ANNALES
POLONICI MATHEMATICI
108.2 (2013)

On global regular solutions to the Navier–Stokes equations with heat convection

by Piotr Kacprzyk (Warszawa)

Abstract. Global existence of regular solutions to the Navier–Stokes equations for velocity and pressure coupled with the heat convection equation for temperature in a cylindrical pipe is shown. We assume the slip boundary conditions for velocity and the Neumann condition for temperature. First we prove long time existence of regular solutions in $[kT,(k+1)T]$. Having T sufficiently large and imposing some decay estimates on $\|f(t)\|_{L_2(\Omega)}$, $\|f_{,x_3}(t)\|_{L_2(\Omega)}$ we continue the local solutions step by step up to a global one.

1. Introduction. We consider the problem

$$
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = \alpha(\theta) f && \text{in } \Omega\times\mathbb{R}_+,\\
&\operatorname{div} v = 0 && \text{in } \Omega\times\mathbb{R}_+,\\
&\theta_{,t} + v\cdot\nabla\theta - \chi\Delta\theta = 0 && \text{in } \Omega\times\mathbb{R}_+,\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S\times\mathbb{R}_+,\\
&v\cdot\bar n = 0,\quad \bar n\cdot\nabla\theta = 0 && \text{on } S\times\mathbb{R}_+,\\
&v|_{t=0} = v_0,\quad \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\alpha\in C^2(\mathbb{R})$, $\Omega\subset\mathbb{R}^3$ is a bounded domain, $S=\partial\Omega$, $v=(v_1(x,t), v_2(x,t), v_3(x,t))\in\mathbb{R}^3$ is the velocity of the fluid, $\theta=\theta(x,t)\in\mathbb{R}$ is the temperature, $p=p(x,t)\in\mathbb{R}$ is the pressure, $f=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ the external force, $\nu>0$ is the constant viscosity coefficient, $\chi>0$ is the constant heat coefficient. We introduce the Cartesian system $x=(x_1,x_2,x_3)$ such that the cylinder Ω is parallel to the x_3 axis. We assume that $S = S_1\cup S_2$, where S_1 is the part of the boundary which is parallel to the x_3 axis and S_2 is perpendicular to x_3. Hence

2010 *Mathematics Subject Classification*: Primary 76D03; Secondary 76D05, 35Q30.
Key words and phrases: Navier–Stokes equations, existence of regular solutions, global existence, slip boundary conditions.

DOI: 10.4064/ap108-2-3

$$S_1 = \{x \in \mathbb{R}^3 : \varphi(x_1, x_2) = c_0,\ -a \le x_3 \le a\},$$
$$S_2(-a) = \{x \in \mathbb{R}^3 : \varphi(x_1, x_2) < c_0,\ x_3 = -a\},$$
$$S_2(a) = \{x \in \mathbb{R}^3 : \varphi(x_1, x_2) < c_0,\ x_3 = a\},$$

where a, c_0 are given positive numbers and $\varphi(x_1, x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$. Next $\bar{n}$ is the unit outward vector normal to S and $\bar{\tau}_\alpha$, $\alpha = 1, 2$, are tangent vectors to S.

By $\mathbb{T}(v, p)$ we denote the stress tensor

$$\mathbb{T}(v, p) = \nu \mathbb{D}(v) - pI,$$

where I is the unit matrix and

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}$$

is the dilatation tensor.

The aim of this paper is to prove the existence of global regular solutions to problem (1.1) without smallness restrictions on the initial velocity and the initial temperature. Since (1.1) is a coupling of the Navier–Stokes equations with the heat equation the aim cannot be achieved without any restrictions, because the regularity problem for the Navier–Stokes equations is up to now an open problem. However, there are already some results on regularity of weak solutions to the Navier–Stokes equations (see [7, 13, 14, 15, 16]). These results describe solutions which are close either to two-dimensional (see [7, 13]) or to axially-symmetric (see [14, 15, 16]) solutions. In the first case the L_2 norm of the derivative of the initial velocity with respect to x_3 must be sufficiently small. The existence of global regular solutions which are either two-dimensional or axially symmetric was proved in [5]. Looking for solutions which are close to two-dimensional solutions requires that the domain Ω be a cylinder.

Therefore to prove the existence of global regular solutions to the Navier–Stokes equations which are close to 2d-solutions we need analytical ($\|v_{0,x_3}\|_{L_2(\Omega)}$ to be small) and geometrical (Ω a cylinder) restrictions.

In this paper we generalize results on the Navier–Stokes equations to the system (1.1) (see also [8, 9]). Moreover, the results from [8, 9] are extended because the global existence is proved.

The problem of existence of global regular solutions to the Navier–Stokes equations with slip boundary conditions which are close to two-dimensional solutions has a long history. In the first paper [12] in this direction, long time existence of regular solutions to the Navier–Stokes equations was proved by using a complicated technique of Besov spaces. A simplified and more elegant revision of the proof from [12] was given in [7], where Sobolev spaces were used only.

In [6] the long time solution from [7] was prolonged in time up to infinity.

Finally in [13] the global existence of regular solutions to the Navier–Stokes equations was proved step by step in time up to infinity.

Problem (1.1) is a nontrivial generalization of the Navier–Stokes equations. The proof of the existence of a global regular solution to problem (1.1) is divided into two steps. In the first step we consider problem (1.1) in the interval $[kT,(k+1)T]$, where $k\in\mathbb{N}\cup\{0\}$ and $T>0$ is a given number. Hence problem (1.1) takes the form

$$
\begin{aligned}
&v_{,t}+v\cdot\nabla v-\operatorname{div}\mathbb{T}(v,p)=\alpha(\theta)f && \text{in } \Omega^{T(k+1)}=\Omega\times(kT,(k+1)T),\\
&\operatorname{div} v=0 && \text{in } \Omega^{T(k+1)},\\
&\theta_{,t}+v\cdot\nabla\theta-\chi\Delta\theta=0 && \text{in } \Omega^{T(k+1)},\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha=0,\quad \alpha=1,2, && \text{on } S^{T(k+1)}=S\times(kT,(k+1)T),\\
&v\cdot\bar n=0,\quad \bar n\cdot\nabla\theta=0 && \text{on } S^{T(k+1)},\\
&v|_{t=kT}=v(kT),\quad \theta|_{t=kT}=\theta(kT) && \text{in } \Omega,
\end{aligned}
\tag{1.2}
$$

where $v(kT)$, $\theta(kT)$ are treated as given.

Let us introduce the quantities

$$
\begin{aligned}
G_1(k,T)&=\|f\|_{L_\infty(kT,(k+1)T;H^1(\Omega))}+\|v(kT)\|_{H^1(\Omega)}\\
&\quad+\|\theta(kT)\|_{H^1(\Omega)}+\|v_{,x_3}(kT)\|_{H^1(\Omega)}+\|\theta_{,x_3}(kT)\|_{H^1(\Omega)},\\
\eta(k,T)&=\|f_{,x_3}\|_{L_2(\Omega\times(kT,(k+1)T))}+\|f_3\|_{L_2(S_2\times(kT,(k+1)T))}\\
&\quad+\|v_{,x_3}(kT)\|_{L_2(\Omega)}+\|\theta_{,x_3}(kT)\|_{L_2(\Omega)}.
\end{aligned}
$$

Then the following local regularity result holds.

Theorem 1.1. *Assume that G_1 is finite and η is sufficiently small. Then there exists a solution (v,θ,p) to problem* (1.2) *such that*

$$
\begin{aligned}
&v,v_{,x_3},\theta,\theta_{,x_3}\in W_2^{2,1}(\Omega\times(kT,(k+1)T)),\\
&\nabla p,\nabla p_{,x_3}\in L_2(\Omega\times(kT,(k+1)T))
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=0}^{1}\big(\|\partial_{x_3}^i v\|_{W_2^{2,1}(\Omega\times(kT,(k+1)T))}+\|\partial_{x_3}^i\theta\|_{W_2^{2,1}(\Omega\times(kT,(k+1)T))}&\\
+\|\nabla\partial_{x_3}^i p\|_{L_2(\Omega\times(kT,(k+1)T))}\big)\le\beta(G_1(k,T))\equiv A(k,T),&
\end{aligned}
\tag{1.3}
$$

where β is an increasing positive function.

From the form of G_1 we see that $A(k,T)$ does not increase with T.

The proof of Theorem 1.1 is divided into the following stages. First the a priori estimate (1.3) is shown by applying the energy method, estimates for the Stokes system (2.2) and the parabolic problem (2.4), using smallness of the quantity η. This smallness does not imply smallness of the initial data

$v(kT)$, $\theta(kT)$ but only smallness of its derivatives with respect to the variable along the axis of the cylinder in the L_2-norm. The estimate is proved in Section 4. The final result is formulated in Lemma 4.8. Having the a priori estimate the existence is proved by using the Leray–Schauder fixed point theorem.

In the second step we prove

MAIN THEOREM. *Assume that*

$$\begin{aligned} &\|f(t)\|_{H^1(\Omega)} \le e^{-\delta t}\|f(0)\|_{H^1(\Omega)},\\ &\|f_{,x_3}(t)\|_{L_2(\Omega)} \le e^{-\delta t}\|f_{,x_3}(0)\|_{L_2(\Omega)},\\ &f_3|_{S_2} = 0, \qquad \delta > 0. \end{aligned} \tag{1.4}$$

Then

$$\|\alpha((k+1)T)\|_{H^1(\Omega)} \le \|\alpha(kT)\|_{H^1(\Omega)}, \tag{1.5}$$

where α replaces v, θ, $v_{,x_3}$, $\theta_{,x_3}$, for any $k \in \mathbb{N}\cup\{0\}$. Moreover, Theorem 1.1 *implies global existence of solutions to problem* (1.1).

Estimates (1.5) are proved in Section 6. The crucial elements of the proof are decays (1.4) and the fact that the quantity $A(k,T)$ does not increase with T. The last property explicitly appears in the proof of Lemma 4.8.

Now we justify the dividing of the proof of global existence of regular solutions to (1.1) into two steps: Theorem 1.1 and Main Theorem. Setting $k=0$ in Theorem 1.1 we actually have global existence because the quantity $A(k,T)$ does not increase with T. Then estimate (1.3) implies strong decays of v, θ and p in time. To omit this restriction we need the Main Theorem.

2. Notation and auxiliary results. To simplify considerations we introduce the following notation:

$$\begin{aligned}
&|u|_{p,Q} = \|u\|_{L_p(Q)}, && Q\in\{\Omega^T, S^T, \Omega, S\},\ p\in[1,\infty],\\
&\|u\|_{s,Q} = \|u\|_{H^s(Q)}, && Q\in\{\Omega, S\},\ s\in\mathbb{R}_+\cup\{0\},\\
&\|u\|_{s,Q^T} = \|u\|_{W_2^{s,s/2}(Q^T)}, && Q\in\{\Omega, S\},\ s\in\mathbb{R}_+\cup\{0\},\\
&|u|_{p,q,Q^T} = \|u\|_{L_q(0,T;L_p(Q))}, && Q\in\{\Omega, S\},\ p,q\in[1,\infty],\\
&\|u\|_{s,q,Q^T} = \|u\|_{W_q^{s,s/2}(Q^T)}, && Q\in\{\Omega, S\},\ s\in\mathbb{R}_+\cup\{0\},\ q\in[1,\infty],\\
&\|u\|_{s,q,Q} = \|u\|_{W_q^{s}(Q)}, && Q\in\{\Omega, S\},\ s\in\mathbb{R}_+\cup\{0\},\ q\in[1,\infty].
\end{aligned}$$

We denote by c a generic constant which changes its magnitude from formula to formula. By $\bar{c}(\sigma)$, $\beta(\sigma)$ we denote generic functions which are always positive and increasing. Finally, we do not distinguish scalar and vector-valued functions in notation.

We denote

$$\Omega^{Tk} = \Omega \times ((k-1)T, kT), \qquad \Omega^{T_2,T_1} = \Omega \times (T_1, T_2).$$

We introduce the quantities

$$(2.1) \qquad \begin{aligned} &h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3}, \quad \varphi = \theta_{,x_3}, \\ &w = v_3, \quad \chi = v_{2,x_1} - v_{1,x_2}. \end{aligned}$$

Moreover, we introduce the space

$$V_2^k(\Omega^T) = \Big\{ u : \|u\|_{V_2^k(\Omega^T)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u\|_{H^k(\Omega)} + \Big(\int_0^T \|\nabla u(t)\|^2_{H^k(\Omega)}\, dt \Big)^{1/2} < \infty \Big\}, \qquad k \in \mathbb{N} \cup \{0\}.$$

Let us consider the Stokes problem

$$(2.2) \qquad \begin{aligned} &v_{,t} - \operatorname{div} \mathbb{T}(v,p) = f && \text{in } \Omega^T, \\ &\operatorname{div} v = 0 && \text{in } \Omega^T, \\ &\bar n \cdot \mathbb{D}(v) \cdot \bar\tau_\alpha = g_\alpha, \quad \alpha = 1,2, && \text{on } S^T, \\ &v \cdot \bar n = d && \text{on } S^T, \\ &v|_{t=0} = v_0 && \text{in } \Omega. \end{aligned}$$

THEOREM 2.1 (see [1]). *Let $f \in L_q(\Omega^T)$, $d \in W_q^{2-1/q,1-1/2q}(S^T)$, $v_0 \in W_q^{2-2/q}(\Omega)$, $g_\alpha \in W_q^{1-1/q,1/2-1/2q}(S^T)$, $\alpha = 1,2$, $q \in (1,\infty)$. Assume the compatibility conditions $\bar n \cdot \mathbb{D}(v_0) \cdot \bar\tau_\alpha|_S = g_\alpha|_{t=0}$, $\alpha = 1,2$, in $W_q^{1-3/q}(S)$ for $q > 3$ and $\bar n_0 \cdot v_0|_S = d|_{t=0}$ in $W_q^{2-3/q}(S)$ for $q > 3/2$. Then there exists a unique solution (v,p) to problem* (2.2) *such that $v \in W_q^{2,1}(\Omega^T)$, $\nabla p \in L_q(\Omega^T)$ and*

$$(2.3) \qquad \|v\|_{W_q^{2,1}(\Omega^T)} + \|\nabla p\|_{L_q(\Omega^T)} \le c\Big(\|f\|_{L_q(\Omega^T)} + \|v_0\|_{W_q^{2-2/q}(\Omega)} + \|d\|_{W_q^{2-1/q,1-1/2q}(S^T)} + \sum_{\alpha=1}^2 \|g_\alpha\|_{W_q^{1-1/q,1/2-1/2q}(S^T)} \Big).$$

Next we consider the following problem:

$$(2.4) \qquad \begin{aligned} &\theta_{,t} - \Delta\theta = f && \text{in } \Omega^T, \\ &\bar n \cdot \nabla\theta = d && \text{on } S^T, \\ &\theta|_{t=0} = \theta_0 && \text{in } \Omega. \end{aligned}$$

THEOREM 2.2 (see [4, Ch. 4]). *Let $f \in L_q(\Omega^T)$, $\theta_0 \in W_q^{2-2/q}(\Omega)$, $d \in W_q^{1-1/q,1/2-1/2q}(S^T)$, $q \in (1,\infty)$. Then there exists a unique solution θ to* *problem* (2.4) *such that* $\theta \in W_q^{2,1}(\Omega^T)$ *and*

$$\|\theta\|_{W_q^{2,1}(\Omega^T)} \le c(\|f\|_{L_q(\Omega^T)} + \|\theta_0\|_{W_q^{2-2/q}(\Omega)} + \|d\|_{W_q^{1-1/q,1/2-1/2q}(S^T)}). \tag{2.5}$$

REMARK. In view of the considerations in [13] we know that the constants c in (2.3) and (2.5) do not depend on T.

THEOREM 2.3 (Korn inequality, see [14]). *Assume that* $\Omega \subset \mathbb{R}^n$ *is not invariant with respect to any rotation. Assume that*

$$\|\mathbb{D}(u)\|_{L_2(\Omega)} < \infty, \quad u \cdot \bar{n}|_S = 0, \quad \operatorname{div} u = 0. \tag{2.6}$$

Then $\|u\|_{H^1(\Omega)} \le c\|\mathbb{D}(u)\|_{L_2(\Omega)}$.

We now show estimates for the temperature. Applying the classical De Giorgi methods (see also [4]) to problem (2.4) we get

LEMMA 2.4. *Assume that* $\theta(0) \ge c_1 > 0$. *Then solutions to* (2.4) *satisfy*

$$\theta(t) \ge c_1, \quad t \ge 0. \tag{2.7}$$

Proof. Let $(\theta - c_1)_- = \min\{0, \theta - c_1\}$. Multiplying $(1.1)_3$ by $(\theta - c_1)_-$ and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\theta - c_1)_-^2\, dx + \chi \int_\Omega |\nabla(\theta - c_1)_-|^2\, dx = 0.$$

Integrating with respect to time we have

$$\|(\theta - c_1)_-\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\nabla(\theta - c_1)_-\|^2_{L_2(\Omega^T)} \le \|(\theta - c_1)_-(0)\|^2_{L_2(\Omega)}.$$

Since $(\theta - c_1)_-(0) = 0$ we conclude the proof. ■

LEMMA 2.5. *Assume that* $\theta_0 \in L_2(\Omega)$. *Then solutions to problem* (2.4) *satisfy*

$$\|\theta\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\theta\|^2_{L_2(\Omega^T)} \le \|\theta_0\|^2_{L_2}. \tag{2.8}$$

Proof. Multiplying $(1.1)_3$ by θ and integrating over Ω using $(1.1)_{2,4,5}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta^2\, dx + \chi \int_\Omega |\nabla\theta|^2\, dx = 0. \tag{2.9}$$

Finally integrating with respect to time we obtain (2.8). ■

LEMMA 2.6. *Assume that* $\theta(0) \le c_2$. *Then solutions to problem* (2.4) *satisfy*

$$\theta(t) \le c_2, \quad t \ge 0. \tag{2.10}$$

Proof. Let $(\theta - c_2)_+ = \max\{0, \theta - c_2\}$. Multiplying $(1.1)_3$ by $(\theta - c_2)_+$ and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\theta - c_2)_+^2\, dx + \chi \int_\Omega |\nabla(\theta - c_2)_+|^2\, dx = 0.$$

Integrating with respect to time we have

$$\|(\theta - c_2)_+\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\nabla(\theta - c_2)_+\|^2_{L_2(\Omega^T)} \le \|(\theta - c_2)_+(0)\|^2_{L_2(\Omega)}.$$

Since $(\theta - c_2)_+(0) = 0$ we conclude the proof. ∎

LEMMA 2.7. *Assume that* $f \in L_2(\mathbb{R}_+, L_{6/5}(\Omega))$, $v(0) \in L_2(\Omega)$ *and* $c_1 \le \theta(t) \le c_2$, $\alpha \in C^0(\mathbb{R})$. *Then*

$$\|v(t)\|_{L_2(\Omega)} \le c(\|f\|_{L_2(\mathbb{R}_+,L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \tag{2.11}$$

for any $t > 0$. *Next*

$$\begin{aligned} \|v\|_{V_2^0(\Omega\times(kT,t))} &\le c(\|f\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|v(kT)\|_{L_2(\Omega)}) \\ &\le c(\|f\|_{L_2(\mathbb{R}_+;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \equiv l_1, \quad t \in (kT,(k+1)T). \end{aligned} \tag{2.12}$$

Proof. Multiplying $(1.1)_1$ by v and integrating over Ω using the boundary conditions, the Korn inequality, Lemmas 2.4, 2.6 and the fact that $\alpha \in C^0(\mathbb{R})$ we obtain

$$\frac{d}{dt}\|v\|^2_{L_2(\Omega)} + \|v\|^2_{H^1(\Omega)} \le c(\varepsilon\|v\|^2_{L_6(\Omega)} + \bar{c}(1/\varepsilon)\|f\|^2_{6/5,\Omega}). \tag{2.13}$$

Integrating (2.13) with respect to time we obtain (2.11). Finally integrating (2.13) with respect to time from kT to $t \in (kT,(k+1)T)$ yields (2.12). ∎

3. Basic formulations. To prove the existence of global solutions to problem (1.1) we follow [10, 13]. Therefore we need problems for quantities (2.1). First we have

LEMMA 3.1 (see [10]). *The quantities* h, q *are solutions to the problem*

$$\begin{aligned} &h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + \alpha_{,\theta}\varphi f + \alpha g && \text{in } \Omega^{T(k+1)},\\ &\operatorname{div} h = 0 && \text{in } \Omega^{T(k+1)},\\ &\bar{n}\cdot h = 0 && \text{on } S_1^{T(k+1)},\\ &\bar{n}\cdot\mathbb{D}(h)\cdot\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^{T(k+1)},\\ &h_i = 0, \quad i = 1,2, && \text{on } S_2^{T(k+1)},\\ &h_{3,x_3} = 0 && \text{on } S_2^{T(k+1)},\\ &h|_{t=kT} = h(kT) && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where v *and* θ *are treated as given.*

LEMMA 3.2 (see [10]). *The function* $\chi = v_{2,x_1} - v_{1,x_2}$ *is a solution to the problem*

$$
(3.2)\qquad
\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi = F_3 && \textit{in } \Omega^{T(k+1)},\\
&\chi = -v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j)\\
&\qquad + v\cdot\bar\tau_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \textit{on } S_1^{T(k+1)},\\
&\chi_{,x_3} = 0 && \textit{on } S_2^{T(k+1)},\\
&\chi|_{t=kT} = \chi(kT) && \textit{in } \Omega,
\end{aligned}
$$

where v, h, w are assumed to be given and

$$
\bar n|_{S_1} = \frac{(\varphi_{,x_1},\varphi_{,x_2},0)}{\sqrt{\varphi_{,x_1}^2+\varphi_{,x_2}^2}},\qquad \bar\tau_1|_{S_1} = \frac{(-\varphi_{,x_2},\varphi_{,x_1},0)}{\sqrt{\varphi_{,x_1}^2+\varphi_{,x_2}^2}},\qquad \bar\tau_2|_{S_1} = (0,0,1)\equiv\bar e_3,
$$

$$
\bar n|_{S_2} = \bar e_3,\qquad \bar\tau_1|_{S_2} = \bar e_1,\qquad \bar\tau_2|_{S_2} = \bar e_2,
$$

where $w = v_3$, $\bar e_1 = (1,0,0)$, $\bar e_2 = (0,1,0)$ and $F_3 = (f_{2,x_1} - f_{1,x_2})\alpha + \alpha_{,\theta}(\theta_{,x_1}f_2 - \theta_{,x_2}f_1)$.

Finally differentiating $(1.1)_3$ with respect to x_3 we get

$$
(3.3)\qquad
\begin{aligned}
&\varphi_{,t} + v\cdot\nabla\varphi + h\cdot\nabla\theta - \chi\Delta\varphi = 0 && \text{in } \Omega^{T(k+1)},\\
&\bar n\cdot\nabla\varphi = 0 && \text{on } S_1^{T(k+1)},\\
&\varphi = 0 && \text{on } S_2^{T(k+1)},\\
&\varphi|_{t=kT} = \varphi(kT) && \text{in } \Omega,
\end{aligned}
$$

where v, h, θ are treated as given.

4. Estimates. First we examine problem (3.2). The aim is to obtain an energy type estimate for solutions to (3.2). Since (3.2) has non-homogeneous Dirichlet boundary condition such approach is not possible. To make it possible we introduce a function $\tilde\chi$ as a solution of the problem

$$
(4.1)\qquad
\begin{aligned}
&\tilde\chi_{,t} - \nu\Delta\tilde\chi = 0 && \text{in } \Omega^{T(k+1)},\\
&\tilde\chi = \chi_* && \text{on } S_1^{T(k+1)},\\
&\tilde\chi_{,x_3} = 0 && \text{on } S_2^{T(k+1)},\\
&\tilde\chi|_{t=kT} = kT && \text{in } \Omega.
\end{aligned}
$$

Introducing the new function $\chi' = \chi - \tilde\chi$ we see that it is a solution to the problem

$$
(4.2)\qquad
\begin{aligned}
&\chi'_{,t} + v\cdot\nabla\chi' - h_3(v_{2,x_1} - v_{1,x_2}) + h_2 w_{,x_1} - h_1 w_{,x_2}\\
&\qquad - \nu\Delta\chi' = F_3 - v\cdot\nabla\tilde\chi && \text{in } \Omega^{T(k+1)},\\
&\chi' = 0 && \text{on } S_1^{T(k+1)},\\
&\chi'_{,x_3} = 0 && \text{on } S_2^{T(k+1)},\\
&\chi'|_{t=kT} = \chi(kT) && \text{in } \Omega.
\end{aligned}
$$

LEMMA 4.1. *Assume that* $h \in L_5(\Omega^{T(k+1)})$, $F_3 \in L_{10/7}(\Omega^{T(k+1)})$, $v' \in L_\infty(kT,(k+1)T;L_3(S_1))$, $v'=(v_1,v_2)$, $v\in W_r^{s,s/2}(\Omega^{T(k+1)})$ *with* $5/r-3/2 < s$, $\chi(kT)\in L_2(\Omega)$. *Assume also that* v *is a weak solution to problem* (1.1) *satisfying* (2.12). *Then every solution* χ *to problem* (3.2) *satisfies the inequality*

$$\begin{aligned}(4.3)\qquad |\chi(t)|_{2,\Omega}^2+\int_{kT}^t\|\chi(t')\|_{1,\Omega}^2\,dt' &\le c(l_1^2(\|v'\|^2_{L_\infty(kT,t;L_3(S_1))}+|h|^2_{5,\Omega^{t,kT}})\\ &\quad+\|v'\|^2_{s,r,\Omega^{t,kT}}+|F_3|^2_{10/7,\Omega^{t,kT}}+|\chi(kT)|^2_{2,\Omega}),\end{aligned}$$

where F_3 *is given by* (3.2), $l_1(t)$ *is given by* (2.12) *and* $t\in(kT,(k+1)T)$.

Proof. Multiplying $(4.2)_1$ by χ' and integrating the result over Ω we get

$$\begin{aligned}\frac12\frac{d}{dt}|\chi'(t)|^2_{2,\Omega}+\nu|\nabla\chi'|^2_{2,\Omega}&=\int_\Omega(v_{2,x_1}-v_{1,x_2})h_3\chi'\,dx\\ &\quad-\int_\Omega(h_2w_{,x_1}-h_1w_{,x_2})\chi'\,dx+\int_\Omega v\cdot\nabla\tilde\chi\chi'\,dx+\int_\Omega F_3\chi'\,dx.\end{aligned}$$

Utilizing the Poincaré inequality and integrating with respect to time yields

$$\begin{aligned}(4.4)\qquad |\chi'(t)|^2_{2,\Omega}&+\int_{kT}^t\|\chi'(t')\|^2_{1,\Omega}\,dt'\\ &\le c\Big(\int_{\Omega^{t,kT}}|h_3|\,|\nabla v'|\,|\chi'|\,dx\,dt'\\ &\quad+\int_{\Omega^{t,kT}}|h'|\,|\nabla' w|\,|\chi'|\,dx\,dt'+\Big|\int_{kT}^t\int_\Omega v(t')\cdot\nabla\tilde\chi(t')\chi'(t')\,dx\,dt'\Big|\\ &\quad+\int_{\Omega^{t,kT}}|F_3|\,|\chi'|\,dx\,dt'+|\chi(kT)|^2_{2,\Omega}\Big).\end{aligned}$$

We estimate the first term on the r.h.s. of (4.4) by $|h_3|_{5,\Omega^{t,kT}}|\nabla v'|_{2,\Omega^{t,kT}}\cdot|\chi'|_{10/3,\Omega^{t,kT}}$ and the second by $|\nabla' w|_{2,\Omega^{t,kT}}|h'|_{5,\Omega^{t,kT}}|\chi'|_{10/3,\Omega^{t,kT}}$. The third term on the r.h.s. of (4.4) can be expressed in the form

$$\Big|-\int_{kT}^t\int_\Omega v(t')\cdot\nabla\chi'(t')\tilde\chi(t')\,dx\,dt'\Big|\equiv I$$

and estimated as follows:

$$\begin{aligned}I&\le\varepsilon\|\chi'\|^2_{L_2(kT,t;H^1(\Omega))}+\|v\|^2_{L_2(kT,t;H^1(\Omega))}|\tilde\chi|^2_{3,\infty,\Omega^{t,kT}}\\ &\le\varepsilon\|\chi'\|^2_{L_2(kT,t;H^1(\Omega))}+l_1^2|\tilde\chi|^2_{3,\infty,\Omega^{t,kT}}.\end{aligned}$$

We bound the fourth integral on the r.h.s. of (4.4) by
$$|\chi'|_{10/3,\Omega^{t,kT}}|F_3|_{10/7,\Omega^{t,kT}}.$$
Utilizing the above estimates in (4.4) we obtain

$$|\chi'|_{2,\Omega}^2+\int_{kT}^{t}\|\chi'(t')\|_{1,\Omega}^2\,dt'\le c(\varepsilon(|\chi'|_{10/3,\Omega^{t,kT}}^2+|\nabla\chi'|_{2,\Omega^{t,kT}}^2)$$
$$+l_1^2|\tilde\chi|_{3,\infty,\Omega^{t,kT}}^2+|h|_{5,\Omega^{t,kT}}^2|\nabla v|_{2,\Omega^{t,kT}}^2+|F_3|_{10/7,\Omega^{t,kT}}^2+|\chi(kT)|_{2,\Omega}^2).$$
Applying the transformation $\chi'=\chi-\tilde\chi$, for sufficiently small ε we have

$$|\chi(t)|_{2,\Omega}^2+\int_{kT}^{t}\|\chi(t')\|_{1,\Omega}^2\,dt'\le c\Big(l_1^2|\tilde\chi|_{3,\infty,\Omega^{t,kT}}^2+|h|_{5,\Omega^{t,kT}}^2|\nabla v|_{2,\Omega^{t,kT}}^2$$
$$+|\tilde\chi(t)|_{2,\infty,\Omega^{t,kT}}^2+\int_{kT}^{t}\|\tilde\chi(t')\|_{1,\Omega}^2\,dt'+|F_3|_{10/7,\Omega^{t,kT}}^2+|\chi(kT)|_{2,\Omega}^2\Big).$$
Now using the inequalities
$$|u|_{10/3,\Omega^{t,kT}}\le c(|u|_{2,\infty,\Omega^{t,kT}}+\|u\|_{L_2(kT,t;W_2^1(\Omega))})\le c\|u\|_{s,r,\Omega^{t,kT}},$$
where $5/r-3/2<s$, $r\le 2$, we obtain

$$|\chi(t)|_{2,\Omega}^2+\int_{kT}^{t}\|\chi(t')\|_{1,\Omega}^2\,dt'\le c(l_1^2|\tilde\chi|_{3,\infty,\Omega^{t,kT}}^2+l_1^2|h|_{5,\Omega^{t,kT}}^2$$
$$+\|\tilde\chi\|_{s,r,\Omega^{t,kT}}^2+|F_3|_{10/7,\Omega^{t,kT}}^2+|\chi(kT)|_{2,\Omega}^2).$$
Finally, using the inequalities
$$\|\tilde\chi\|_{s,r,\Omega^{t,kT}}\le c\|\chi_*\|_{s-1/r,r,S_1^{t,kT}}\le c\|v'\|_{s,r,\Omega^{t,kT}}$$
and
$$|\tilde\chi|_{3,\infty,\Omega^{t,kT}}\le|v'|_{3,\infty,S_1^{t,kT}},$$
we obtain (4.3). ∎

Having energy type estimates for χ and h we are able to consider the problem

$$\begin{aligned}&v_{1,x_2}-v_{2,x_1}=\chi &&\text{in } \Omega',\\ &v_{1,x_1}+v_{2,x_2}=-h_3 &&\text{in } \Omega',\\ &v'\cdot n'=0 &&\text{on } S_1',\end{aligned}\tag{4.5}$$

where $\Omega'=\Omega\cap\{x_3=\text{const}\in(-a,a)\}$, $S_1'=S_1\cap\{x_3=\text{const}\in(-a,a)\}$, and x_3,t are treated as parameters.

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Then every solution* $v'=(v_1,v_2)$ *to problem* (4.5) *satisfies*

(4.6)
$$\sup_{kT\le t'\le t}\|v'(t')\|_{1,\Omega}^2+\|v'\|_{L_2(kT,t;H^2(\Omega))}^2 \le c\Big(l_1^2(|h|_{5,\Omega^{t,kT}}^2+1)+\|v'\|_{s,r,\Omega^{t,kT}}^2+|F_3|_{10/7,\Omega^{t,kT}}^2 + |\chi(kT)|_{2,\Omega}^2+\sup_{kT\le t'\le t}|h(t')|_{2,\Omega}^2+\int_{kT}^t\|h(t')\|_{1,\Omega}^2\,dt'\Big),$$

whenever $5/r-3/2<s$, $r\le 2$.

Proof. For solutions to problem (4.5) we get the estimates

$$\|v'\|_{1,\Omega'}^2\le c(|\chi|_{2,\Omega'}^2+|h_3|_{2,\Omega'}^2),\qquad \|v'\|_{2,\Omega'}^2\le c(\|\chi\|_{1,\Omega'}^2+\|h_3\|_{1,\Omega'}^2),$$

where $v'=(v_1,v_2)$. Integrating the above estimates with respect to x_3 and the second one also with respect to time, and adding them, we obtain

$$\sup_{kT\le t'\le t}\int_{-a}^a\|v'(x_3,t')\|_{1,\Omega'}^2\,dx_3+\int_{kT}^t\int_{-a}^a\|v'(x_3,t')\|_{2,\Omega'}^2\,dx_3\,dt' \le c\Big(\sup_{kT\le t'\le t}\int_{-a}^a|\chi(t')|_{2,\Omega'}^2\,dx_3+\sup_{kT\le t'\le t}\int_{-a}^a|h_3(t')|_{2,\Omega'}^2\,dx_3 +\int_{kT}^t\|\chi(t')\|_{1,\Omega}^2\,dt'+\int_{kT}^t\|h_3(t')\|_{1,\Omega}^2\,dt'\Big).$$

Adding to both sides the expression $\sup_{kT\le t'\le t}|h'|_{2,\Omega}^2+\int_{kT}^t\|h'(t')\|_{1,\Omega}^2\,dt'$, we obtain

$$\sup_{kT\le t'\le t}\|v'\|_{1,\Omega}^2+\int_{kT}^t\|v'\|_{2,\Omega}^2\,dt'\le c\Big(\sup_{kT\le t'\le t}|\chi(t')|_{2,\Omega}^2+\int_{kT}^t\|\chi(t')\|_{1,\Omega}^2\,dt' +\sup_{kT\le t'\le t}|h|_{2,\Omega}^2+\int_{kT}^t\|h(t')\|_{1,\Omega}^2\,dt'\Big).$$

Utilizing (4.3) to estimate the first norm in the last inequality and the inequality

$$|v'|_{3,\infty,S_1^{t,kT}}\le\varepsilon\|v'\|_{L_\infty(kT,t;H^1(\Omega))}+c(1/\varepsilon)|v|_{2,\infty,\Omega^{t,kT}},$$

we obtain (4.6). ∎

Next, we examine problems (3.1) and (3.3).

LEMMA 4.3. *Assume that v is a weak solution to problem* (1.1) *and*

$$f\in L_2(kT,(k+1)T;L_3(\Omega)),\qquad g\in L_2(\Omega^{T(k+1)}),$$
$$f_3\in L_2(S_2^{T(k+1)}),\qquad h(kT),\varphi(kT)\in L_2(\Omega),$$
$$v,\theta\in L_2(kT,(k+1)T;W_3^1(\Omega)).$$

Then φ, h *defined by* (2.1), *which are solutions to problem* (3.3), (3.1), *respectively, satisfy*

$$
\begin{aligned}
(4.7)\qquad |\varphi(t)|^2_{2,\Omega} + |h(t)|^2_{2,\Omega} &+ \int\limits_{kT}^{t} (\|\varphi(t')\|^2_{1,\Omega} + \|h(t')\|^2_{1,\Omega})\,dt' \\
&\le c\exp(|\nabla v|^2_{3,2,\Omega^{t,kT}} + |\nabla\theta|^2_{3,2,\Omega^{t,kT}} + |f|^2_{3,2,\Omega^{t,kT}}) \\
&\quad \times (|g|^2_{2,\Omega^{t,kT}} + |f_3|^2_{2,S_2^{t,kT}} + |h(kT)|^2_{2,\Omega} + |\varphi(kT)|^2_{2,\Omega}) \\
&\equiv c\exp(\eta_1(t))\eta^2(t)
\end{aligned}
$$

whenever $t \in (kT, (k+1)T)$.

Proof. Multiplying $(3.3)_1$ by φ and integrating over Ω and by parts yields

$$
\begin{aligned}
(4.8)\qquad \frac{d}{dt}|\varphi|^2_{2,\Omega} + |\nabla\varphi|^2_{2,\Omega} &\le \int\limits_{\Omega} |h\cdot\nabla\theta\varphi|\,dx \\
&\le c(\varepsilon|\varphi|^2_{6,\Omega} + \bar{c}(1/\varepsilon)|\nabla\theta|^2_{3,\Omega}|h|^2_{2,\Omega}).
\end{aligned}
$$

Next multiplying $(3.1)_1$ by h and integrating over Ω and by parts using the Korn inequality, we get

$$
\begin{aligned}
(4.9)\qquad \frac{d}{dt}|h|^2_{2,\Omega} + \nu\|h\|^2_{1,\Omega} &\le \varepsilon\|h\|^2_{1,\Omega} + \bar{c}(1/\varepsilon)|\nabla v|^2_{3,\Omega}|h|^2_{2,\Omega} \\
&\quad + c(|\varphi|^2_{2,\Omega}\|f\|^2_{L_3(\Omega)} + |g|^2_{6/5,\Omega} + |f_3|^2_{2,S_2}).
\end{aligned}
$$

Assuming ε is sufficiently small and adding (4.8) and (4.9) we have

$$
\begin{aligned}
&\frac{d}{dt}(|\varphi|^2_{2,\Omega} + |h|^2_{2,\Omega}) + \|\varphi\|^2_{1,\Omega} + \|h\|^2_{1,\Omega} \\
&\quad \le c(|\nabla v|^2_{3,\Omega} + |f|^2_{3,\Omega} + |\nabla\theta|^2_{3,\Omega})(|\varphi|^2_{2,\Omega} + |h|^2_{2,\Omega}) + c(|g|^2_{6/5,\Omega} + |f_3|^2_{2,S_2}).
\end{aligned}
$$

Integrating the above inequality with respect to time from $t = kT$ to $t \in (kT, (k+1)T]$ we obtain (4.7). ∎

Now we increase the regularity of v. From (4.6) we derive

$$
\begin{aligned}
(4.10)\qquad &\|v'\|^2_{L_\infty(kT,t;H^1(\Omega))} + \|v'\|^2_{L_2(kT,t;H^2(\Omega))} \\
&\quad \le c(|h|^2_{5,\Omega^{t,kT}} + \|h\|^2_{V_2^0(\Omega^{t,kT})} + \|v'\|^2_{s,r,\Omega^{t,kT}} + |F_3|^2_{10/7,\Omega^{t,kT}} + l_1^2 \\
&\qquad + |f_3|^2_{2,S_2^{t,kT}} + |\chi(kT)|^2_{2,\Omega})
\end{aligned}
$$

and

$$
\begin{aligned}
|F_3|_{10/7,\Omega^{t,kT}} &\le c(|\nabla' f|_{10/7,\Omega^{t,kT}} + |f|_{7/2,\infty,\Omega^{t,kT}}|\nabla\theta|_{2,\Omega^{t,kT}}) \\
&\le c(c_2)(|\nabla f|_{10/7,\Omega^{t,kT}} + |f|_{7/2,\infty,\Omega^{t,kT}}),
\end{aligned}
$$

where we used (2.8) and (2.10).

Using the notation

$$
\begin{aligned}
&H_1(kT,t) = |h|_{5,\Omega^{t,kT}} + \|h\|_{V_2^0(\Omega^{t,kT})},\\
&K_1(kT,t) = |\nabla' f|_{10/7,\Omega^{t,kT}} + |f|_{7/2,\infty,\Omega^{t,kT}},\\
&D_1(kT) = |\chi(kT)|_{2,\Omega},
\end{aligned}
$$

we write (4.10) in the form

$$
\begin{aligned}
(4.11)\qquad &\|v'\|_{V_2^1(\Omega^{t,kT})}\\
&\le c(H_1(kT,t) + K_1(kT,t) + D_1(kT) + \|v'\|_{s,r,\Omega^{t,kT}} + l_1).
\end{aligned}
$$

In view of (4.11) we are in a position to consider the Stokes system

$$
(4.12)\qquad
\begin{aligned}
&v_{,t} - \operatorname{div}\mathbb{T}(v,p) = -v'\cdot\nabla v - v_3 h + \alpha(\theta) f && \text{in } \Omega^{T(k+1)},\\
&\operatorname{div} v = 0 && \text{in } \Omega^{T(k+1)},\\
&v\cdot\bar n = 0, \quad \bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0, \quad \alpha = 1,2 && \text{on } S^{T(k+1)},\\
&v|_{t=kT} = v(kT) && \text{in } \Omega.
\end{aligned}
$$

LEMMA 4.4. *Assume that* $H_2(kT,t) < \infty$, $K_2(kT,t) + D_2(kT,t) \equiv A_1(kT,t) < \infty$, $v' \in W_r^{s,s/2}(\Omega^{t,kT})$, $5/r - 3/2 < s$, $r \le 2$ *and* $v(kT) \in W_2^1(\Omega)$, *where* H_2, K_2, D_2 *are given by* (4.14). *Then every solution* (v,p) *of* (4.12) *satisfies*

$$
(4.13)\qquad \|v\|_{2,2,\Omega^{t,kT}} + |\nabla p|_{2,\Omega^{t,kT}} \le \bar c(l_1)H_2^2 + \beta(A_1).
$$

where

$$
(4.14)\qquad
\begin{aligned}
&H_2 = H_1, \qquad K_2 = K_1 + |f|_{2,\Omega^{t,kT}},\\
&D_2 = D_1 + \|v(kT)\|_{1,2,\Omega}.
\end{aligned}
$$

Proof. By Theorem 2.1 with $q = 5/3$ and (4.11) we have

$$
\begin{aligned}
(4.15)\qquad \|v\|_{2,5/3,\Omega^{t,kT}} + |\nabla p|_{5/3,\Omega^{t,kT}} &\le c(\|v'\|_{V_2^1(\Omega^{t,kT})}|\nabla v|_{2,\Omega^{t,kT}}\\
&\quad + |v_3|_{10/3,\Omega^{t,kT}}|h|_{10/3,\Omega^{t,kT}} + |f|_{5/3,\Omega^{t,kT}} + \|v(kT)\|_{4/5,5/3,\Omega})\\
&\le \bar c(l_1)(H_2(kT,t) + K_2(kT,t) + D_2(kT) + l_1),
\end{aligned}
$$

where we used that

$$
\|v'\|_{s,r,\Omega^{t,kT}} \le \varepsilon\|v'\|_{2,5/3,\Omega^{t,kT}} + c(1/\varepsilon)|v'|_{2,\Omega^{t,kT}},
$$

whenever $5/r - 3/2 < s$, $r \le 2$. In view of (4.14) we obtain for solutions to problem (4.12) the inequality

$$
\begin{aligned}
(4.16)\qquad \|v\|_{2,2,\Omega^{t,kT}} + |\nabla p|_{2,\Omega^{t,kT}} &\le c(|v'|_{10,\Omega^{t,kT}}|\nabla v|_{5/2,\Omega^{t,kT}}\\
&\quad + |v_3|_{5,\Omega^{t,kT}}|h|_{10/3,\Omega^{t,kT}} + |f|_{2,\Omega^{t,kT}} + \|v(kT)\|_{1,2,\Omega}).
\end{aligned}
$$

From (4.11), (4.14) and (4.15) we obtain the estimate

$$
(4.17)\qquad \|v'\|_{V_2^1(\Omega^{t,kT})} \le \bar c(l_1)(H_2(kT,t) + K_2(kT,t) + D_2(kT) + l_1).
$$

Using (4.14) and (4.17) in (4.16) yields

$$\|v\|_{2,2,\Omega^{t,kT}} + |\nabla p|_{2,\Omega^{t,kT}} \le c((H_2(kT,t) + K_2(kT,t) + D_2(kT) + l_1)^2 + |f|_{2,\Omega^{t,kT}} + \|v(kT)\|_{1,2,\Omega}). \ \blacksquare$$

Next we consider the problem

$$\begin{aligned} &\theta_{,t} - \Delta\theta = -v'\nabla\theta - v_3\varphi && \text{in } \Omega^{T(k+1)},\\ &\bar n\cdot\nabla\theta = 0 && \text{on } S^{T(k+1)},\\ &\theta|_{t=kT} = \theta(kT) && \text{in } \Omega. \end{aligned} \tag{4.18}$$

Lemma 4.5. *Assume* $H_2(kT,t)<\infty$, $A_2(kT,t)\equiv A_1(kT,t)+\|\theta(kT)\|_{1,2,\Omega} < \infty$, $\varphi\in L_{10/3}(\Omega^{t,kT})$ *and* $\theta(kT), v(kT)\in W^1_2(\Omega)$. *Then every solution* θ *of* (4.18) *satisfies*

$$\|\theta\|_{2,2,\Omega^{t,kT}} \le \bar c(l_1,c_2)(H_2^2(kT,t) + A_2^2(kT,t) + A_2(kT,t) + |\varphi|_{10/3,\Omega^{t,kT}}). \tag{4.19}$$

Proof. In view of Theorem 2.2 we have

$$\|\theta\|_{2,5/3,\Omega^{t,kT}} \le c(|v'|_{10,\Omega^{t,kT}}|\nabla\theta|_{2,\Omega^{t,kT}}|v_3|_{10/3,\Omega^{t,kT}}|\varphi|_{10/3,\Omega^{t,kT}} + \|\theta(kT)\|_{4/5,5/3,\Omega}). \tag{4.20}$$

Using (2.8) and (4.17) in (4.20) we obtain

$$\|\theta\|_{2,5/3,\Omega^{t,kT}} \le \bar c(l_1,c_2)(H_2(kT,t) + A_1(kT,t) + |\varphi|_{10/3,\Omega^{t,kT}} + \|\theta(kT)\|_{4/5,5/3,\Omega}). \tag{4.21}$$

In view of (4.20) we obtain for solutions to problem (4.18) the inequality

$$\|\theta\|_{2,2,\Omega^{t,kT}} \le c(|v'|_{10,\Omega^{t,kT}}|\nabla\theta|_{5/2,\Omega^{t,kT}} + |v_3|_{5,\Omega^{t,kT}}|\varphi|_{10/3,\Omega^{t,kT}} + \|\theta(kT)\|_{1,2,\Omega}). \tag{4.22}$$

Using (4.17) and (4.21) in (4.22) implies (4.19). ■

Finally we consider the problem

$$\begin{aligned} &\varphi_{,t} - \chi\Delta\varphi = -v\cdot\nabla\varphi - h\cdot\nabla\theta && \text{in } \Omega^{T(k+1)},\\ &\bar n\cdot\nabla\varphi = 0 && \text{on } S_1^{T(k+1)},\\ &\varphi = 0 && \text{on } S_2^{T(k+1)},\\ &\varphi|_{t=kT} = \varphi(kT) && \text{in } \Omega. \end{aligned} \tag{4.23}$$

Lemma 4.6. *Assume* $H_3(kT,t)=H_2(kT,t)+|h|_{3,\infty,\Omega^{t,kT}}<\infty$, $A_3(kT,t) = A_2(kT,t) + |\varphi(kT)|_{2,\Omega} < \infty$, $A_4(kT,t) = A_3(kT,t) + \|\varphi(kT)\|_{1,2,\Omega}$. *Then every solution* φ *of* (4.23) *satisfies*

$$\|\varphi\|_{V_2^0(\Omega^{t,kT})} \le \bar c(c_2)(H_3(kT,t) + A_3(kT,t)), \tag{4.24}$$

$$\|\varphi\|_{2,2,\Omega^{t,kT}} \le \bar c(l_1,c_2)(H_4^3(kT,t) + \beta(A_4(kT,t))). \tag{4.25}$$

Proof. Multiplying (4.23) by φ, integrating over $\Omega\times(kT,t)$, using the boundary conditions and initial conditions we obtain

$$\|\varphi\|_{V_2^0(\Omega^{t,kT})}\le\varepsilon|\varphi|^2_{2,\Omega^{t,kT}}+\bar c(1/\varepsilon)|h|^2_{3,\infty,\Omega^{t,kT}}|\nabla\theta|^2_{2,\Omega^{t,kT}}+c|\varphi(kT)|^2_{2,\Omega}.$$

Assuming that ε is sufficiently small and using (2.8) and (2.10) we have

$$\|\varphi\|_{V_2^0(\Omega^{t,kT})}\le\bar c(c_2)|h|_{3,\infty,\Omega^{t,kT}}+c|\varphi(kT)|^2_{2,\Omega}. \tag{4.26}$$

Hence, (4.24) holds.

In view of (4.26) we derive from (4.19) the inequality

$$\|\theta\|_{2,2,\Omega^{t,kT}}\le\bar c(l_1,c_2)(H_3^2(kT,t)+A_3^2(kT,t)+H_3(kT,t)+A_3(kT,t)). \tag{4.27}$$

Applying Theorem 2.2 to (4.23) yields

$$\begin{aligned}\|\varphi\|_{2,5/3,\Omega^{t,kT}}&\le c(|v|_{10,\Omega^{t,kT}}|\nabla\varphi|_{2,\Omega^{t,kT}}+|h|_{10/3,\Omega^{t,kT}}|\nabla\theta|_{10/3,\Omega^{t,kT}}\\&\quad+\|\varphi(kT)\|_{4/5,5/3,\Omega})\\&\le\bar c(l_1,c_2)(H_3^2+\beta(A_3))+c\|\varphi(kT)\|_{4/5,5/3,\Omega},\end{aligned} \tag{4.28}$$

where in the second inequality we used (4.13), (4.24) and (4.27).

Applying again Theorem 2.2 and employing (4.13), (4.27) and (4.28) we obtain

$$\begin{aligned}\|\varphi\|_{2,2,\Omega^{t,kT}}&\le c(|v|_{10,\Omega^{t,kT}}|\nabla\varphi|_{5/2,\Omega^{t,kT}}+|h|_{5,\Omega^{t,kT}}|\nabla\theta|_{10/3,\Omega^{t,kT}}\\&\quad+\|\varphi(kT)\|_{1,2,\Omega})\le\bar c(l_1,c_2)(H_4^3(kT,t)+\beta(A_4)).\end{aligned} \tag{4.29}$$ ∎

Lemma 4.7. *Assume $v\in W_2^{2,1}(\Omega^{t,kT})$, $g\in L_2(\Omega^{t,kT})$, $\varphi\in L_{10/3}(\Omega^{t,kT})$, $h(kT)\in L_2(\Omega)$ and $f\in L_5(\Omega^{t,kT})$. Then every solution (h,q) of* (3.1) *satisfies*

$$\begin{aligned}\|h\|_{2,2,\Omega^{t,kT}}+|\nabla q|_{2,\Omega^{t,kT}}&\le c(\beta(\|v\|_{2,2,\Omega^{t,kT}})|h|_{2,\Omega^{t,kT}}\\&\quad+|\varphi|_{10/3,\Omega^{t,kT}}|f|_{5,\Omega^{t,kT}}+\|h(kT)\|_{1,2,\Omega}+|g|_{2,\Omega^{t,kT}}),\end{aligned} \tag{4.30}$$

where β an increasing positive function.

Proof. From (3.1) we get

$$\begin{aligned}\|h\|_{2,2,\Omega^{t,kT}}+|\nabla q|_{2,\Omega^{t,kT}}&\le c(|v\cdot\nabla h|_{2,\Omega^{t,kT}}+|h\cdot\nabla v|_{2,\Omega^{t,kT}}\\&\quad+|\varphi|_{10/3,\Omega^{t,kT}}|f|_{5,\Omega^{t,kT}}+\|h(kT)\|_{1,2,\Omega}+|g|_{2,\Omega^{t,kT}}).\end{aligned} \tag{4.31}$$

Using the Hölder inequality in (4.31) we obtain

$$\begin{aligned}&\|h\|_{2,2,\Omega^{t,kT}}+|\nabla q|_{2,\Omega^{t,kT}}\\&\quad\le c(|v|_{10,\Omega^{t,kT}}|\nabla h|_{5/2,\Omega^{t,kT}}+|h|_{5,\Omega^{t,kT}}|\nabla v|_{10/3,\Omega^{t,kT}}\\&\quad\quad+|\varphi|_{10/3,\Omega^{t,kT}}|f|_{5,\Omega^{t,kT}}+\|h(kT)\|_{1,2,\Omega}+|g|_{2,\Omega^{t,kT}}).\end{aligned}$$

Now using the interpolations

$$|v|_{10,\Omega^{t,kT}}|\nabla h|_{5/2,\Omega^{t,kT}}\le\varepsilon\|h\|_{2,2,\Omega^{t,kT}}+\varphi_1(|v|_{10,\Omega^{t,kT}})|h|_{2,\Omega^{t,kT}}$$

and

$$|\nabla v|_{10/3,\Omega^{t,kT}}|h|_{5,\Omega^{t,kT}} \le \varepsilon\|h\|_{2,2,\Omega^{t,kT}} + \varphi_2(|\nabla v|_{10/3,\Omega^{t,kT}})|h|_{2,\Omega^{t,kT}},$$

where φ_1, φ_2 are increasing positive functions, we obtain (4.27). ∎

Finally, we derive an a priori estimate for solutions to problem (1.1). Let

$$\begin{aligned} G_1(k,t) &= \|v(kT)\|_{1,2,\Omega} + \|\theta(kT)\|_{1,2,\Omega} + \|h(kT)\|_{1,2,\Omega} + \|\varphi(kT)\|_{1,2,\Omega} \\ &\quad + \|f\|_{L_\infty(kT,t;H^1(\Omega))} + |g|_{2,\Omega^{t,kT}}, \\ \eta(k,t) &= |g|_{2,\Omega^{t,kT}} + |f_3|_{2,S_2^{t,kT}} + |h(kT)|_{2,\Omega} + |\varphi(kT)|_{2,\Omega}. \end{aligned}$$

LEMMA 4.8. *Assume that $G_1(k,t) < \infty$, $\eta(k,t)$ is sufficiently small, $t \in [kT,(k+1)T]$, $k \in \mathbb{N}_0$. Then there exists a constant $A(k,t) > 0$ such that solutions (v,p,θ) of* (1.1), *(h,q) of* (3.1), *and φ of* (3.3) *satisfy*

$$\begin{aligned} (4.32)\qquad \|v\|_{2,2,\Omega^{t,kT}} + \|h\|_{2,2,\Omega^{t,kT}} + \|\theta\|_{2,2,\Omega^{t,kT}} + \|\varphi\|_{2,2,\Omega^{t,kT}} \\ + |\nabla p|_{2,\Omega^{t,kT}} + |\nabla q|_{2,\Omega^{t,kT}} \le A(k,t) \equiv \alpha(G_1(k,t)), \end{aligned}$$

where α is an increasing positive function.

Proof. Using (4.7), (4.13), (4.27) in (4.30) and employing the imbedding

$$H_4(kT,t) \le c\|h\|_{2,2,\Omega^{t,kT}}$$

we obtain from (4.30) the inequality

$$(4.33)\qquad \|h\|_{2,2,\Omega^{t,kT}} \le \beta(\|h\|_{2,2,\Omega^{t,kT}}, G_1(k,t))\eta(k,t) + \beta(G_1(k,t)),$$

where β is an increasing positive function and $t \in [kT,(k+1)T]$.

For $\eta(k,t)$ sufficiently small and a fixed point argument we obtain from (4.33) the estimate

$$\|h\|_{2,2,\Omega^{t,kT}} \le \beta(G_1(k,t)).$$

Then from (4.13), (4.25) and (4.27) we obtain (4.32). ∎

5. Existence. We prove the existence of solutions by the Leray–Schauder fixed point theorem (see [4]). Define

$$\begin{aligned} \|(v,\theta)\|_{\mathcal{M}(\Omega^{T(k+1)})} &= \|v\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))} + \|\theta\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))} \\ &\quad + \|v_{,x_3}\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))} \\ &\quad + \|\theta_{,x_3}\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}, \\ \mathcal{M}(\Omega^{T(k+1)}) &= \{(v,\theta) : \|(v,\theta)\|_{\mathcal{M}(\Omega^{T(k+1)})} < \infty\}, \\ \|(v,\theta)\|_{\mathcal{N}(\Omega^{T(k+1)})} &= \|v\|_{W_2^{2,1}(\Omega^{T(k+1)})} + \|\theta\|_{W_2^{2,1}(\Omega^{T(k+1)})} \\ &\quad + \|v_{,x_3}\|_{W_2^{2,1}(\Omega^{T(k+1)})} + \|\theta_{,x_3}\|_{W_2^{2,1}(\Omega^{T(k+1)})}, \\ \mathcal{N}(\Omega^{T(k+1)}) &= \{(v,\theta) : \|(v,\theta)\|_{\mathcal{N}(\Omega^{T(k+1)})} < \infty\}. \end{aligned}$$

Lemma 5.1. *We have:*

- $(\mathcal{M}(\Omega^{T(k+1)}), \| \ \|_{\mathcal{M}(\Omega^{T(k+1)})})$ *is a Banach space.*
- $(\mathcal{N}(\Omega^{T(k+1)}), \| \ \|_{\mathcal{N}(\Omega^{T(k+1)})})$ *is a Banach space.*
- $\|u\|_{\mathcal{M}(\Omega^{T(k+1)})} \le c\|u\|_{\mathcal{N}(\Omega^{T(k+1)})}$ *for* $u \in \mathcal{N}(\Omega^{T(k+1)})$ *and the imbedding* $\mathcal{N}(\Omega^{T(k+1)}) \subset \mathcal{M}(\Omega^{T(k+1)})$ *is compact.*

Proof. This follows from imbeddings between Sobolev spaces. ∎

Let us consider the problems

$$
\begin{aligned}
& v_{,t} - \operatorname{div} \mathbb{T}(v,p) = -\lambda[\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta}) f], \\
& \operatorname{div} v = 0, \\
& \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha|_{S^{T(k+1)}} = 0, \quad \alpha = 1,2, \\
& \bar{n} \cdot v|_{S^{T(k+1)}} = 0, \\
& v|_{t=kT} = v(kT),
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
& \theta_{,t} - \chi \Delta \theta = -\lambda \tilde{v} \cdot \nabla \tilde{\theta}, \\
& \bar{n} \cdot \nabla \theta|_{S^{T(k+1)}} = 0, \\
& \theta|_{t=kT} = \theta(kT),
\end{aligned}
\tag{5.2}
$$

where $\lambda \in [0,1]$ is a parameter and $\tilde{v}, \tilde{\theta}$ are treated as given functions.

Lemma 5.2. *Assume that* $0 < c_1 \le \tilde{\theta} \le c_2$,

$$(\tilde{v}, \tilde{\theta}) \in \mathcal{M}(\Omega^{T(k+1)}), \qquad f \in L_2(\Omega^{T(k+1)}), \qquad v(kT) \in W_2^1(\Omega).$$

Then there exists a unique solution (v,p) *to problem* (5.1) *such that*

$$v \in W_2^{2,1}(\Omega^{T(k+1)}) \subset L_4(kT,(k+1)T; W^1_{12/5}(\Omega))$$

and

$$
\begin{aligned}
\|v\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))} &\le c\|v\|_{W_2^{2,1}(\Omega^{T(k+1)})} \\
&\le c(\lambda \|\tilde{v}, \tilde{\theta}\|^2_{\mathcal{M}(\Omega^{T(k+1)})} \lambda c_3 \|f\|_{L_2(\Omega^{T(k+1)})} + \|v(kT)\|_{W_2^1(\Omega)}).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
\|\tilde{v} \cdot \nabla \tilde{v}\|_{L_2(\Omega^{T(k+1)})} &\le c\|\tilde{v}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))} \|\nabla \tilde{v}\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega^{T(k+1)}))} \\
&\le c\|\tilde{v}\|^2_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))} \le c\|\tilde{v}, \tilde{\theta}\|^2_{\mathcal{M}(\Omega^{T(k+1)})}.
\end{aligned}
$$

and

$$\|\alpha(\tilde{\theta}) f\|_{L_2(\Omega^{T(k+1)})} \le c_3 \|f\|_{L_2(\Omega^{T(k+1)})}.$$

By Theorem 2.1 the proof is complete. ∎

Lemma 5.3. *Assume that* $0 < c_1 \le \tilde{\theta} \le c_2$,

$$(\tilde{v}, \tilde{\theta}) \in \mathcal{M}(\Omega^{T(k+1)}), \qquad \theta(kT) \in W_2^1(\Omega).$$

Then there exists a unique solution θ *to problem* (5.2) *such that*

$$\theta \in W_2^{2,1}(\Omega^T) \subset L_4(kT,(k+1)T;W^1_{12/5}(\Omega))$$

and

$$\begin{aligned}\|\theta\|_{L_4(0,T;W_{12/5}(\Omega))} &\le c\|\theta\|_{W_2^{2,1}(\Omega^{T(k+1)})}\\ &\le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})} + \|\theta(kT)\|_{W_2^1(\Omega)}).\end{aligned}$$

Proof. We have

$$\begin{aligned}\|\tilde v\cdot\nabla\tilde\theta\|_{L_2(\Omega^{T(k+1)})} &\le c\|\tilde v\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde\theta\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\ &\le c\|\tilde v\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|\tilde\theta\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\\ &\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})}. \ \blacksquare\end{aligned}$$

LEMMA 5.4. *Assume that* $0 < c_1 \le \tilde\theta \le c_2$, $h(kT) \in W_2^1(\Omega)$,

$$(\tilde v,\tilde\theta) \in \mathcal M(\Omega^{T(k+1)}), \quad f \in L_4(\Omega^{T(k+1)}), \quad g \in L_2(\Omega^{T(k+1)}),$$

where $g = f_{,x_3}$. *Let* (v,p) *be the unique solution to problem* (5.1) *and* $h = v_{,x_3}$, $q = p_{,x_3}$. *Then*

$$h \in W_2^{2,1}(\Omega^{T(k+1)}) \subset L_4(kT,(k+1)T;W^1_{12/5}(\Omega))$$

and

$$\begin{aligned}&\|h\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))} \le c\|h\|_{W_2^{2,1}(\Omega^{T(k+1)})}\\ &\quad\le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})} + \lambda c_3\|(\tilde v,\tilde\theta)\|_{\mathcal M(\Omega^{T(k+1)})}\|f\|_{L_4(\Omega^{T(k+1)})}\\ &\qquad + \lambda\|g\|_{L_2(\Omega^{T(k+1)})} + \|h(kT)\|_{W_2^1(\Omega)}).\end{aligned}$$

Proof. The function h is a solution to the problem

$$\begin{aligned}&h_{,t} - \operatorname{div}\mathbb T(h,q) = \lambda[-\tilde v\cdot\nabla\tilde h - \tilde h\cdot\nabla\tilde v\\ &\qquad + \alpha_{,\theta}(\tilde\theta)\tilde\theta_{,x_3}f + \alpha(\tilde\theta)g] && \text{in } \Omega^{T(k+1)},\\ &\operatorname{div} h = 0 && \text{in } \Omega^{T(k+1)},\\ &\bar n\cdot h = 0, \quad \bar n\cdot\mathbb D(h)\cdot\bar\tau_\alpha = 0, \quad \alpha = 1,2 && \text{on } S_1^{T(k+1)},\\ &h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^{T(k+1)},\\ &h|_{t=0} = h(kT) && \text{in } \Omega,\end{aligned}$$

where $\tilde h = \tilde v_{,x_3}$, $\tilde\varphi = \tilde\theta_{,x_3}$. We have

$$\begin{aligned}\|\tilde v\cdot\nabla\tilde h\|_{L_2(\Omega^T)} &\le c\|\tilde v\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde h\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\ &\le c\|\tilde v\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|\tilde h\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\\ &\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})},\end{aligned}$$

$$\begin{aligned}\|\tilde h\cdot\nabla\tilde v\|_{L_2(\Omega^{T(k+1)})}&\le c\|\tilde h\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde v\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\&\le c\|\tilde h\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|\tilde v\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\\&\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})},\end{aligned}$$

$$\begin{aligned}\|\alpha_{,\theta}(\tilde\theta)\tilde\theta_{,x_3}f\|_{L_2(\Omega^{T(k+1)})}&\le c_3\|\tilde\theta\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|f\|_{L_4(\Omega^{T(k+1)})}\\&\le c_3\|(\tilde v,\tilde\theta)\|_{\mathcal M(\Omega^{T(k+1)})}\|f\|_{L_4(\Omega^{T(k+1)})}\end{aligned}$$

and

$$\|\alpha(\tilde\theta)g\|_{L_2(\Omega^{T(k+1)})}\le c_3\|g\|_{L_2(\Omega^{T(k+1)})}\le c_3\|g\|_{L_2(\Omega^{T(k+1)})}.$$

Thus by Theorem 2.1 the proof is complete. ∎

LEMMA 5.5. *Assume that* $0<c_1\le\tilde\theta\le c_2$, $\varphi(kT)\in W^1_2(\Omega)$, $(\tilde v,\tilde\theta)\in\mathcal M(\Omega^{T(k+1)})$. *Let* θ *be the unique solution to problem* (5.2) *and* $\varphi=\theta_{,x_3}$. *Then*

$$\varphi\in W^{2,1}_2(\Omega^{T(k+1)})\subset L_4(kT,(k+1)T;W^1_{12/5}(\Omega))$$

and

$$\begin{aligned}\|\varphi\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}&\le c\|\varphi\|_{W^{2,1}_2(\Omega^{T(k+1)})}\\&\le c(\lambda\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})}+\|\varphi(kT)\|_{W^1_2(\Omega)}).\end{aligned}$$

Proof. The function φ is a solution to the problem

$$\begin{aligned}&\varphi_{,t}-\chi\Delta\varphi=-\lambda[\tilde h\cdot\nabla\tilde\theta+\tilde v\cdot\nabla\tilde\varphi] &&\text{in } \Omega^{T(k+1)},\\&\bar n\cdot\nabla\varphi=0 &&\text{on } S^{T(k+1)},\\&\varphi=0 &&\text{on } S^{T(k+1)},\\&\varphi|_{t=kT}=\varphi(kT) &&\text{in } \Omega,\end{aligned}$$

where $\tilde\varphi=\tilde\theta_{,x_3}$. We have

$$\begin{aligned}\|\tilde h\cdot\nabla\tilde\theta\|_{L_2(\Omega^{T(k+1)})}&\le\|\tilde h\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde\theta\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\&\le c\|\tilde h\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|\tilde\theta\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\\&\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})}\end{aligned}$$

and

$$\begin{aligned}\|\tilde v\cdot\nabla\tilde\varphi\|_{L_2(\Omega^{T(k+1)})}&\le\|\tilde h\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde\varphi\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\&\le c\|\tilde v\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|\tilde\varphi\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\\&\le c\|(\tilde v,\tilde\theta)\|^2_{\mathcal M(\Omega^{T(k+1)})}.\ \blacksquare\end{aligned}$$

From Lemmas 5.1–5.5 it follows that if $(\tilde v,\tilde\theta)\in\mathcal M(\Omega^{T(k+1)})$, then there exists a unique solution (v,θ) to problems (5.1)–(5.2) such that $(v,\theta)\in\mathcal M(\Omega^{T(k+1)})$.

To prove the existence of solutions to the problem (1.1) we apply the Leray–Schauder fixed point theorem. Therefore we introduce the mapping $\phi : [0,1] \times \mathcal{M}(\Omega^{T(k+1)}) \to \mathcal{M}(\Omega^{T(k+1)})$, $(\lambda, \tilde{v}, \tilde{\theta}) \mapsto \phi(\lambda, \tilde{v}, \tilde{\theta}) = (v, \theta)$, where (v, θ) is the solution to problems (5.1)–(5.2).

For $\lambda = 0$ we have the existence of a unique solution. For $\lambda = 1$ every fixed point is a solution to problem (1.1).

LEMMA 5.6. *Let the assumptions of Lemmas* 5.2–5.5 *be satisfied and* $f \in L_\infty(\Omega^{T(k+1)})$. *Then the mappings*

$$\phi(\lambda, \cdot) : \mathcal{M}(\Omega^{T(k+1)}) \to \mathcal{M}(\Omega^{T(k+1)}), \quad \lambda \in [0,1],$$

are completely continuous.

Proof. By Lemmas 5.1–5.5 the mappings $\phi(\lambda, \cdot)$, $\lambda \in [0,1]$, are compact. It follows that bounded sets in $\mathcal{M}(\Omega^{T(k+1)})$ are transformed into bounded sets in $\mathcal{M}(\Omega^{T(k+1)})$. Let $(\tilde{v}_i, \tilde{\theta}_i) \in \mathcal{M}(\Omega^{T(k+1)})$, $i = 1, 2$, be two given elements. Then (v_i, θ_i), $i = 1, 2$, are solutions to the problems

$$\tag{5.3} \begin{aligned} & v_{i,t} - \operatorname{div} \mathbb{T}(v_i, p_i) = -\lambda(\tilde{v}_i \cdot \nabla \tilde{v}_i + \alpha(\tilde{\theta}_i) f), \\ & \operatorname{div} v_i = 0 \\ & \bar{n} \cdot \mathbb{D}(v_i) \cdot \bar{\tau}_\alpha|_{S^{T(k+1)}} = 0, \\ & \bar{n} \cdot v_i|_{S^{T(k+1)}} = 0, \\ & v_i|_{t=kT} = v(kT), \qquad i = 1, 2, \end{aligned}$$

and

$$\tag{5.4} \begin{aligned} & \theta_{i,t} - \chi \Delta \theta_i = -\lambda \tilde{v}_i \cdot \nabla \tilde{\theta}_i, \\ & \bar{n} \cdot \nabla \theta_i|_{S^{T(k+1)}} = 0, \\ & \bar{n} \cdot \nabla \theta_i|_{S^{T(k+1)}} = 0, \\ & \theta_i|_{t=kT} = \theta(kT), \qquad i = 1, 2. \end{aligned}$$

To show continuity we introduce the differences

$$\tag{5.5} V = v_1 - v_2, \qquad P = p_1 - p_2, \qquad \mathcal{T} = \theta_1 - \theta_2,$$

which are solutions to the problems

$$\tag{5.6} \begin{aligned} & V_{,t} - \operatorname{div} \mathbb{T}(V, P) = -\lambda[\tilde{V} \cdot \nabla \tilde{v}_1 + \tilde{v}_2 \cdot \nabla \tilde{V} + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2)) f], \\ & \operatorname{div} V = 0, \\ & \bar{n} \cdot \mathbb{D}(V) \cdot \bar{\tau}_\alpha|_{S^{T(k+1)}} = 0, \qquad \alpha = 1, 2, \\ & V \cdot \bar{n}|_{S^{T(k+1)}} = 0, \\ & V|_{t=kT} = 0, \end{aligned}$$

and

$$
(5.7)\qquad
\begin{aligned}
&\mathcal{T}_t - \chi\Delta\mathcal{T} = -\lambda[\tilde{V}\cdot\nabla\tilde{\theta}_1 + \tilde{v}_2\cdot\nabla\tilde{\mathcal{T}}],\\
&\bar{n}\cdot\nabla\mathcal{T}|_{S^{T(k+1)}} = 0,\\
&\mathcal{T}|_{t=kT} = 0,
\end{aligned}
$$

where $\tilde{V} = \tilde{v}_1 - \tilde{v}_2$, $\tilde{\mathcal{T}} = \tilde{\theta}_1 - \tilde{\theta}_2$. For problems (5.6), (5.7) we have

$$
(5.8)\qquad
\begin{aligned}
&\|V\|_{W_2^{2,1}(\Omega^{T(k+1)})} + \|\mathcal{T}\|_{W_2^{2,1}(\Omega^{T(k+1)})}\\
&\quad\le c[\|\tilde{V}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{v}_1\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{v}_2\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{V}\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + c_3\|\tilde{\mathcal{T}}\|_{L_\infty(kT,(k+1)T;W_4^1(\Omega))}\|f\|_{L_4(\Omega^{T(k+1)})}\\
&\qquad + \|\tilde{v}_2\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{\mathcal{T}}\|_{L_\infty(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{V}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{\theta}_1\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}]\\
&\quad\le c(\|\tilde{V}\|_{\mathcal{M}(\Omega^{T(k+1)})} + \|\tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^{T(k+1)})}).
\end{aligned}
$$

Let $h_i = v_{i,x_3}$, $q_i = p_{i,x_3}$, $\varphi_i = \theta_{i,x_3}$, $\tilde{h}_i = \tilde{v}_{i,x_3}$, $\tilde{\varphi}_i = \tilde{\theta}_{i,x_3}$. The functions h_i, φ_i, $i = 1,2$, are solutions to the problems

$$
\begin{aligned}
&h_{i,t} - \operatorname{div}\mathbb{T}(h_i,q_i)\\
&\quad = -\lambda[\tilde{h}_i\cdot\nabla\tilde{v}_i + \tilde{v}_i\cdot\nabla\tilde{h}_i + \alpha_{,\theta}(\tilde{\theta}_i)\tilde{\varphi}_i f + \alpha(\tilde{\theta}_i)g] && \text{in } \Omega^{T(k+1)},\\
&\operatorname{div} h_i = 0 && \text{in } \Omega^{T(k+1)},\\
&\bar{n}\cdot h_i = 0,\quad \bar{n}\cdot\mathbb{D}(h)\cdot\bar{\tau}_\alpha = 0,\quad \alpha = 1,2,\quad i = 1,2, && \text{on } S_1^{T(k+1)},\\
&h_i = 0,\quad i = 1,2, && \text{on } S_2^{T(k+1)},\\
&h_{i3,x_3} = 0 && \text{on } S_2^{T(k+1)},\\
&h_i|_{t=kT} = h(kT) && \text{in } \Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi_{i,t} - \chi\Delta\varphi_i = -\lambda[\tilde{h}_i\cdot\nabla\tilde{\theta}_i + \tilde{v}_i\cdot\nabla\tilde{\varphi}_i] && \text{in } \Omega^{T(k+1)},\\
&\bar{n}\cdot\nabla\varphi_i = 0 && \text{on } S_1^{T(k+1)},\\
&\varphi_i = 0 && \text{on } S_2^{T(k+1)},\\
&\varphi_i|_{t=kT} = \varphi(kT) && \text{in } \Omega.
\end{aligned}
$$

We introduce the differences

$$
H = h_1 - h_2,\qquad Q = q_1 - q_2,\qquad R = \varphi_1 - \varphi_2,
$$

which are solutions to the problems

$$
\begin{aligned}
&H_{,t} - \operatorname{div}\mathbb{T}(H,Q) = -\lambda[\tilde{H}\cdot\nabla\tilde{v}_1 + \tilde{h}_2\cdot\nabla\tilde{V} + \tilde{V}\cdot\nabla\tilde{h}_1\\
&\qquad + \tilde{v}_2\cdot\nabla\tilde{H} + (\alpha_{,\theta}(\tilde{\theta}_1) - \alpha_{,\theta}(\tilde{\theta}_2))\tilde{\varphi}_1 f + \alpha_{,\theta}(\tilde{\theta}_2)\tilde{R}f\\
&\qquad + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2))g] && \text{in } \Omega^{T(k+1)},\\
&\operatorname{div} H = 0 && \text{in } \Omega^{T(k+1)},\\
&\bar{n}\cdot\mathbb{D}(H)\cdot\bar{\tau}_\alpha|_{S_1^{T(k+1)}} = 0, \quad \alpha=1,2, \quad \bar{n}\cdot H|_{S_1^{T(k+1)}} = 0,\\
&H_i|_{S_2^{T(k+1)}} = 0, \quad i=1,2, \quad H_{3,x_3}|_{S_2^{T(k+1)}} = 0,\\
&H|_{t=kT} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&R_{,t} - \chi\Delta R = -\lambda[\tilde{H}\cdot\nabla\tilde{\theta}_1 + \tilde{h}_2\cdot\nabla\tilde{\mathcal{T}} + \tilde{V}\cdot\nabla\tilde{\varphi}_1 + \tilde{v}_2\cdot\nabla\tilde{R}] && \text{in } \Omega^{T(k+1)},\\
&\bar{n}\cdot\nabla R = 0 && \text{on } S_1^{T(k+1)},\\
&R = 0 && \text{on } S_2^{T(k+1)},\\
&R|_{t=kT} = 0 && \text{in } \Omega,
\end{aligned}
$$

where $\tilde{H} = \tilde{h}_1 - \tilde{h}_2$, $\tilde{R} = \tilde{\varphi}_1 - \tilde{\varphi}_2$. Then we have

$$
\begin{aligned}
&\|H\|_{W_2^{2,1}(\Omega^{T(k+1)})} + \|R\|_{W_2^{2,1}(\Omega^{T(k+1)})}\\
&\quad\le c[\|\tilde{H}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{v}_1\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{h}_2\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{V}\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{V}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{h}_1\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{v}_2\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{H}\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + c_3\|\tilde{\mathcal{T}}\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|\tilde{\varphi}_1\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|f\|_{L_\infty(\Omega^{T(k+1)})}\\
&\qquad + c_3\|\tilde{R}\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|f\|_{L_\infty(\Omega^{T(k+1)})}\\
&\qquad + c_3\|\tilde{\mathcal{T}}\|_{L_4(kT,(k+1)T;W^1_{12/5}(\Omega))}\|g\|_{L_4(\Omega^{T(k+1)})}\\
&\qquad + \|\tilde{H}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{\theta}_1\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{h}_2\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}|\nabla\tilde{\mathcal{T}}\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{V}\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{\varphi}_1\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}\\
&\qquad + \|\tilde{v}_2\|_{L_4(kT,(k+1)T;L_{12}(\Omega))}\|\nabla\tilde{R}\|_{L_4(kT,(k+1)T;L_{12/5}(\Omega))}]\\
&\quad\le c(\|\tilde{H}\|_{\mathcal{M}(\Omega^{T(k+1)})}] + \|\tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^{T(k+1)})}).
\end{aligned}
$$

Now from (5.8) and Lemma 5.1 we obtain

$$
\|(V,\mathcal{T})\|_{\mathcal{M}(\Omega^{T(k+1)})} \le c\|(\tilde{V},\tilde{\mathcal{T}})\|_{\mathcal{M}(\Omega^{T(k+1)})}.
$$

So continuity of ϕ follows. ∎

Proof of Theorem 1.1. In view of the above considerations the assumptions of the Leray–Schauder fixed point theorem are satisfied. Hence the main theorem is proved. ∎

6. Global existence. To prove the global existence of solutions to the problem (1.1) we have to show that the constant A appearing in Theorem 1.1 does not depend on k. For this purpose we have to show that

$$
\begin{aligned}
&\|v((k+1)T)\|_{H^1(\Omega)} \le \|v(kT)\|_{H^1(\Omega)},\\
&\|\theta((k+1)T)\|_{H^1(\Omega)} \le \|\theta(kT)\|_{H^1(\Omega)},\\
&\|h((k+1)T)\|_{H^1(\Omega)} \le \|h(kT)\|_{H^1(\Omega)},\\
&\|\varphi((k+1)T)\|_{H^1(\Omega)} \le \|\varphi(kT)\|_{H^1(\Omega)}.
\end{aligned} \tag{6.1}
$$

To show (6.1) we need

LEMMA 6.1. *Asume that there exists a local solution to problem* (1.2) *in the interval* $[kT,(k+1)T]$ *and there exists* $\delta>0$ *such that*

$$
\begin{aligned}
&\|f(t)\|_{H^1(\Omega)} \le \|f(kT)\|_{H^1(\Omega)} e^{-\delta(t-kT)},\\
&\|g(t)\|_{L_2(\Omega)} \le \|g(kT)\|_{L_2(\Omega)} e^{-\delta(t-kT)}, \qquad t\in(kT,(k+1)T].
\end{aligned} \tag{6.2$_1$}
$$

Then

$$
\begin{aligned}
\|v(k+1)T\|^2_{H^1(\Omega)} \le{}& ce^{-\delta T+\alpha(A(k,T))}\|f(kT)\|^2_{L_2(\Omega)}\\
&+ e^{-c_1T+\alpha(A(k,T))}\|v(kT)\|^2_{H^1(\Omega)},
\end{aligned} \tag{6.2$_2$}
$$

where $c_1>2\delta$, *the constants* c, c_1 *are independent of* T, *and* α *is an increasing positive function of* $A(k,T)$, *which in view of* $(6.2)_1$ *and* (4.32) *does not increase with* T.

Proof. Multiplying $(1.2)_1$ by $\operatorname{div}\mathbb{T}(v,p)$ and integrating the result over Ω yields

$$
\begin{aligned}
&\int_\Omega v_{,t}\cdot \operatorname{div}\mathbb{T}(v,p)\,dx - \int_\Omega |\operatorname{div}\mathbb{T}(v,p)|^2\,dx\\
&\quad= -\int_\Omega v\cdot\nabla v\cdot \operatorname{div}\mathbb{T}(v,p)\,dx + \int_\Omega \alpha(\theta) f\cdot \operatorname{div}\mathbb{T}(v,p)\,dx.
\end{aligned} \tag{6.3}
$$

Integrating the first integral by parts leads to

$$
\begin{aligned}
\int_\Omega v_{i,t}T_{ij}(v,p)_{,x_j}\,dx &= \int_\Omega (v_{i,t}T_{ij}(v,p))_{,x_j}\,dx - \int_\Omega v_{i,x_jt}T_{ij}(v,p)\,dx\\
&= -\int_\Omega v_{i,x_jt}D_{ij}(v)\,dx = -\frac12\int_\Omega D_{ij}(v_t)D_{ij}(v)\,dx\\
&= -\frac14\frac{d}{dt}\int_\Omega |\mathbb{D}(v)|^2\,dx,
\end{aligned} \tag{6.4}
$$

where

$$\mathbb{D}(v) = \{D_{ij}(v)\}_{i,j=1,2,3}, \qquad \mathbb{T}(v,p) = \{T_{ij}(v,p)\}_{i,j=1,2,3}.$$

Using (6.4) in (6.3) we obtain

$$\frac{1}{4}\frac{d}{dt}\int_\Omega |\mathbb{D}(v)|^2\,dx + \frac{1}{2}\int_\Omega |\operatorname{div}\mathbb{T}(v,p)|^2\,dx \le c\Big(\int_\Omega |v\cdot\nabla v|^2\,dx + \int_\Omega f^2\,dx\Big).$$

Using Theorem 2.1 and the Korn inequality we have

$$\frac{d}{dt}\bigg(\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx\bigg) + c_1\frac{1}{4}\int_\Omega |\mathbb{D}(v)|\,dx \le c_2\|v\|^2_{L_\infty(\Omega)}\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx + c_3\|f\|^2_{L_2(\Omega)}, \tag{6.5}$$

where c_1, c_2, c_3 do not depend on f. From (6.5) we have

$$\frac{d}{dt}\bigg(\frac{1}{4}\int_\Omega |\mathbb{D}(v)|^2\,dx e^{c_1t - c_2\int_{kT}^t \|v(t')\|^2_{L_\infty(\Omega)}\,dt'}\bigg) \le c_3\|f\|^2_{L_2(\Omega)}e^{c_1t - c_2\int_{kT}^t \|v(t')\|^2_{L_\infty(\Omega)}\,dt'} \tag{6.6}$$

for all $t\in(kT,(k+1)T)$. Integrating (6.6) with respect to time yields

$$\frac{1}{4}\int_\Omega |\mathbb{D}(v(t))|^2\,dx \le e^{-c_1t + c_2\int_{kT}^t \gamma\,dt'}c_3\int_{kT}^t \|f\|^2_{L_2(\Omega)}e^{c_1t' - c_2\int_{kT}^{t'}\gamma\,dt''}\,dt' + e^{-c_1(t-kT)+c_2\int_{kT}^t\gamma\,dt'}\frac{1}{4}\int_\Omega |\mathbb{D}(v(kT))|^2\,dx, \tag{6.7}$$

where $\gamma = \|v(t)\|^2_{L_\infty(\Omega)}$. Since we assume $(6.2)_1$, (6.7) implies

$$\frac{1}{4}\int_\Omega |\mathbb{D}(v(t))|^2\,dx \le c_3e^{-c_1t + c_2\int_{kT}^t\gamma\,dt'}\|f(kT)\|^2_{L_2(\Omega)}\int_{kT}^t e^{(c_1-2\delta)t'}\,dt + e^{-c_1(t-kT)+c_2\int_{kT}^t\gamma\,dt'}\frac{1}{4}\int_\Omega |\mathbb{D}(v(kT))|^2\,dx.$$

Hence

$$\frac{1}{4}\int_\Omega |\mathbb{D}(v(t))|^2\,dx \le c_3e^{-2\delta(t-kT)+c_2\int_{kT}^t\gamma\,dt'}\frac{1}{c_1-2\delta}\|f(kT)\|^2_{L_2(\Omega)} + e^{-c_1(t-kT)+c_2\int_{kT}^t\gamma\,dt'}\frac{1}{4}\int_\Omega |\mathbb{D}(v(kT))|^2\,dx. \tag{6.8}$$

Setting $t = (k+1)T$ and using the Korn inequality we obtain from (6.8) the relation

$$\begin{aligned}(6.9)\qquad &\|v((k+1)T)\|^2_{H^1(\Omega)}\\ &\le c_3 e^{-2\delta T + c_2\int_{kT}^{(k+1)T}\|v(t)\|^2_{L_\infty(\Omega)}\,dt}\frac{1}{c_1-2\delta}\|f(kT)\|^2_{L_2(\Omega)}\\ &\quad + e^{-c_1T+c_2\int_{kT}^{(k+1)T}\|v(t)\|^2_{L_\infty(\Omega)}\,dt}\|v(kT)\|^2_{H^1(\Omega)}.\end{aligned}$$

From (6.9) we obtain $(6.2)_2$. ■

Lemma 6.2. *Assume that there exists a local solution (v,p,θ) to problem* (1.2) *in the interval $[kT,(k+1)T]$. Then*

$$\begin{aligned}(6.10)\qquad \|\theta((k+1)T)\|^2_{H^1(\Omega)} &\le e^{-c_1T+\alpha(A(k,T))}\|\nabla\theta(kT)\|^2_{L_2(\Omega)}\\ &\quad + \|\theta(kT)\|^2_{L_2(\Omega)},\end{aligned}$$

where the constant c_1 is independent of T, and α is an increasing positive function of $A(k,T)$, which in view of $(6.2)_1$ *and* (4.32) *does not increase with T.*

Proof. Multiplying $(1.2)_3$ by θ and integrating the result over Ω yields

$$(6.11)\qquad \frac{d}{dt}\|\theta\|^2_{L_2(\Omega)} \le 0.$$

Next multiplying $(1.1)_3$ by $\Delta\theta$ and integrating the result over Ω and by parts yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2_{L_2(\Omega)} + \chi\|\Delta\theta\|^2_{L_2(\Omega)} \le \varepsilon\|\Delta\theta\|^2_{L_2(\Omega)} + c(1/\varepsilon)\int_\Omega |v\cdot\nabla\theta|^2\,dx.$$

Using Theorem 2.2 we obtain

$$(6.12)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2_{L_2(\Omega)} + c_1\|\nabla\theta\|^2_{L_2(\Omega)} \le c_2\|v\|^2_{L_\infty(\Omega)}\|\nabla\theta\|^2_{L_2(\Omega)}.$$

where c_1, c_2 do not depend on T. From (6.12) we have

$$(6.13)\qquad \frac{d}{dt}\big(\|\nabla\theta\|^2 e^{c_1t-c_2\int_{kT}^t\gamma\,dt'}\big) \le 0,$$

for all $t\in(kT,(k+1)T)$, where $\gamma = \|v(t)\|^2_{L_\infty(\Omega)}$. Integrating (6.13) with respect to time yields

$$(6.14)\qquad \|\nabla\theta(t)\|^2_{L_2(\Omega)} \le e^{-c_1(t-kT)+c_2\int_{kT}^t\gamma\,dt'}\|\nabla\theta(kT)\|^2_{L_2(\Omega)}.$$

Integrating (6.11) with respect to time we obtain

$$(6.15)\qquad \|\theta(t)\|^2_{L_2(\Omega)} \le \|\theta(kT)\|^2_{L_2(\Omega)}.$$

Adding (6.14), (6.15) and setting $t=(k+1)T$ we have

$$\|\theta((k+1)T)\|^2_{H^1(\Omega)} \le e^{-c_1T+c_2\int_{kT}^{(k+1)T}\gamma dt'}\|\nabla\theta(kT)\|^2_{L_2(\Omega)} + \|\theta(kT)\|^2_{L_2(\Omega)}. \ ■$$

LEMMA 6.3. *Assume that there exists a local solution (v,p,θ) to problem (1.2) in $[kT,(k+1)T]$. Then*

$$\begin{aligned}(6.16)\qquad &\|\varphi((k+1)T)\|^2_{H^1(\Omega)}\\ &\le e^{-c_1T+\alpha(A(k,T))}(\|\nabla\theta(kT)\|^2_{L_2(\Omega)}\alpha(A(k,T))+\|\varphi(kT)\|^2_{H^1(\Omega)}),\end{aligned}$$

where the constant c_1 is independent of T and α is an increasing positive function of $A(k,T)$, which in view of $(6.2)_1$ and (4.32) does not increase with T.

Proof. Multiplying (3.3) by $\Delta\varphi$ and integrating the result over Ω and by parts yields

$$\begin{aligned}(6.17)\qquad &\frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2_{L_2(\Omega)}+\chi\|\Delta\varphi\|^2_{L_2(\Omega)}\\ &\le\varepsilon\|\Delta\varphi\|^2_{L_2(\Omega)}+c(1/\varepsilon)(\|h\|^2_{L_\infty(\Omega)}\|\nabla\theta\|^2_{L_2(\Omega)}+\|v\|^2_{L_\infty(\Omega)}\|\nabla\varphi\|^2_{L_2(\Omega)}).\end{aligned}$$

Using Theorem 2.2, from (6.17) we have

$$\begin{aligned}(6.18)\qquad &\frac{d}{dt}\bigg(\frac{1}{2}\|\nabla\varphi\|^2_{L_2(\Omega)}e^{c_1't-c_2'\int_{kT}^t\|v(t')\|^2_{L_\infty(\Omega)}\,dt'}\bigg)\\ &\qquad\le\|h\|^2_{L_\infty(\Omega)}\|\nabla\theta\|^2_{L_2(\Omega)}e^{c_1't-c_2'\int_{kT}^t\|v(t')\|^2_{L_\infty(\Omega)}\,dt'}\end{aligned}$$

for all $t\in(kT,(k+1)T)$. Integrating (6.18) with respect to time yields

$$\begin{aligned}(6.19)\qquad &\frac{1}{2}\|\nabla\varphi(t)\|^2_{L_2(\Omega)}\\ &\le e^{-c_1't+c_2'\int_{kT}^t\gamma\,dt'}\int_{kT}^t\|h\|^2_{L_\infty(\Omega)}\|\nabla\theta\|^2_{L_2(\Omega)}e^{c_1't'-c_2'\int_{kT}^{t'}\gamma\,dt''}\,dt'\\ &\quad+e^{-c_1'(t-kT)+c_2'\int_{kT}^t\gamma\,dt'}\|\nabla\varphi(kT)\|^2_{L_2(\Omega)},\end{aligned}$$

where $\gamma=\|v(t)\|^2_{L_\infty(\Omega)}$. From (6.19) we get

$$\begin{aligned}(6.20)\qquad &\|\nabla\varphi(t)\|^2_{L_2(\Omega)}\le e^{-c_1't+c_2'\int_{kT}^t\gamma\,dt'}\\ &\cdot\int_{kT}^t\|h(t')\|^2_{L_\infty(\Omega)}e^{-c_1(t'-kT)+c_2\int_{kT}^{t'}\gamma\,dt''+c_1't'-c_2'\int_{kT}^{t'}\gamma\,dt''}\,dt'\|\nabla\theta(kT)\|^2_{L_2(\Omega)}\\ &+e^{-c_1'(t-kT)+c_2'\int_{kT}^t\gamma\,dt'}\|\nabla\varphi(kT)\|^2_{L_2(\Omega)}.\end{aligned}$$

We assume that c_1 in (6.12) is equal to c_1' in (6.18). Then from (6.20) we get

$$(6.21)\quad \|\nabla\varphi(t)\|_{L_2(\Omega)}^2 \le e^{-c_1t+c_2'\int_{kT}^t\gamma\,dt'}\int_{kT}^t\|h(t')\|_{L_\infty(\Omega)}^2\,dt' e^{c_1kT+c_3\int_{kT}^{(k+1)T}\gamma\,dt} \cdot\|\nabla\theta(kT)\|_{L_2(\Omega)}^2 + e^{-c_1(t-kT)+c_2'\int_{kT}^t\gamma\,dt'}\|\nabla\varphi(kT)\|_{L_2(\Omega)}^2.$$

Setting $t=(k+1)T$ and using the Poincaré inequality we get (6.16). ∎

LEMMA 6.4. *Assume that there exists a local solution (v,p,θ) to problem* (1.2) *in $[kT,(k+1)T]$ and there exists $\delta>0$ such that*

$$(6.22)\quad \begin{aligned}&\|f(t)\|_{H^1(\Omega)}\le\|f(kT)\|_{H^1(\Omega)}e^{-\delta(t-kT)},\\ &\|g(t)\|_{L_2(\Omega)}\le\|g(kT)\|_{L_2(\Omega)}e^{-\delta(t-kT)},\qquad t\in(kT,(k+1)T],\end{aligned}$$

and $f_3|_{S_2}=0$. Then

$$(6.23)\quad \|h((k+1)T)\|_{H^1(\Omega)}^2\le e^{-c_1T+\alpha(A(k,T))}\|h(kT)\|_{H^1(\Omega)}^2 + ce^{-\delta T+\alpha(A(k,T))}(\alpha(A(k,T))\|f(kT)\|_{L_2(\Omega)}^2+\|g(kT)\|_{L_2(\Omega)}^2),$$

where the constants c, c_1 are independent of T, $c_1>2\delta$, and α is an increasing positive function of $A(k,T)$, which in view of (6.22) *and* (4.32) *does not increase with T.*

Proof. Multiplying $(3.1)_1$ by $\operatorname{div}\mathbb{T}(h,q)$ and integrating the result over Ω yields

$$(6.24)\quad \int_\Omega h_t\cdot\operatorname{div}\mathbb{T}(h,q)\,dx-\int_\Omega|\operatorname{div}\mathbb{T}(h,q)|^2\,dx = -\int_\Omega(v\cdot\nabla h+h\cdot\nabla v)\operatorname{div}\mathbb{T}(h,q)\,dx + \int_\Omega(\alpha_{,\theta}\varphi f+\alpha g)\operatorname{div}\mathbb{T}(h,q)\,dx.$$

Repeating the considerations from Lemma 6.1 we have

$$(6.25)\quad \int_\Omega h_t\cdot\operatorname{div}\mathbb{T}(h,q)\,dx=-\frac{1}{4}\frac{d}{dt}\int_\Omega|\mathbb{D}(h)|^2\,dx+\int_{S_2}h_{3,t}T_{33}(h,q)\,dS_2,$$

where the last term equals

$$\int_{S_2}h_{3,t}(2h_{3,x_3}+q)\,dS_2=\int_{S_2}h_{3,t}f_3\,dS_2.$$

Since we do not know how to cope with this term, we have assumed that

$$(6.26)\quad f_3|_{S_2}=0.$$

Then (6.25) implies

$$\frac{d}{dt}\frac{1}{4}\int_\Omega |\mathbb{D}(h)|^2\,dx + \int_\Omega |\operatorname{div}\mathbb{T}(h,q)|^2\,dx \le \varepsilon\int_\Omega |\operatorname{div}\mathbb{T}(h,q)|^2\,dx + c(1/\varepsilon)(\|v\|^2_{L_\infty(\Omega)} + \|\nabla v\|^2_{L_3(\Omega)})\|h\|^2_{H^1(\Omega)} + c(1/\varepsilon)(\|\varphi\|^2_{L_\infty(\Omega)}\|f\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega)}). \tag{6.27}$$

Repeating the considerations from the proof of Lemma 6.1 we obtain

$$\int_\Omega |\mathbb{D}(h(t))|^2\,dx \le c_3 e^{-c_1 t + c_2\int_{kT}^t(\gamma+\beta)\,dt'}\Big(\int_{kT}^t \|h\|^2_{L_\infty(\Omega)} e^{(c_1-2\delta)t'}\,dt'\|f(kT)\|^2_{L_2(\Omega)} + \int_{kT}^t e^{(c_1-2\delta)t'}\,dt'\|g(kT)\|^2_{L_2(\Omega)}\Big) + e^{-c_1(t-kT)+c_2\int_{kT}^t(\gamma+\beta)\,dt'}\int_\Omega |\mathbb{D}(h(kT))|^2\,dx, \tag{6.28}$$

where $\gamma = \|v\|_{L_\infty(\Omega)}$ and $\beta = \|v\|_{L_3(\Omega)}$. We estimate

$$\int_{kT}^t \|h(t')\|^2_{L_\infty(\Omega)} e^{(c_1-2\delta)t'}\,dt' = \int_{kT}^{t'} \|h(t'')\|^2_{L_\infty(\Omega)}\,dt' e^{(c_1-2\delta)t'}\Big|_{kT}^t + (2\delta - c_1)\int_{kT}^t\int_{kT}^{t'} \|h(t'')\|^2_{L_\infty(\Omega)}\,dt'' e^{(c_1-2\delta)t'}\,dt' \le \int_{kT}^t \|h(t')\|^2_{L_\infty(\Omega)}\,dt' e^{(c_1-2\delta)t}. \tag{6.29}$$

Using (6.29) in (6.28), setting $t = (k+1)T$ and using the Korn inequality we obtain (6.23). ∎

To prove the global existence of solution to problem (1.1) we have to prove inequalities (6.1) for any $k \in \mathbb{N}_0$. For this purpose we use Lemmas 6.1–6.4. Let us introduce some assumptions. Assume that for some $\delta > 0$,

$$\begin{aligned} &\|f(t)\|_{H^1(\Omega)} \le \|f(kT)\|_{H^1(\Omega)} e^{-\delta(t-kT)},\\ &\|g(t)\|_{L_2(\Omega)} \le \|g(kT)\|_{L_2(\Omega)} e^{-\delta(t-kT)}. \end{aligned} \tag{6.30}$$

Moreover, assume that

$$
\begin{aligned}
&ce^{-\delta T+\alpha(A(T))}\|f(0)\|_{L_2(\Omega)}^2+e^{-c_1T+\alpha(A(T))}\|v(0)\|_{H^1(\Omega)}\le\|v(0)\|_{H^1(\Omega)}^2,\\
&e^{-c_1T+\alpha(A(T))}\|\nabla\theta(0)\|_{L_2(\Omega)}^2+\|\theta(0)\|_{L_2(\Omega)}^2\le\|\theta(0)\|_{H^1(\Omega)}^2,\\
&e^{-c_1T+\alpha(A(T))}(\|\nabla\theta(0)\|_{L_2(\Omega)}^2\alpha(A(T))+\|\varphi(0)\|_{H^1(\Omega)}^2)\le\|\varphi(0)\|_{H^1(\Omega)}^2,\\
&e^{-c_1T+\alpha(A(T))}\|h(0)\|_{H^1(\Omega)}^2+ce^{-\delta T+\alpha(A(T))}(\alpha(A(T))\|f(0)\|_{L_2(\Omega)}^2\\
&\qquad\qquad+\|g(0)\|_{L_2(\Omega)}^2)\le\|h(0)\|_{H^1(\Omega)}^2,
\end{aligned}
\tag{6.31}
$$

where $c_1 > 2\delta$, α is a polynomial and $A(T) = A(0, T)$.

Proof of Main Theorem. Take $k = 0$. Then in view of assumptions (6.30), (6.31) we obtain

$$
\begin{aligned}
&\|v(T)\|_{H^1(\Omega)}\le\|v(0)\|_{H^1(\Omega)},\\
&\|\theta(T)\|_{H^1(\Omega)}\le\|\theta(0)\|_{H^1(\Omega)},\\
&\|\varphi(T)\|_{H^1(\Omega)}\le\|\varphi(0)\|_{H^1(\Omega)},\\
&\|h(T)\|_{H^1(\Omega)}\le\|h(0)\|_{H^1(\Omega)}
\end{aligned}
\tag{6.32}
$$

and so $A(1, T) \le A(0, T)$.

Take $k = 1$. Then in view of (6.31) we can repeat the proof of Theorem 1.1 in the interval $[T, 2T]$. The assumptions (6.1) imply

$$
\begin{aligned}
&\|v(2T)\|_{H^1(\Omega)}\le\|v(T)\|_{H^1(\Omega)}\le\|v(0)\|_{H^1(\Omega)},\\
&\|\theta(2T)\|_{H^1(\Omega)}\le\|\theta(T)\|_{H^1(\Omega)}\le\|\theta(0)\|_{H^1(\Omega)},\\
&\|\varphi(2T)\|_{H^1(\Omega)}\le\|\varphi(T)\|_{H^1(\Omega)}\le\|\varphi(0)\|_{H^1(\Omega)},\\
&\|h(2T)\|_{H^1(\Omega)}\le\|h(T)\|_{H^1(\Omega)}\le\|h(0)\|_{H^1(\Omega)},
\end{aligned}
\tag{6.33}
$$

so $A(2, T) \le A(1, T)$. Repeating the above considerations we prove the Main Theorem. ∎

REMARK. Since we assumed the decay estimates

$$
\|f(t)\|_{H^1(\Omega)}\le\|f(0)\|_{H^1(\Omega)}e^{-\delta t},\qquad\|g(t)\|_{L_2(\Omega)}\le\|g(0)\|_{L_2(\Omega)}e^{-\delta t},
\tag{6.34}
$$

we find that for any $k \in \mathbb{N}_0$, $A(k, T)$ decreases with time. Therefore for T sufficiently large, $-c_1T + \alpha(A(k, T))$ is as small as we need.

Instead of the decay estimates (6.34) it is sufficient to assume that all time integrals of norms of f increase less than linearly. Then the above assertion holds too.

Acknowledgements. The paper is partially supported by Polish Grant No. NN 201 396 937.

References

[1] W. Alame, *On existence of solutions for the nonstationary Stokes system with boundary slip conditions*, Appl. Math. (Warsaw) 32 (2005), 195–223.

[2] P. Kacprzyk, *Global regular nonstationary flow for the Navier–Stokes equations in a cylindrical pipe*, Appl. Math. (Warsaw) 34 (2007), 289–307.

[3] P. Kacprzyk, *Long time existence of solutions to 2D Navier–Stokes equations with inflow-outflow and heat convection*, Topol. Methods Nonlinear Anal. 37 (2011), 342–355.

[4] O. Ladyzhenskaya, V. Solonnikow and N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967 (in Russian).

[5] O. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Flow*, Nauka, Moscow, 1970 (in Russian).

[6] B. Nowakowski and W. M. Zajączkowski, *Global existence of solutions to Navier–Stokes equations in cylindrical domains*, Appl. Math. (Warsaw) 36 (2009), 169–182.

[7] J. Rencławowicz and W. M. Zajączkowski, *Large time regular solutions to the Navier–Stokes equations in cylindrical domains*, Topol. Methods Nonlinear Anal. 32 (2008), 69–87.

[8] J. Socała and W. M. Zajączkowski, *Long time estimate of solutions to 3d Navier–Stokes equations coupled with heat convection*, Appl. Math. (Warsaw) 39 (2012), 23–41.

[9] J. Socała and W. M. Zajączkowski, *Long time existence of regular solutions to 3d Navier–Stokes equations coupled with heat convection*, Appl. Math. (Warsaw) 39 (2012), 231–242.

[10] W. M. Zajączkowski, *Global regular nonstationary flow for the Navier-Stokes equations in a cylindrical pipe*, Topol. Methods Nonlinear Anal. 26 (2005), 221–286.

[11] W. M. Zajączkowski, *Global special regular solutions to the Navier–Stokes equations in a cylindrical domain without the axis of symmetry*, Topol. Methods Nonlinear Anal. 24 (2004), 60–105.

[12] W. M. Zajączkowski, *Long time existence of regular solutions to Navier–Stokes equations in cylindrical domains under boundary slip conditions*, Studia Math. 169 (2005), 243–285.

[13] W. M. Zajączkowski, *On global regular solutions to the Navier–Stokes equations in cylindrical domains*, Topol. Methods Nonlinear Anal. 37 (2011), 55–85.

[14] W. M. Zajączkowski, *Global special regular solutions to the Navier–Stokes equations in a cylindrical domain under boundary slip conditions*, Gakuto Int. Ser. Math. Sci. Appl. 21 (2004), 180 pp.

[15] W. M. Zajączkowski, *Global special regular solutions to the Navier–Stokes equations in axially symmetric domains under boundary slip conditions*, Dissertationes Math. 432 (2005), 138 pp.

[16] W. M. Zajączkowski, *Global regular solutions to the Navier–Stokes equations in a cylinder*, in: Banach Center Publ. 74, Inst. Math., Polish Acad. Sci., 2006, 235–255.

Piotr Kacprzyk
Institute of Mathematics and Cryptology
Cybernetics Faculty
Military University of Technology
Kaliskiego 2
00-908 Warszawa, Poland
E-mail: pk_wat@wp.pl

Received 2.2.2012
and in final form 12.4.2012 (2686)