|\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_j}\varphi_k(x)\,dx\Big|^p\Big]^{1/p}\le Cr^{d(1/q-1/\alpha)}.$$

Thus

$$\Big[\sum_{k\in K_n}\Big\|\frac{\partial f}{\partial x_j}\chi_{I_k^r}\Big\|_q^p\Big]^{1/p}\le Cr^{d(1/q-1/\alpha)}.$$

Letting n go to infinity, we obtain

$${}_r\Big\|\frac{\partial f}{\partial x_j}\Big\|_{q,p}\le Cr^{d(1/q-1/\alpha)}.$$

Finally,

$$\Big\|\frac{\partial f}{\partial x_j}\Big\|_{q,p,\alpha}\le C,\qquad j=\{1,\dots,d\}. \ \blacksquare$$

$W^1((L^q,l^p)^\alpha_{c,0})$ has the following approximation property.

Proposition 3.5. *We have*

$$\lim_{m\to\infty}\|f-(f\omega_m)*\rho_m\|_{W^1((L^q,l^p)^\alpha)}=0,\qquad f\in W^1((L^q,l^p)^\alpha_{c,0})$$

where ρ_m *and* ω_m *are as in Notations* 2.2.

Proof. Let $f\in W^1((L^q,l^p)^\alpha_{c,0})$. By Proposition 2.6(b), $((f\omega_m)*\rho_m)_{m\ge1}$ converges to f in $(L^q,l^p)^\alpha$. For any $(j,m)\in\{1,\dots,d\}\times\mathbb{N}^*$ we have

$$\frac{\partial}{\partial x_j}((f\omega_m)*\rho_m)=\Big(\frac{\partial f}{\partial x_j}\omega_m\Big)*\rho_m+\Big(f\frac{\partial\omega_m}{\partial x_j}\Big)*\rho_m,$$

so

$$\begin{aligned}\Big\|\frac{\partial f}{\partial x_j}-\frac{\partial}{\partial x_j}((f\omega_m)*\rho_m)\Big\|_{q,p,\alpha}&\\ \le\Big\|\frac{\partial f}{\partial x_j}-\frac{\partial f}{\partial x_j}*\rho_m\Big\|_{q,p,\alpha}&+\Big\|\Big(\frac{\partial f}{\partial x_j}-\frac{\partial f}{\partial x_j}\omega_m\Big)*\rho_m\Big\|_{q,p,\alpha}\\ +\Big\|\Big(f\frac{\partial\omega_m}{\partial x_j}\Big)*\rho_m\Big\|_{q,p,\alpha}&\end{aligned}$$

and therefore, by Proposition 2.3(b),

$$\begin{aligned}&\Big\|\frac{\partial f}{\partial x_j}-\frac{\partial}{\partial x_j}((f\omega_m)*\rho_m)\Big\|_{q,p,\alpha}\\ &\le\Big\|\frac{\partial f}{\partial x_j}-\frac{\partial f}{\partial x_j}*\rho_m\Big\|_{q,p,\alpha}+\Big\|\frac{\partial f}{\partial x_j}\chi_{\mathbb{R}^d\setminus J_0^m}\Big\|_{q,p,\alpha}+\frac{1}{m}\Big\|\frac{\partial\omega}{\partial x_j}\Big\|_\infty\|f\chi_{J_0^{2m}\setminus J_0^m}\|_{q,p,\alpha}.\end{aligned}$$

Thus

$$\lim_{m\to\infty}\Big\|\frac{\partial f}{\partial x_j}-\frac{\partial}{\partial x_j}((f\omega_m)*\rho_m)\Big\|_{q,p,\alpha}=0. \ \blacksquare$$

Notice that, by the result above, $W^1((L^q,l^p)^\alpha_{c,0})$ is the closure in $W^1((L^q,l^p)^\alpha)$ of $\mathcal{D}$ and therefore of $W^{1,\alpha}$ if $\alpha<\infty$.

4. Boundedness of singular integrals. In [F-L-Y] several results on the boundedness of singular integrals in Morrey spaces were given. In this section we shall establish an analoguous result in $(L^q, l^p)^\alpha$ for Riesz potential operators and deduce from it Sobolev type inequalities.

PROPOSITION 4.1. *Suppose that* $1 < q \leq \alpha < p \leq \infty$, $0 < \gamma < 1/\alpha - 1/p$, $1/q^* = 1/q - \gamma$ *and* $1/\alpha^* = 1/\alpha - \gamma$. *Then, for any* f *in* $(L^q, l^p)^\alpha$, $I_\gamma f$ *belongs to* $(L^{q^*}, l^p)^{\alpha^*}$ *and* $\|I_\gamma f\|_{q^*,p,\alpha^*} \leq C\|f\|_{q,p,\alpha}$ *where* C *is a real number not depending on* f.

Proof. (a) Let $f \in (L^q, l^p)^\alpha$ be non-negative and $(x, r) \in \mathbb{R}^d \times (0, \infty)$. We have

$$f = \sum_{n \geq 0} f_{x,r,n}$$

where

$$f_{x,r,0} = f\chi_{J_x^{2r}}, \quad f_{x,r,n} = f\chi_{T_{x,r,n}} \quad \text{with} \quad T_{x,r,n} = J_x^{2^{n+1}r} \setminus J_x^{2^n r} \quad \text{for } n \geq 1.$$

f being non-negative, the monotone convergence theorem gives

$$I_\gamma f = \sum_{n \geq 0} I_\gamma f_{x,r,n}.$$

By the Hardy–Littlewood–Sobolev theorem for fractional integration there is a real number A not depending on f or r such that

$$\|I_\gamma f_{x,r,0}\|_{q^*} \leq A\|f_{x,r,0}\|_q = A\|f\chi_{J_x^{2r}}\|_q.$$

Therefore

$$\begin{aligned}
\|(I_\gamma f)\chi_{J_x^r}\|_{q^*} &\leq \sum_{n \geq 0} \|(I_\gamma f_{x,r,n})\chi_{J_x^r}\|_{q^*} \\
&\leq A\|f\chi_{J_x^{2r}}\|_q + \sum_{n \geq 1} \bigg[\int\limits_{J_x^r} \bigg(\int\limits_{T_{x,r,n}} \frac{f(y)}{|z-y|^{d(1-\gamma)}}\, dy \bigg)^{q^*} dz \bigg]^{1/q^*}
\end{aligned}$$

Notice that for $n \geq 1$, $z \in J_x^r$ and $y \in J_x^{2^{n+1}r} \setminus J_x^{2^n r}$, we have

$$|z - y| \geq \frac{2^n r}{2} - \frac{r}{2} = \frac{(2^n - 1)r}{2} \geq \frac{2^{n-1} r}{2}.$$

Thus we get

$$\begin{aligned}
\|(I_\gamma f)\chi_{J_x^r}\|_{q^*} &\leq A\|f\chi_{J_x^{2r}}\|_q + \sum_{n \geq 1} \frac{2^{d(1-\gamma)} r^{d/q^*}}{(2^{n-1} r)^{d(1-\gamma)}} \int\limits_{T_{x,r,n}} f(y)\, dy \\
&\leq A\|f\chi_{J_x^{2r}}\|_q + 2^{2d(1-\gamma)} \sum_{n \geq 1} \frac{(2^d - 1)^{1-1/q}}{2^{nd(1/q-\gamma)}} \|f\chi_{J_x^{2^{n+1}r}}\|_q \\
&\leq B_{q,\infty,\alpha}\|f\|_{q,\infty,\alpha} r^{d(1/q-1/\alpha)}
\end{aligned}$$

with

$$B_{q,\infty,\alpha} = \left[A + 2^{2d(1-\gamma)} \sum_{n\geq 1} \frac{(2^d - 1)}{2^{nd(1/\alpha-\gamma)}}\right] 2^{d(1/q-1/\alpha)} < \infty$$

because $1/\alpha - \gamma \geq 1/\alpha - \gamma - 1/p > 0$.

(b) Let $f \in (L^q, l^p)^\alpha$. By Proposition 2.2(f), $f \in (L^q, l^\infty)^\alpha$, that is, $\|f\|_{q,\infty,\alpha} < \infty$.

Since $|f|$ is a non-negative element of $(L^q, l^p)^\alpha$, by the results in (a) we have

$$\|(I_\gamma(|f|)\chi_{J_x^r}\|_q \leq B_{q,\infty,\alpha}\|f\|_{q,\infty,\alpha} r^{d(1/q-1/\alpha)} < \infty, \quad (x,r) \in \mathbb{R}^d \times (0,\infty).$$

This implies that for almost every $z \in \mathbb{R}^d$, $I_\gamma(|f|)(z) < \infty$ and therefore $I_\gamma f(z) = \int_{\mathbb{R}^d} \frac{f(y)}{|z-y|^{d(1-\gamma)}}\,dy$ converges and satisfies $|I_\gamma f(z)| \leq I_\gamma(|f|)(z)$.

Consequently, for any $(x,r) \in \mathbb{R}^d \times (0,\infty)$ we have

$$(\star) \quad \|(I_\gamma f)\chi_{J_x^r}\|_{q^*} \leq A\|f\chi_{J_x^{2r}}\|_q + 2^{2d(1-\gamma)} \sum_{n\geq 1} \frac{(2^d-1)^{1-1/q}}{2^{nd(1/q-\gamma)}} \|f\chi_{J_x^{2^{n+1}r}}\|_q,$$

$$(\star\star) \quad \|(I_\gamma f)\chi_{J_x^r}\|_{q^*} \leq B_{q,\infty,\alpha}\|f\|_{q,\infty,\alpha} r^{d(1/q-1/\alpha)}.$$

Now, $(\star\star)$ ends the proof for $p = \infty$. In the case $p < \infty$, $(\star)$ implies

$$\begin{aligned}\Big(\int_{\mathbb{R}^d} [\|(I_\gamma f)\chi_{J_x^r}\|_{q^*}]^p\,dx\Big)^{1/p} &\leq A\Big(\int_{\mathbb{R}^d} [\|f\chi_{J_x^{2r}}\|_q]^p\,dx\Big)^{1/p} \\ &\quad + 2^{2d(1-\gamma)} \sum_{n\geq 1} \frac{(2^d-1)^{1-1/q}}{2^{nd(1/q-\gamma)}} \Big(\int_{\mathbb{R}^d} [\|f\chi_{J_x^{2^{n+1}r}}\|_q]^p\,dx\Big)^{1/p} \\ &\leq B_{q,p,\alpha} |||f|||_{q,p,\alpha} r^{d(1/q+1/p-1/\alpha)}\end{aligned}$$

with

$$B_{q,p,\alpha} = \left[A + 2^{2d(1-\gamma)} \sum_{n\geq 1} \frac{(2^d-1)^{1-1/q}}{2^{nd(1/\alpha-1/p-\gamma)}}\right] 2^{d(1/q+1/p-1/\alpha)} < \infty$$

because $1/\alpha - 1/p - \gamma > 0$.

Thus, by Proposition 2.2(b),

$$\|I_\gamma f\|_{q^*,p,\alpha^*} \leq C\|f\|_{q,p,\alpha}$$

where C is a real number not depending on f. ■

The proposition above has the following consequence.

Corollary 4.1. *Suppose that* $1 < q \leq \alpha < p \leq \infty$, $0 < \gamma < 1/\alpha - 1/p$, $1/q^* = 1/q - \gamma$ *and* $1/\alpha^* = 1/\alpha - \gamma$. *Then for any* f *in* $(L^q, l^p)^\alpha_{c,0}$, $I_\gamma f$ *belongs to* $(L^{q^*}, l^p)^{\alpha^*}_{c,0}$.

Proof. Let $f \in (L^q, l^p)^\alpha_{c,0}$. There is a sequence $(g_n)_{n\geq 1} \subset L^\alpha$ converging to f in $(L^q, l^p)^\alpha$. By the Hardy–Littlewood–Sobolev inequality and Proposition 4.1, we have

$$I_\gamma g_n \in L^{\alpha^*}, \qquad n \geq 1,$$

and

$$0 = \lim_{n\to\infty} \|I_\gamma(g_n - f)\|_{q^*,p,\alpha^*} = \lim_{n\to\infty} \|I_\gamma g_n - I_\gamma f\|_{q^*,p,\alpha^*}.$$

Therefore $I_\gamma f \in (L^{q^*}, l^p)^{\alpha^*}_{c,0}$. ■

The results above give the following Sobolev inequality.

PROPOSITION 4.2. *Suppose that* $1 < q \leq \alpha < p \leq \infty$, $1/d < 1/\alpha - 1/p$, $1/q^* = 1/q - 1/d$ *and* $1/\alpha^* = 1/\alpha - 1/d$. *Then* $W^1((L^q, l^p)^\alpha_{c,0}) \subset (L^{q^*}, l^p)^{\alpha^*}_{c,0}$ *and there is a real number* C *such that*

$$\|f\|_{q^*,p,\alpha^*} \leq C\| \, |\nabla f| \, \|_{q,p,\alpha}, \qquad f \in W^1((L^q, l^p)^\alpha_{c,0}).$$

Proof. (a) Let $\varphi \in \mathcal{D}$. It is known (see [St]) that

$$|\varphi| \leq A \sum_{j=1}^{d} I_{1/d}\left(\left|\frac{\partial \varphi}{\partial x_j}\right|\right)$$

where A is a real number not depending on φ. Therefore, by Proposition 4.1,

$$\|\varphi\|_{q^*,p,\alpha^*} \leq C \sum_{j=1}^{d} \left\|\frac{\partial \varphi}{\partial x_j}\right\|_{q,p,\alpha}$$

where C is a real number not depending on φ.

(b) Let $f \in W^1((L^q, l^p)^\alpha_{c,0})$. For any integer $m \geq 1$, we set $\varphi_m = (f\omega_m) * \rho_m$ where ρ_m and ω_m are defined as in Notations 2.2. Then $(\varphi_m)_{m\geq 1}$ is a sequence of elements of $\mathcal{D}$ which converges to f in $W^1((L^q, l^p)^\alpha)$ (see Proposition 3.5) and therefore is a Cauchy sequence. Furthermore, by the result in (a) we have

$$\|\varphi_m - \varphi_n\|_{q^*,p,\alpha^*} \leq C \sum_{j=1}^{d} \left\|\frac{\partial \varphi_m}{\partial x_j} - \frac{\partial \varphi_n}{\partial x_j}\right\|_{q,p,\alpha}, \qquad m, n \in \mathbb{N}^*.$$

Thus $(\varphi_m)_{m\geq 1}$ is a Cauchy sequence and therefore converges in $(L^{q^*}, l^p)^{\alpha^*}$ to an element which is nothing other than f. So $f \in (L^{q^*}, l^p)^{\alpha^*}$ and

$$\|f\|_{q^*,p,\alpha^*} = \lim_{m\to\infty} \|\varphi_m\|_{q^*,p,\alpha^*} \leq C \lim_{m\to\infty} \sum_{j=1}^{d} \left\|\frac{\partial \varphi_m}{\partial x_j}\right\|_{q,p,\alpha}$$
$$= C \sum_{j=1}^{d} \left\|\frac{\partial f}{\partial x_j}\right\|_{q,p,\alpha}. \ ■$$

As in the classical case, from the above Sobolev inequality we may deduce a Kondrashov–Rellich compactness theorem in $W^1((L^q, l^p)^\alpha)$. For its proof we shall need the following results.

LEMMA 4.2.

(a) *Suppose that* $1 \le q \le \alpha \le p \le \infty$, $1 \le q^* \le \alpha^* \le p^* \le \infty$, $0 < t < 1$,

$$\frac{1}{\widetilde{q}} = \frac{1-t}{q} + \frac{t}{q^*}, \qquad \frac{1}{\widetilde{\alpha}} = \frac{1-t}{\alpha} + \frac{t}{\alpha^*}, \qquad \frac{1}{\widetilde{p}} = \frac{1-t}{p} + \frac{t}{p^*}.$$

Then there exists a real C *such that*

$$\|f\|_{\widetilde{q},\widetilde{p},\widetilde{\alpha}} \le C\|f\|_{q,p,\alpha}^{1-t}\|f\|_{q^*,p^*,\alpha^*}^{t}, \qquad f \in L^0.$$

(b) *Suppose that* $1 < q \le \alpha < p \le \infty$, $1/q^* = 1/q - 1/d > 0$, $1/\alpha^* = 1/\alpha - 1/d$, $0 < t < 1$,

$$\frac{1}{\widetilde{q}} = \frac{1-t}{q} + \frac{t}{q^*} \quad \textit{and} \quad \frac{1}{\widetilde{\alpha}} = \frac{1-t}{\alpha} + \frac{t}{\alpha^*}.$$

Then there exists a real number C *such that*

$$\|f\|_{\widetilde{q},p,\widetilde{\alpha}} \le C\|f\|_{q,p,\alpha}^{1-t}\| \, |\nabla f| \, \|_{q,p,\alpha}^{t}, \qquad f \in W^1((L^q, l^p)^\alpha_{c,0}).$$

Proof. (a) Let $f \in L^1_{\mathrm{loc}}$.

(i) From the Hölder inequality we obtain, for any $(x, r) \in \mathbb{R}^d \times (0, \infty)$,

$$\|f\chi_{J_x^r}\|_{\widetilde{q}} \le \|f\chi_{J_x^r}\|_q^{1-t}\|f\chi_{J_x^r}\|_{q^*}^{t}$$

and therefore

$$r^{d(1/\widetilde{\alpha}-1/\widetilde{q}-1/\widetilde{p})}\|f\chi_{J_x^r}\|_{\widetilde{q}} \le [r^{d(1/\alpha-1/q-1/p)}\|f\chi_{J_x^r}\|_q]^{1-t}[r^{d(1/\alpha^*-1/q^*-1/p^*)}\|f\chi_{J_x^r}\|_{q^*}]^t.$$

(ii) *First case*: $p = p^* = \infty$. The result in (i) immediately yields

$$\|f\|_{\widetilde{q},\infty,\widetilde{\alpha}} \le \|f\|_{q,\infty,\alpha}^{1-t}\|f\|_{q^*,\infty,\alpha^*}^{t},$$

that is,

$$\|f\|_{\widetilde{q},\widetilde{p},\widetilde{\alpha}} \le \|f\|_{q,p,\alpha}^{1-t}\|f\|_{q^*,p^*,\alpha^*}^{t}.$$

(iii) *Second case*: $p^* < \infty = p$. Using the result obtained in (i) we get

$$r^{d(1/\widetilde{\alpha}-1/\widetilde{q}-1/\widetilde{p})}\|f\chi_{J_x^r}\|_{\widetilde{q}} \le \|f\|_{q,\infty,\alpha}^{1-t}[r^{d(1/\alpha^*-1/q^*-1/p^*)}\|f\chi_{J_x^r}\|_{q^*}]^t, \qquad x \in \mathbb{R}^d,\ r > 0,$$

and therefore, as $\widetilde{p} = p^* t^{-1} < \infty$,

$$r^{d(1/\widetilde{\alpha}-1/\widetilde{q}-1/\widetilde{p})}\Big\{\int_{\mathbb{R}^d} \|f\chi_{J_x^r}\|_{\widetilde{q}}^{\widetilde{p}}\,dx\Big\}^{1/\widetilde{p}} \le \|f\|_{q,\infty,\alpha}^{1-t} r^{d(1/\alpha^*-1/q^*-1/p^*)t}\Big\{\int_{\mathbb{R}^d} \|f\chi_{J_x^r}\|_{q^*}^{t\widetilde{p}}\,dx\Big\}^{1/\widetilde{p}}, \qquad r > 0,$$

that is,

$$r^{d(1/\widetilde{\alpha}-1/\widetilde{q}-1/\widetilde{p})}\Big\{\int_{\mathbb{R}^d}\|f\chi_{J_x^r}\|_{\widetilde{q}}^{\widetilde{p}}\,dx\Big\}^{1/\widetilde{p}} \le \|f\|_{q,\infty,\alpha}^{1-t} r^{d(1/\alpha^*-1/q^*-1/p^*)t}\Big\{\int_{\mathbb{R}^d}\|f\chi_{J_x^r}\|_{q^*}^{p^*}\,dx\Big\}^{t/p^*}, \quad r>0.$$

Taking the supremum with respect to $r>0$, we obtain

$$|||f|||_{\widetilde{q},\widetilde{p},\widetilde{\alpha}} \le \|f\|_{q,p,\alpha}^{1-t}|||f|||_{q^*,p^*,\alpha^*}^{t}.$$

In the case $p<\infty=p^*$, the inequality above is obtained by a similar argument.

(iv) *Third case*: $p<\infty$ and $p^*<\infty$. By the result in (i) and the Hölder inequality we get

$$r^{d(1/\widetilde{\alpha}-1/\widetilde{q}-1/\widetilde{p})}\Big\{\int_{\mathbb{R}^d}\|f\chi_{J_x^r}\|_{\widetilde{q}}^{\widetilde{p}}\,dx\Big\}^{1/\widetilde{p}} \le \Big\{r^{d(1/\alpha-1/q-1/p)p}\int_{\mathbb{R}^d}\|f\chi_{J_x^r}\|_{q}^{p}\,dx\Big\}^{(1-t)/p} \times r^{d(1/\alpha^*-1/q^*-1/p^*)t}\Big\{\int_{\mathbb{R}^d}\|f\chi_{J_x^r}\|_{q^*}^{p^*}\,dx\Big\}^{t/p^*}, \quad r>0,$$

and therefore

$$|||f|||_{\widetilde{q},\widetilde{p},\widetilde{\alpha}} \le |||f|||_{q,p,\alpha}^{1-t}|||f|||_{q^*,p^*,\alpha^*}^{t}.$$

An application of Proposition 2.2(b) ends the proof.

(b) is an immediate consequence of (a) and Proposition 4.2. ∎

Proposition 4.3. *Suppose that* $1<q\le\alpha<p\le\infty$, $1/d<1/\alpha-1/p$, $1/q^*=1/q-1/d$, $1/\alpha^*=1/\alpha-1/d$, $0<t<1$,

$$\frac{1}{\widetilde{q}}=\frac{1-t}{q}+\frac{t}{q^*},\qquad \frac{1}{\widetilde{\alpha}}=\frac{1-t}{\alpha}+\frac{t}{\alpha^*}$$

and H *is a bounded subset of* $W^1((L^q,l^p)_{c,0}^{\alpha})$ *satisfying*

$$\lim_{\rho\to\infty}\sup_{f\in H}\|f-f\chi_{J_0^\rho}\|_{q,p,\alpha}=0.$$

Then H *is a relatively compact subset of* $(L^{\widetilde{q}},l^p)^{\widetilde{\alpha}}$.

Proof. (a) By Lemma 4.2(b) there is a real number C such that for any $f\in H$,

$$\|f\|_{\widetilde{q},p,\widetilde{\alpha}} \le C\|f\|_{q,p,\alpha}^{1-t}\|\,|\nabla f|\,\|_{q,p,\alpha}^{t}.$$

Therefore

$$\sup_{f\in H}\|f\|_{\widetilde{q},p,\widetilde{\alpha}}\leq C\sup_{f\in H}\left[\|f\|_{q,p,\alpha}+\sum_{j=1}^{d}\left\|\frac{\partial f}{\partial x_j}\right\|_{q,p,\alpha}\right]<\infty.$$

Thus H is a bounded subset of $(L^{\widetilde{q}},l^p)^{\widetilde{\alpha}}$.

(b) It is clear that $\tau_u f - f \in W^1((L^q,l^p)^{\alpha}_{c,0})$ for any (u,f) in $\mathbb{R}^d\times H$. Therefore, by Lemma 4.2(a), Proposition 3.2, Proposition 4.2 and Proposition 2.4, there are $C_1,C_2,C_3,C_4>0$ such that, for any $(u,f)\in\mathbb{R}^d\times H$,

$$\begin{aligned}\|\tau_u f-f\|_{\widetilde{q},p,\widetilde{\alpha}}&\leq C_1\|\tau_u f-f\|^{1-t}_{q,p,\alpha}\|\tau_u f-f\|^{t}_{q^*,p,\alpha^*}\\&\leq C_2|u|^{1-t}\|\,|\nabla f|\,\|^{1-t}_{q,p,\alpha}\|\,|\nabla(\tau_u f-f)|\,\|^{t}_{q,p,\alpha}\\&\leq C_3|u|^{1-t}\|\,|\nabla f|\,\|_{q,p,\alpha}\\&\leq C_4|u|^{1-t}\left[\|f\|_{q,p,\alpha}+\sum_{j=1}^{d}\left\|\frac{\partial f}{\partial x_j}\right\|_{q,p,\alpha}\right].\end{aligned}$$

Thus

$$\sup_{f\in H}\|\tau_u f-f\|_{\widetilde{q},p,\widetilde{\alpha}}\leq C_4|u|^{1-t}\sup_{f\in H}\left[\|f\|_{q,p,\alpha}+\sum_{j=1}^{d}\left\|\frac{\partial f}{\partial x_j}\right\|_{q,p,\alpha}\right]$$

and

$$\lim_{u\to 0}\sup_{f\in H}\|\tau_u f-f\|_{\widetilde{q},p,\widetilde{\alpha}}=0.$$

(c) Let $\theta\in\mathcal{C}^{\infty}$ satisfy $\chi_{\mathbb{R}^d\setminus J_0^1}\leq\theta\leq\chi_{\mathbb{R}^d\setminus J_0^{1/2}}$ and

$$\theta_R(x)=\theta(x/R),\qquad x\in\mathbb{R}^d,\ R>0.$$

It is clear that, for any $(f,R)\in H\times(0,\infty)$,

$$|f\chi_{\mathbb{R}^d\setminus J_0^R}|\leq|f\theta_R|\leq|f\chi_{\mathbb{R}^d\setminus J_0^{R/2}}|$$

and therefore, by Lemma 4.2(a) and Proposition 4.2, there are $C_1,C_2,C_3>0$ not depending on (f,R) such that

$$\begin{aligned}\|f\chi_{\mathbb{R}^d\setminus J_0^R}\|_{\widetilde{q},p,\widetilde{\alpha}}&\leq C_1\|f\chi_{\mathbb{R}^d\setminus J_0^{R/2}}\|^{1-t}_{q,p,\alpha}\|f\theta_R\|^{t}_{q^*,p,\alpha^*}\\&\leq C_1\|f\chi_{\mathbb{R}^d\setminus J_0^{R/2}}\|^{1-t}_{q,p,\alpha}\|f\|^{t}_{q^*,p,\alpha^*}\\&\leq C_2\|f\chi_{\mathbb{R}^d\setminus J_0^{R/2}}\|^{1-t}_{q,p,\alpha}\|\,|\nabla f|\,\|^{t}_{q,p,\alpha}.\\&\leq C_3\|f\chi_{\mathbb{R}^d\setminus J_0^{R/2}}\|^{1-t}_{q,p,\alpha}\left[\|f\|_{q,p,\alpha}+\sum_{j=1}^{d}\left\|\frac{\partial f}{\partial x_j}\right\|_{q,p,\alpha}\right]^{t}.\end{aligned}$$

Thus,

$$\sup_{f\in H}\|f\chi_{\mathbb{R}^d\setminus J_0^R}\|_{\widetilde{q},p,\widetilde{\alpha}} \le C_3 \sup_{f\in H}\|f\chi_{\mathbb{R}^d\setminus J_0^{R/2}}\|_{q,p,\alpha}^{1-t} \sup_{f\in H}\bigg[\|f\|_{q,p,\alpha}+\sum_{j=1}^d\bigg\|\frac{\partial f}{\partial x_j}\bigg\|_{q,p,\alpha}\bigg]^t$$

and

$$\lim_{R\to\infty}\sup_{f\in H}\|f\chi_{\mathbb{R}^d\setminus J_0^R}\|_{\widetilde{q},p,\widetilde{\alpha}} = 0.$$

An application of Proposition 2.7 ends the proof. ∎

In the case where $q=\alpha$, the proposition above is read as follows.

PROPOSITION 4.4. *Suppose that* $1<\alpha<\infty$, $1/\alpha^*=1/\alpha-1/d$, $0<t<1$, $1/\widetilde{\alpha}=(1-t)/\alpha+t/\alpha^*$ *and* H *is a bounded subset of* $W^{1,\alpha}$ *satisfying*

$$\lim_{\rho\to\infty}\sup_{f\in H}\|f-f\chi_{J_0^\rho}\|_\alpha = 0.$$

Then H *is a relatively compact subset of* $L^{\widetilde{\alpha}}$.

This result improves on Theorem 10 of [H-H] because it does not use the hypothesis $\lim_{R\to\infty}\sup_{f\in H}\|\,|\nabla f|\chi_{\mathbb{R}^d\setminus J_0^R}\|_\alpha=0$.

Proposition 4.1 has the following generalization.

PROPOSITION 4.5. *Suppose that*

- $1\le q\le\alpha<p\le\infty$, $0\le\gamma<1/\alpha-1/p$, $1/q^*=1/q-\gamma$, $1/\alpha^*=1/\alpha-\gamma$.
- T *is a bounded linear map of* L^q *into* L^{q^*} *such that, for any* f *in* L^q *with compact support* K *and any* x *in* $\mathbb{R}^d\setminus K$,

$$|Tf(x)|\le A\int_{\mathbb{R}^d}\frac{|f(y)|}{|x-y|^{d(1-\gamma)}}\,dy$$

where A *is a real number not depending on* f *and* x.

Then T *admits a unique bounded linear extension defined on* $(L^q,l^p)^\alpha_{c,0}$.

Proof. (a) Let $f\in L^q\cap L^\alpha$. Using the notations in the proof of Proposition 4.1, for any (x,r) in $\mathbb{R}^d\times(0,\infty)$ we have

$$f=\sum_{n\ge0}f_{x,r,n}\qquad\text{in } L^q,$$

$$Tf=\sum_{n\ge0}Tf_{x,r,n}\qquad\text{in } L^{q^*};$$

furthermore,

$$\|Tf_{x,r,n}\|_{q^*} \le A\Big\{\int_{J_x^r}\Big[\int_{T_{x,r,n}} \frac{|f(y)|}{|z-y|^{d(1-\gamma)}}\,dy\Big]^{q^*}dz\Big\}, \qquad n \ge 1,$$

$$\|(Tf_{x,r,0})\chi_{J_x^r}\|_{q^*} \le B\|f\chi_{J_x^{2r}}\|_q$$

where B is a real number not depending on (f,x,r). An argument similar to the proof of Proposition 4.1 leads easily to

$$\|Tf\|_{q^*,p;\alpha^*} \le C\|f\|_{q,p;\alpha}$$

where C is a real number not depending on f.

(b) Notice that $(L^q,l^p)^\alpha_{c,0}$ is the closure of $L^q \cap L^\alpha$ in $(L^q,l^p)^\alpha$. Therefore the result follows from (a). ■

Remark 4.3. Let $\mathcal{S}$ denote the Schwartz space of test functions on $\mathbb{R}^d$ and let $j \in \{1,\dots,d\}$. It is well known (see [Gr]) that the Riesz transform R_j defined by

$$R_jf(x) = \lim_{\varepsilon\to 0^+} \frac{\Gamma\big(\frac{d+1}{2}\big)}{\pi^{(d+1)/2}} \int_{|x-y|\ge\varepsilon} f(y)\frac{x_j-y_j}{|x-y|^{d+1}}\,dy,$$

$$x=(x_1,\dots,x_d)\in\mathbb{R}^d,\ f\in\mathcal{S}.$$

extends to a bounded linear operator on L^q for $1<q<\infty$. Furthermore, for any f in L^q with compact support K and any x in $\mathbb{R}^d \setminus K$ we have

$$|R_jf(x)| \le \frac{\Gamma\big(\frac{d+1}{2}\big)}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^d}\,dy.$$

Therefore, as a particular case of Proposition 4.5, we have the following result.

Corollary 4.4. *Suppose $1<q\le\alpha<p\le\infty$. Then the Riesz transforms R_j ($j\in\{1,\dots,d\}$) extend to bounded linear operators on $(L^q,l^p)^\alpha_{c,0}$.*

5. Application. We suppose $d\ge 3$.

(a) Let $\varphi\in\mathcal{D}$. The boundedness properties of the operators $I_{1/d}$ and R_j ($j\in\{1,\dots,d\}$) yield

$$\phi_j = R_j[I_{1/d}(\varphi)] \in \bigcap_{s>d/(d-1)} L^s, \qquad j\in\{1,\dots,d\}.$$

As $2>d/(d-1)$, we can use the Fourier transform to obtain $c_d\sum_{j=1}^d \partial\phi_j/\partial x_j = \varphi$ where c_d is a real number depending only on d (for a similar formula see [St, p. 125].

(b) Let $1 < q \leq \alpha < p \leq \infty$ with $1/p < 1/\alpha - 1/d$ and let $f \in (L^q, l^p)^{\alpha}_{c,0}$. We are interested in the equation

$$(E_f) \qquad \operatorname{div} F = f.$$

Fix an integer $n \geq 1$ and put $f_n = \rho_n * (\omega_n f)$ where ρ_n and ω_n are as in Notations 2.2. As $f_n \in \mathcal{D}$, the result in (a) implies that the equation

$$(E_{f_n}) \qquad \operatorname{div} F = f_n$$

admits a solution $F_n = (F_{n_j})_{1\leq j\leq d}$ with

$$F_{n_j} = c_d R_j[I_{1/d} f_n] \in \bigcap_{s>d/(d-1)} L^s, \qquad j \in \{1, \dots, d\}.$$

Using Proposition 2.3, Proposition 4.1, Corollary 4.1 and Corollary 4.4, we find that

- $(f_n)_{n\geq1}$ converges to f in $(L^q, l^p)^{\alpha}_{c,0}$,
- for any $j \in \{1, \dots, d\}$, $(F_{nj})_{n\geq1}$ converges to $F_j = c_d R_j[I_{1/d} f]$ in $(L^{q^*}, l^p)^{\alpha^*}_{c,0}$,

with $1/q^* = 1/q - 1/d$ and $1/\alpha^* = 1/\alpha - 1/d$.

Therefore, for any φ in $\mathcal{D}$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \operatorname{div} F(x)\varphi(x)\,dx &= -\sum_{j=1}^{d} \int_{\mathbb{R}^d} F_j(x) \frac{\partial\varphi}{\partial x_j}(x)\,dx \\
&= \lim_{n\to\infty} \left[-\sum_{j=1}^{d} \int_{\mathbb{R}^d} F_{nj}(x) \frac{\partial\varphi}{\partial x_j}(x)\,dx \right] = \lim_{n\to\infty} \int_{\mathbb{R}^d} \left(\sum_{j=1}^{d} \frac{\partial F_{nj}}{\partial x_j}(x) \right) \varphi(x)\,dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\varphi(x)\,dx = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx,
\end{aligned}$$

that is, equation (E_f) admits the solution $F = (F_j)_{1\leq j\leq d}$ in $[(L^{q^*}, l^p)^{\alpha^*}_{c,0}]^d$.

It is worth noting the link between the above result and Proposition 1.1 in the light of Proposition 1.2.

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