ANNALES
POLONICI MATHEMATICI
111.2 (2014)

Parallel hypersurfaces

by Barbara Opozda (Kraków) and Udo Simon (Berlin)

Dedicated to the memory of Franki Dillen

Abstract. We investigate parallel hypersurfaces in the context of relative hypersurface geometry, in particular including the cases of Euclidean and Blaschke hypersurfaces. We describe the geometric relations between parallel hypersurfaces in terms of deformation operators, and we apply the results to the parallel deformation of special classes of hypersurfaces, e.g. quadrics and Weingarten hypersurfaces.

1. Introduction. Our paper deals with two classical topics: parallel hypersurfaces and Weingarten hypersurfaces. Our aim is twofold:

• We investigate parallel hypersurfaces in $\mathbb{R}^{n+1}$ in the context of relative hypersurface geometry; in particular this includes Euclidean and unimodular-affine Blaschke theory. We start with the following situation: We consider an immersion $x : M \to \mathbb{R}^{n+1}$ of a differentiable manifold and equip this hypersurface with a relative normal y and a conormal Y; then we define the notion of a parallel map $\tilde{x} : M \to \mathbb{R}^{n+1}$. Later, in more special cases, we consider a one-parameter family of hypersurfaces

$$x_t := x + ty, \qquad t \in \mathbb{R},$$

that are parallel to the given hypersurface immersion x. We show that parallelity in both cases can be described in terms of respective deformation operators L and L_t. In Section 3 we establish a list of invariants and show how the parallel deformation depends on L and L_t. This part is more or less algebraic and is the same for all relative normalizations of a given hypersurface.

It is of particular importance to realize that the concept of extrinsic relative curvature theory, defined via the relative shape operator, not only includes affine extrinsic curvature theories, but also the Euclidean case; see

2010 *Mathematics Subject Classification*: 53A15, 53A07, 53B25, 53C42.
Key words and phrases: parallel submanifolds, relative hypersurfaces, Weingarten hypersurfaces, quadrics, parallel shape operator.

DOI: 10.4064/ap111-2-1

© Instytut Matematyczny PAN, 2014

e.g. [9, Section 6.4.2]. Thus our relative results hold true in different special hypersurface geometries.

Considering so called minimal (hyper)surfaces, we point out that our results hold in the Euclidean and Blaschke cases, as the Euler–Lagrange equations in both theories are equivalent to the vanishing of the trace of the corresponding shape operator. Recall that the notion of *minimal* in unimodular theory is due to Blaschke, but it is not really appropriate, as, following Calabi, locally strongly convex surfaces satisfying the unimodular Euler–Lagrange equations are *maximal*; see e.g. [5].

• There appear different types of Weingarten hypersurfaces in the literature depending on the choice of curvature relations that are studied. We list four types of such relations:

- H: Relations between the normalized elementary symmetric functions H_k of the principal curvatures.
- P: Relations between the normalized elementary symmetric functions P_k of the principal radii of curvature.
- k: Relations between the principal curvatures k_i.
- R: Relations between the principal radii of curvature R_i.

Corresponding subclasses of the foregoing types can be defined by *linear relations* between curvature invariants of the same type; different types might lead to different classes, e.g. the two classes of *linear Weingarten surfaces* in Euclidean $\mathbb{R}^3$ given by

$$\sum_{i=0,1,2} a_i H_i = 0, \qquad a_i \in \mathbb{R},$$

and by

$$\sum_{i=0,1,2} a_i k_i = 0, \qquad a_i \in \mathbb{R},$$

are different.

We list some papers devoted to Weingarten hypersurfaces in the references: [2], [3], [4], [6], [12].

To describe the notion of parallelity in more detail, let us give the following definition:

Definition 1.1. Let

$$x : M \to \mathbb{R}^{n+1} \tag{1}$$

be a hypersurface. A mapping $\tilde{x} : M \to \mathbb{R}^{n+1}$ is called *parallel* to x if their differentials satisfy

$$d\tilde{x}(T_pM) \subset dx(T_pM)$$

for every $p \in M$. In that case there is a $(1,1)$-tensor field L on M for which

$$d\tilde{x} = dx \circ L.$$

The tensor field L is called the *deformation tensor* from x to $\tilde{x}$. The mapping $\tilde{x}$ is an immersion if and only if L is non-singular at each point of M.

Consider a hypersurface immersion (1) of a connected, oriented C^∞-manifold of dimension $n \geq 2$. If a hypersurface is non-degenerate then it has infinitely many non-trivial (non-Euclidean and non-centroaffine) relative normalizations.

As above denote by (Y, y) a relative normalization where Y denotes a conormal field and y a transversal field along the hypersurface. The family of mappings

$$x_t := x + ty$$

is called a *one-parameter family of parallel mappings* with respect to the fixed normalization chosen. We will make this more precise in Section 3 below.

The parallel deformation of a hypersurface with relative normalization leads to hypersurfaces within the same large class of relative hypersurfaces. In Section 3 we make a general study of relative invariants that do not depend on the type of relative normalization. Studying special relative subclasses we learn that the type of relative normalization might be of importance, and the deformation of a special relative class of hypersurfaces might depend on the normalization. The following examples and the theorem illustrate this.

(i) The parallel deformation of a hypersurface with Euclidean normalization leads to hypersurfaces with the same Euclidean normal.
(ii) The parallel deformation of a hypersurface with unimodular affine (so called Blaschke) normalization leads to hypersurfaces with relative normalization, and only for special classes of Blaschke hypersurfaces is the normalization of this class of parallel hypersurfaces again of Blaschke type (see Corollary 6.7).

Theorem 1.2. *Consider a parallel family x_t as above and assume that $x = x_0$ is a centered hyperquadric* (*that is, a quadric with a center*).

- *Normalize x by the Euclidean unit normal field. Assume that x has no umbilics on M. Then $x = x_0$ is the* **only** *hyperquadric in the family x_t.*
- *Normalize x by the affine* (*Blaschke*) *normal field; then* **any** *x_t is a centered hyperquadric.*

We give a proof in Section 4 below. There we study the parallel deformation of relative spheres and quadrics, while we treat relative hypersurfaces with parallel shape operator in Section 5. Section 6 is devoted to the study of parallel deformations of Weingarten hypersurfaces in relative geometry.

A typical result concerns linear Weingarten hypersurfaces. To formulate it, we define:

Definition 1.3. We say that a hypersurface x with relative normalization (Y, y) is *H-linear Weingarten* if there are real numbers $a_0, \dots, a_n$ such that $\sum_{i=1}^{n} a_i^2 \neq 0$ and

$$H_n a_n + \cdots + H_1 a_1 + H_0 a_0 = 0 \tag{2}$$

at each point of M, where $H_0 := 1$ by definition. We call the polynomial

$$W(t) := a_0 t^n + \cdots + a_{n-1} t + a_n \tag{3}$$

the *associated polynomial* for the H-linear Weingarten hypersurface x.

Theorem 1.4. *Let $x : M \to \mathbb{R}^{n+1}$ be a hypersurface immersion with relative normalization (Y, y) and diagonizable shape operator S. Assume that the parallel hypersurface x_t is an immersion. If (x, Y, y) is an H-linear Weingarten hypersurface satisfying* (2) *then (x_t, Y, y) is also H-linear Weingarten. It satisfies the condition*

$$H_n(t)W(t) + \frac{1}{1!}H_{n-1}(t)W'(t) + \cdots + \frac{1}{n!}H_0(t)W^{(n)}(t) = 0. \tag{4}$$

Finally, in Section 6 we show that H-linear Weingarten hypersurfaces (and also some other special surfaces) admit parallel (hyper)surfaces with special extrinsic curvature properties. An example of such a result is the following proposition that we will prove in Section 6.3.

Proposition 1.5. *Let $x : M \to \mathbb{R}^3$ be an H-linear Weingarten surface satisfying $a_2 H_2 + a_1 H_1 = 0$ with $a_1, a_2 \in \mathbb{R}$ and $H_2 \leq 0$ on M. If $a_1 \neq 0$ then there exists $t \in \mathbb{R}$ such that x_t is an immersion and $H_1(t) = 0$ on M.*

2. Basic properties. In this section we summarize basic properties of relative hypersurfaces; we refer to the monographs [5] and [9].

2.1. Relative hypersurfaces. Consider the real vector space $\mathbb{R}^{n+1}$ and its dual vector space $\mathbb{R}^{(n+1)*}$, where the duality is described in terms of a non-degenerate scalar product

$$\langle\ ,\ \rangle : \mathbb{R}^{(n+1)*} \times \mathbb{R}^{n+1} \to \mathbb{R};$$

both spaces can also be considered as real affine spaces. By det and det* we denote an arbitrary fixed pair of dual determinant forms on the vector spaces $\mathbb{R}^{n+1}$ and $\mathbb{R}^{(n+1)*}$, resp., and by $\overline{\nabla}$ we denote the canonical flat connections on both $\mathbb{R}^{n+1}$ and $\mathbb{R}^{(n+1)*}$.

Let M be a connected, oriented, C^∞-differentiable manifold of dimension $n \geq 2$, and $x : M \to \mathbb{R}^{n+1}$ a hypersurface immersion. A *normalization* of x is a pair (Y, z) with $\langle Y, z\rangle = 1$, where $z : M \to \mathbb{R}^{n+1}$ is an arbitrary *transversal field*, and $Y : M \to \mathbb{R}^{(n+1)*}$, satisfying $\langle Y, dx(v)\rangle = 0$ at any $p \in M$ and for all tangent vectors $v \in T_pM$, is a *conormal field* of x. While a transversal field z extends a tangential basis to the ambient space, a conormal field fixes the tangent plane. A *normalized hypersurface* is a triple (x, Y, z).

The transversal field z induces a volume form ω on M by

$$\omega(v_1, \ldots, v_n) := \det(dx(v_1), \ldots, dx(v_n), z),$$

where $(v_1, \ldots, v_n)$ is an arbitrary frame; obviously this *induced volume form* depends on the choice of z (if det is fixed).

2.1.1. *Structure equations.* The geometry of the triple (x, Y, z) can be described in terms of the induced volume form ω and further geometric invariants, defined via the *structure equations* of *Gauß* and *Weingarten*:

$$\overline{\nabla}_v dx(w) = dx(\nabla_v w) + h(v, w)z, \tag{5}$$

$$dz(v) = dx(-S(v)) + \tau(v)z. \tag{6}$$

Here and in the following $u, v, w, \ldots$ denote tangent vectors and fields. The *induced connection* ∇ is torsion free, h is bilinear and symmetric, S is the *shape* or *Weingarten operator* and τ is a 1-form, the *connection form*; the sign in front of S in the Weingarten equation is a convention corresponding to an appropriate choice of the orientation of z. The coefficients in the structure equations depend on the normalization, they are invariant under the affine group of transformations in $\mathbb{R}^{n+1}$.

2.1.2. *Non-degenerate hypersurfaces.* In the following, in general, we restrict to non-degenerate hypersurfaces defined as follows: A hypersurface x is *non-degenerate* if the bilinear form h in the Gauß structure equation is non-degenerate; it is well known that this property is independent of the choice of the normalization as all such symmetric bilinear forms are conformally related, defining a *conformal class* $\mathfrak{C}$. Thus, on a non-degenerate hypersurface, any transversal field z induces a semi-Riemannian *metric* $h \in \mathfrak{C}$ with *Levi-Civita connection* $\nabla(h)$ and *Riemannian volume form* $\omega(h)$.

The non-degeneracy of x is equivalent to the fact that any conormal field Y itself is an immersion $Y : M \to \mathbb{R}^{(n+1)*}$ with transversal position vector Y. The associated *Gauß structure equation* reads

$$\overline{\nabla}_v dY(w) = dY(\nabla^*_v w) + \frac{1}{n-1}\mathrm{Ric}^*(v, w)(-Y), \tag{7}$$

where the *conormal connection* ∇^* is torsion free and *Ricci-symmetric*, i.e. its Ricci tensor Ric^* is symmetric. The Ricci symmetry is equivalent to the existence of a local ∇^*-parallel volume form ω^* on M which is unique modulo a non-zero constant factor. We have

$$\omega^*(v_1, \ldots, v_n) := \det{}^*(dY(v_1), \ldots, dY(v_n), -Y) \tag{8}$$

for an arbitrary local frame. It is well known (see e.g. [10]) that all conormal connections are projectively related; we denote the class of all conormal connections by $\mathfrak{P}^*$. Moreover, for a non-degenerate hypersurface there is a bijective correspondence between the class of conormal fields and the conformal class $\mathfrak{C}$ of semi-Riemannian metrics; that allows one to define a metric via a conormal. This can be seen from the relation

$$h(v,w) = -\langle dY(v), dx(w)\rangle. \tag{9}$$

2.1.3. *Relative normalizations.* A normalization (Y, y) of x is called *relative* if $\tau \equiv 0$ in the structure equation of Weingarten. One can easily prove that any non-degenerate hypersurface admits infinitely many different relative normalizations. A triple (x, Y, y) is called a *relative hypersurface* if x is non-degenerate and its normalization (Y, y) is relative. Note that for such hypersurfaces the shape operator S is h-self adjoint and the induced connection ∇ is also Ricci-symmetric (see [9]). From now on we consider relative hypersurfaces only.

2.1.4. *The cubic form.* An important invariant in relative hypersurface geometry is the cubic form. It is defined as follows:

Denote by K the symmetric *difference tensor* with

$$K(u,v) := \tfrac{1}{2}(\nabla_u v - \nabla^*_u v).$$

Then the *cubic form* C, defined by $C(u,v,w) := h(K(u,v),w)$, is totally symmetric in its three arguments (see [9, Section 4.4.3]).

2.1.5. *Extrinsic curvature functions.* If S is diagonizable with eigenvalues k_i for $i = 1, \dots, n$, we denote by

$$H_l := \binom{n}{l}^{-1} \sum_{1\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l} \tag{10}$$

the *normalized elementary symmetric functions* for $l = 1, \dots, n$, and set $H_0 := 1$. We call k_i a (*relative*) *principal curvature* and the functions H_l extrinsic *higher mean curvature functions.* In case $\operatorname{rk} S = n$ we also consider the *principal radii of curvature* R_i and the corresponding normalized elementary symmetric functions

$$P_l := \binom{n}{l}^{-1} \sum_{1\le i_1<\cdots<i_l\le n} R_{i_1}\cdots R_{i_l},$$

where again $P_0 := 1$.

Except for Section 6, we assume the existence of an eigenbasis of the relative shape operator S at each point of M. In particular this is guaranteed for locally strongly convex relative hypersurfaces, while in the case of a Euclidean normalization we can omit the convexity assumption as then the shape operator S always has an eigenbasis. Below we will consider both types of extrinsic curvature functions: H_l and P_l.

2.2. Euclidean hypersurfaces. The Euclidean space $\mathbb{R}^{n+1}$ and its dual vector space are identified, and now the scalar product denotes the inner product. For a hypersurface the Euclidean unit normal μ and the associated conormal coincide, i.e. $(Y, y) = (\mu, \mu)$. We denote the *first fundamental form* by $I(u, v) := \langle dx(u), dx(v)\rangle$, and by II and III the *second* and the *third fundamental form*; II is an element of the conformal class $\mathfrak{C}$ of relative metrics, namely, in case of a Euclidean normalization, $h = II$ in the structure equations (5). A Euclidean hypersurface x is non-degenerate if and only if $\operatorname{rk} II = n$; this is equivalent to the fact that the Euclidean shape (Weingarten) operator S has maximal rank; unless otherwise stated, we will assume that $\operatorname{rk} S = n$ below.

If $\operatorname{rk} S = n$ then I and III are Riemannian metrics, II is semi-Riemannian (and Riemannian exactly if x is locally strongly convex). Recall that S is self adjoint with respect to all three fundamental forms, and that

$$II(u, v) = I(Su, v) \quad \text{and} \quad III(u, v) = I(Su, Sv).$$

The Levi-Civita connections $\nabla(I)$ and $\nabla(III)$ of I and III, resp., satisfy $\nabla(III)_u v = S^{-1}\nabla(I)_u Sv$ (see e.g. [11]).

From [9] we recall the following characterization; note that the covariant derivative $\nabla(III)\, II$ is totally symmetric.

Proposition 2.1. *Let x be a non-degenerate hypersurface in Euclidean space with absolute Gauß–Kronecker curvature $G := |H_n|$. Then x is a quadric if and only if, in local components,*

$$\nabla(III)_k II_{ij} + \frac{1}{n+2}(II_{ij}\partial_k \ln G + II_{jk}\partial_i \ln G + II_{ki}\partial_j \ln G) \equiv 0. \tag{11}$$

2.3. Blaschke hypersurfaces. Considering $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1*}$ as vector spaces, for each we have a one-dimensional vector space of determinant forms, and the duality of determinant forms is given by a pairing $(\det, \det^*)$.

Consider a non-degenerate hypersurface in affine space $\mathbb{R}^{n+1}$ with a given relative normalization (Y, y). Then any pair $(\det, \det^*)$ induces a pair of volume forms on M by

$$\begin{aligned} \omega(v_1, \dots, v_n) &:= \det(dx(v_1), \dots, dx(v_n), y), \\ \omega^*(v_1, \dots, v_n) &:= \det(dY(v_1), \dots, dY(v_n), Y), \end{aligned}$$

where $(v_1, \dots, v_n)$ is a frame. Any other dual pair $(\det^\sharp, \det^{\sharp *})$ with

$$\det{}^\sharp = c \det \quad \text{and} \quad \det{}^{\sharp *} = c^{-1} \det{}^*$$

analogously induces volume forms $\omega^\sharp$ and $\omega^{\sharp *}$. On any frame, we have

$$\omega(v_1, \dots, v_n) \cdot \omega^*(v_1, \dots, v_n) = \omega^\sharp(v_1, \dots, v_n) \cdot \omega^{\sharp *}(v_1, \dots, v_n).$$

Recall that the volume forms $\det$ and $\det^*$ are parallel with respect to the canonical flat connections in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{(n+1)*}$, resp. *Modulo non-zero con-*

stants there is a unique volume form that is parallel to the canonical flat connection in the vector space considered.

Analogously the induced volume forms ω and ω^* are parallel with respect to the connections ∇ and ∇^*, resp., and their parallelity determines them uniquely modulo non-zero constants.

To characterize so called Blaschke hypersurfaces within the class of relative hypersurfaces in affine space $\mathbb{R}^{n+1}$ without fixing a determinant form in the ambient space, we use the foregoing facts and proceed as follows:

Definition 2.2. A relative hypersurface (x, Y, y) in affine space $\mathbb{R}^{n+1}$ is called a *Blaschke hypersurface* if $\omega = c\omega^*$ for some non-zero constant $c \in \mathbb{R}$.

Fixing an orientation on the ambient space, the induced volume forms will have the same orientation and thus c will be positive. Moreover, we can compare the induced oriented volume forms ω and ω^* with the oriented Riemannian volume form $\omega(h)$ of the relative metric h and characterize a Blaschke hypersurface in affine space $\mathbb{R}^{n+1}$ as follows (compare [9, Sections 4.4.7–4.4.9 and 6.2]):

Lemma 2.3. *A relative hypersurface (x, Y, y) in affine space $\mathbb{R}^{n+1}$ is a Blaschke hypersurface if one of the following conditions is satisfied:*

- *any two of the three oriented volume forms ω, ω^*, $\omega(h)$ coincide modulo constant positive factors;*
- *the* Tchebychev form T *vanishes identically* (*apolarity condition*), *where with the above notations*

$$2nT := d \ln \frac{\omega(v_1, \ldots, v_n)}{\omega^*(v_1, \ldots, v_n)}.$$

Lemma 2.4. *A non-degenerate Blaschke hypersurface is a hyperquadric if and only if $C = 0$.*

3. Parallel hypersurfaces

3.1. The deformation operator. For a relative hypersurface (x, Y, y) on M consider the one-parameter family of mappings

$$x_t := x + ty.$$

If x_t is an immersion we say that t is *admissible.* Consider the differential

$$dx_t = dx + tdy = dx(\mathrm{id} - tS),$$

which shows that the mapping x_t is parallel to x. Moreover, if t is admissible then $(Y_t, y_t) := (Y, y)$ is a relative normalization for x_t. This can be seen as follows:

As $\langle Y, dx_t(v)\rangle = 0$ for all tangent vectors v, and as y is transversal to the tangent plane of x_t, we immediately see that $(Y_t, y_t) := (Y, y)$ is a normalization of x_t, and the following lemma shows that this normalization is again relative if L_t has maximal rank.

LEMMA 3.1.

(i) *x_t is an immersion if and only if* $\operatorname{rk} L_t = n$, *as*

$$dx_t(v) = dx \circ L_t(v)$$

for any tangent vector v. The set of points of M where x_t is an immersion is open in M.

(ii) *If* $\operatorname{rk} L_t = n$ *then the pair $(Y_t, y_t) := (Y, y)$ defines a relative normalization of each* (*admissible*) *hypersurface immersion x_t with relative shape operator $S_t = L_t^{-1} S$.*

Proof. For the proof of (ii) we use the definition of x_t and the Weingarten structure equation to calculate

$$-dx_t(S_t v) = dy_t(v) = dy(v) = -dx(Sv) = -dx_t(L_t^{-1} Sv). \ \blacksquare$$

We call

$$L_t := \mathrm{id} - tS$$

the *deformation operator* of the deformation $x \mapsto x_t$; this operator completely describes the deformation; in particular this means that we will be able to describe the deformation of all intrinsic and extrinsic invariants in terms of L_t.

REMARK 3.2. (i) For the following, we emphasize that, when we consider a one-parameter parallel family $\{x_t\}$, we assume, in general, that t is *admissible*, meaning that $\operatorname{rk} L_t$ is maximal for all $p \in M$; moreover, in general, the mark "t" is used only for admissible parameters t. We drop the assumption that t is admissible in Section 6; there, under appropriate assumptions, we will prove the existence of admissible parameter values t.

(ii) If (x, Y, y) is a Blaschke hypersurface then the parallel deformation $x \mapsto x_t$ in general does not lead to a Blaschke hypersurface $(x_t, Y_t, y_t) = (x_t, Y, y)$ again; see below.

(iii) Note that $L_t = \mathrm{id} - tS$ is positive definite for sufficiently small t in a sufficiently small neighborhood of any point, and if x_t is considered in case $\operatorname{rk} L_t = n$ only, we necessarily have $\det L_t > 0$ for such sufficiently small t. But note that $\det L_t$ is a polynomial in t with only finitely many zeros (at a fixed point of M) and that a parallel deformation might be of interest also when $\det L_t < 0$.

3.2. Structure equations for x_t. As stated above, the geometry of a relative hypersurface (x_t, Y_t, y_t) can be described in terms of its induced volume form and the induced geometric invariants defined via the structure equations of Gauß and Weingarten. For the invariants of a relative hypersurface $(x_t, Y_t, y_t) = (x_t, Y, y)$ we use an obvious notation with appropriate mark "t".

$$\begin{aligned}\overline{\nabla}_v dx_t(w) &= dx_t(\nabla(t)_v w) + h_t(v,w)\, y_t,\\ dy_t(v) &= dx_t(-S_t(v)),\\ \overline{\nabla}_v dY_t(w) &= dY_t(\nabla^*(t)_v w) + \frac{1}{n-1}\mathrm{Ric}\,(t)^*(v,w)(-Y_t);\end{aligned}$$

here and in the following the *-notation marks invariants of the ∇^*-geometry.

3.3. Relation between the invariants of x and x_t. Straightforward computations give the following relations between the invariants of x and x_t, and how they depend on the deformation operator L_t; moreover we list further properties of these invariants.

Considering the relative shape operators S and S_t, we now restrict to the case where S is diagonalizable, and thus also L_t and S_t are diagonalizable. In this case we have (joint) eigenbases for S and S_t. In a few cases (which we explicitly state) we allow that $\mathrm{rk}\, S < n$. We make frequent use of the fact that L_t satisfies the Codazzi equations relative to the connection ∇, which implies the intrinsic properties stated in (i) and (iii) below; for details see [7], [8], [11].

3.4. Invariants of x and x_t. Assume that L_t has maximal rank and S is diagonalizable. The following is a list of relations between invariants of x and x_t.

(i) *Relative metric and its volume form*:

(a) (see [9, Section 4.8.3])

$$\begin{aligned}h_t(u,v) = h(L_t u, v) = h(u, L_t v) &= h(u,v) - t h(u, Sv)\\ &= h(u,v) - t\frac{1}{n-1}\mathrm{Ric}^*(u,v)\\ &= h(u,v) - t\frac{1}{n-1}\mathrm{Ric}^*(t)(u,v).\end{aligned}$$

(b) $\omega(h_t) = \det(L_t)^{n/2}\omega(h)$,
(c) $\nabla(h) - \nabla(h_t) = K - K(t)$.

(ii) *Connections and their parallel volume forms*:

(d) $\nabla(t)_u v = L_t^{-1}\nabla_u L_t v$, briefly $\nabla(t) = L_t^{-1}\nabla L_t$,
(e) $\nabla^*(t) = \nabla^*$,
(f) $\omega(t) = \det(L_t)^n\omega$ and $\omega^*(t) = \omega^*$; recall that parallel volume forms are unique modulo a non-zero constant factor.

(iii) *Intrinsic curvature*:

Denote by R and $R(t)$ the (1,3) curvature operators of the induced connections ∇ and $\nabla(t)$, resp., and denote by R^* and $R(t)^*$ the corresponding operators for ∇^* and $\nabla(t)^*$; then

(g) $L_t R(t)(u,v)w = R(u,v)L_t w$, and thus
(h) $R(t)(u,v)w = R(u,v)w + t[SR(t)(u,v)w - R(u,v)Sw]$,
(j) if $\operatorname{rk} S = n$ then

$$\begin{aligned} S^{-1}R(u,v)Sw &= R^*(u,v)w = R^*(t)(u,v)w \\ &= S_t^{-1}R(t)(u,v)S_t w. \end{aligned}$$

(iv) *Extrinsic curvature*:

(k) $S_t = L_t^{-1}S$,
(l) the operators S, L_t and S_t are self adjoint with respect to the relative metrics h and h_t, resp.
(m) if S has an eigenbasis then the operators S, L_t and finally S_t have the same eigenbasis (see e.g. [11], [8]); thus S, L_t and S_t pairwise commute; the corresponding eigenvalues satisfy:
(n) $k_i(t) = \frac{k_i}{1-tk_i}$ and, for $k_i(t) \neq 0$, $\frac{1}{k_i(t)} = R_i(t) = R_i - t$,
(p) $k_i = \frac{k_i(t)}{1+tk_i(t)}$ and $R_i = R_i(t) + t$.

(v) *Further properties*:

(q) if x is locally strongly convex and $\det L_t > 0$ then x_t is locally strongly convex; then h and also h_t are definite (and positive definite for an appropriate orientation of the normalization);
(r) the pairs (∇, S) and (∇, L_t) satisfy the Codazzi equations (see [9, Section 4.8.1]).

(vi) *Cubic form*:

(s) $K(t)(u,v) + \frac{t}{2}L_t^{-1}(\nabla_u S)v = K(u,v)$,
(t) $C(t)(u,v,w) = C(u,v,L_t w) - \frac{1}{2}th((\nabla_u S)v,w)$.

(vii) *Support function*:

Define the *support function* of x with respect to a fixed $c_0 \in \mathbb{R}$ by $\rho(c_0) := \langle Y, x - c_0\rangle$. Then $\rho(t)(c_0) = \rho(c_0) - t$.

Proof. The proof for $\omega(t)$ in (f) follows from (b) above and [9, Lemma 3.4.4.1.ii]. The proof of (s) uses the definition of K and (e):

$$\begin{aligned} (K(t) - K)(u,v) &= \tfrac{1}{2}(L_t^{-1}\nabla_u L_t v - \nabla_u v) \\ &= \tfrac{1}{2}L_t^{-1}(\nabla_u L_t v - L_t \nabla_u v) \\ &= \tfrac{1}{2}L_t^{-1}(\nabla_u L_t)v = -\tfrac{1}{2}tL_t^{-1}(\nabla_u S)v. \ \blacksquare \end{aligned}$$

3.5. Deformation invariants. If, in the following, a condition is satisfied for some t, it is satisfied for any admissible t. In other words, the foregoing list of invariants implies that the following invariants of a parallel relative family $\{x_t\}$ are independent of t:

(1) $L_t\nabla(t)L_t^{-1} = \nabla$,
(2) $\nabla^*(t) = \nabla(h_t) - K(t) = \nabla(h) - K = \nabla^*$,
(3) $\frac{1}{n-1}\mathrm{Ric}\,(t)^*(u,v) = h_t(S_t u, v) = h(Su,v) = \frac{1}{n-1}\mathrm{Ric}\,^*(u,v)$,
(4) $\det(L_t)^{-n/2}\omega(h_t) = \omega(h)$,
(5) $L_t R(t)(u,v)L_t^{-1}w = R(u,v)w$,
(6) $L_t S_t = S$,
(7) $R_i(t) + t = R_i$ for $i = 1,\ldots,n$,
(8) $x_t + R_i(t)y_t = x + R_i y$, and thus $x_t + P_1(t)y_t = x + P_1 y$,
(9) $\rho(t) + t = \rho$.

Remark 3.3. Following [9, Section 4.6], for $\mathrm{rk}\,S = n$ the mappings

$$y : M \to \mathbb{R}^{n+1}, \qquad Y : M \to \mathbb{R}^{(n+1)*}$$

are immersions themselves and then are interpreted as *relative spherical indicatrices* or *relative Gauß maps.* In this case the symmetric bilinear form $\hat{S}(u,v) := h(Su,v)$ is the joint *spherical metric* of both indicatrices.

If $\mathrm{rk}\,S = n$ then also $\mathrm{rk}\,S_t = n$, thus the spherical metric $\hat{S}_t$ is defined for all admissible t; then relation (3) in the foregoing list of deformation invariants states that *all relative hypersurfaces in the family* $\{x_t\}$ *have the same spherical metric.*

3.6. The deformation $x_t \mapsto x_{t+s}$. So far we only considered the deformation $x \mapsto x_t$. To study the deformations $x_t \mapsto x_{t+s}$ for admissible arguments, it is necessary to extend our notation as follows. Set

$$x_{t+s} := x_t + s y_t, \quad \text{where} \quad y_t = y,$$

then

$$dx_{t+s}(v) = (dx_t + s\,dx_t(-S_t))(v) = dx(\mathrm{id} - (t+s)S)(v).$$

To describe the deformation $x_t \mapsto x_{t+s}$ appropriately, we introduce the operator

$$L(t,s) := \mathrm{id} - sS_t \quad \text{with} \quad L(0,s) := L_s;$$

then

$$dx_{t+s}(v) = dx_t(L(t,s))(v).$$

One easily verifies:

Lemma 3.4.

$$L_t \cdot L(t,s) = L(0,t)\cdot L(t,s) = L(0,t+s) = L_{t+s},$$
$$L(0,t)\cdot L(t,-t) = \mathrm{id}.$$

From the foregoing lemma the set $\{L(t,s)\}$ forms a local one-parameter group.

3.7. Curvature relations. As emphasized above, considering extrinsic curvature in relative hypersurface theory, we assume that S has an eigenbasis. For simplicity assume now that there exists a maximal non-empty open interval

$$\mathfrak{I}(U) := (-\epsilon_1(U), \epsilon_2(U)),$$

where $0 < \epsilon_i(U) \in \mathbb{R}$ for $i = 1, 2$, such that L_t has maximal rank for $t \in \mathfrak{I}(U)$ and $p \in U$, where $U \subset M$ is open.

LEMMA 3.5. *Consider the one-parameter family $\{x_t\}$ on a chart $U \subset M$ and assume that there is a non-empty interval $\mathfrak{I}(U)$ as stated above. Then*

$$S_{t_2+t_1} = (L_{t_2})^{-1}(L_{t_1})^{-1}S = (L_{t_2})^{-1}S_{t_1}.$$

We have the following relations between curvature functions of x_t and x.

PROPOSITION 3.6.

$$\binom{n}{k} P_k(t) = \sum_{j=0}^{k} \binom{n}{j} P_j(t) t^{k-j}, \tag{12}$$

$$\binom{n}{k} P_k(t) = \sum_{j=0}^{k} \binom{n}{j} P_j(-t)^{k-j}, \tag{13}$$

$$\binom{n}{k} H_k(t) = \sum_{j=0}^{n-k} \binom{n}{n-j} H_{n-j}(t) \det L_t \cdot t^{k-j}. \tag{14}$$

Proof. The first relation follows directly from $R_i = R_i(t) + t$. For the last relation apply $H_k = P_{n-k}/P_n = H_n P_{n-k}$ for $k = 0, 1, \dots, n$. ∎

3.8. Parallel relative hypersurfaces in dimension $n = 2$. As before, we assume that S is diagonalizable. We consider a chart U for x without umbilics. Consequently, S_t is without umbilics for any admissible t. On U we choose a curvature line parametrization such that S has a diagonal representation, and h, h_t and S_t have the following matrix representations:

$$h : \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix}, \quad h_t : \begin{pmatrix} (1-k_1t)h_{11} & 0 \\ 0 & (1-k_2t)h_{22} \end{pmatrix},$$

$$S_t : \begin{pmatrix} \frac{k_1}{1-tk_1} & 0 \\ 0 & \frac{k_2}{1-tk_2} \end{pmatrix}.$$

3.9. Parallel deformation in special geometries. We study families of parallel hypersurfaces in special geometries, namely in the most important relative geometries (see [9]). The parallel deformation of a centroaffine hypersurface is of no interest as its transversal field is given by its position vector. Thus here we restrict to the study of parallel hypersurfaces in Euclidean and in Blaschke geometry.

3.9.1. *Parallel hypersurfaces in Euclidean geometry.* It is trivial to verify that the parallel deformation of a hypersurface with Euclidean normalization gives a family of parallel hypersurfaces with the same parallel Euclidean normal μ. We refer to [11] and recall that, for $x_t = x + t\mu$, the first and the second fundamental forms are related by

$$I(t)(u,v) = I(L_t u, L_t v), \tag{15}$$

$$II(t)(u,v) = II(L_t u, v) = II(u, L_t v), \tag{16}$$

for all t (which here and later again means *for all admissible* t). As the pair $(\nabla(I), L_t)$ satisfies the Codazzi equations, we have (see [11])

$$\nabla(I(t))_u v = L_t^{-1} \nabla(I)_u L_t v \quad \text{for all } t.$$

The Weingarten operator always has an eigenbasis, thus at the same time it is an eigenbasis of L_t for all t. As the one-parameter family $\{x_t\}$ has parallel normals, the third fundamental forms coincide: $III(t) = III$ for all t. In an obvious short notation we have

$$S^{-1} \cdot \nabla(I) \cdot S = \nabla(III) = \nabla(III(t)) = S_t^{-1} \cdot \nabla(I(t)) \cdot S_t.$$

REMARK 3.7. The relation (15) and the definition of L_t finally give

$$I(t) = I + 2t \cdot II + t^2 \cdot III, \tag{17}$$

$$II(t) = II - t \cdot III. \tag{18}$$

In dimension $n = 2$, the foregoing relations and

$$K(t)I(t) - 2H(t)II(t) + III(t) = 0$$

imply

$$\begin{aligned} 0 &= \mathrm{id} - 2(t + P_1(t)) \cdot S + (t^2 + 2tP_1(t) + P_2(t)) \cdot S^2 \\ &= \mathrm{id} - 2P_1 \cdot S + P_2 \cdot S^2. \end{aligned}$$

3.9.2. *Parallel hypersurfaces in Blaschke's hypersurface theory.* While the parallel deformation of a Euclidean hypersurface gives a family of Euclidean hypersurfaces with the same Euclidean normal, the situation is different in Blaschke's hypersurface theory. This is seen from the following lemma.

LEMMA 3.8. *Let (x, Y, y) be a non-degenerate hypersurface with Blaschke normalization, and let $x_\tau = x + \tau y$, for $\tau \neq 0$ fixed and admissible, be a hypersurface in the parallel relative family $\{x_t\}$. Then the following statements are equivalent:*

(i) $(x_\tau, Y_\tau, y_\tau) = (x_\tau, Y, y)$ *is again a Blaschke hypersurface,*

(ii) $\det L_\tau = \text{const} \neq 0$ *on* M,

(iii) *the relative Gauß–Kronecker curvatures coincide modulo a non-zero constant on* M*:*

$$H_n(\tau) = \text{const} \cdot H_n \neq 0.$$

Proof. From Subsection 3.4 we know that

$$\omega(\tau)^* = \omega^* \quad \text{and} \quad \omega(h(\tau)) = (\det(L_\tau)^{n/2}) \cdot \omega(h).$$

The characterization of Blaschke's geometry within relative geometry states that x_τ is a Blaschke hypersurface if and only if $\omega(\tau)^* = c\omega(h(\tau))$ with $0 < c \in \mathbb{R}$ (see Lemma 2.3). Both statements imply that x_τ is a Blaschke hypersurface if and only if $\det L_\tau = \text{const}$. Now (iii) follows from 3.4(k). ∎

4. Proper relative spheres and quadrics. We recall the characterization of two important classes of hypersurfaces in relative geometry. As above, the parameter t is used only for admissible t.

Remark 4.1. (i) *Quadrics.* In relative geometry, a non-degenerate quadric can be characterized by the vanishing of the trace-free part of the difference tensor K; it is well known that the trace-free part of K is the same tensor in any relative geometry, independent of the relative normalization chosen on the hypersurface x. Recall that K itself is trace-free exactly in Blaschke's unimodular affine hypersurface theory (*apolarity*), and recall the characterization of quadrics in Blaschke's hypersurface geometry, given in Section 2.3 (see [9, Sections 5.1 and 7.1]). As above, a quadric with center is called a centered quadric.

(ii) *Relative spheres.* A relative hypersurface (x, Y, y) is a *proper relative sphere with center* c_0 if the position vector $x - c_0$ of the hypersurface and the relative normal y satisfy $y = \lambda(x - c_0)$. A proper relative sphere can be characterized by $S = H \cdot \mathrm{id}$ with relative mean curvature $H := H_1 = \text{const} \neq 0$ (see [9]).

Remark 4.2. (a) x is a proper relative sphere with center c_0 if and only if $\rho(c_0) = \text{const} \neq 0$. This is obvious from [9, Section 4.13.1].

(b) x is a proper relative sphere with center if and only if x_t is a proper relative sphere with center; this follows from 3.4(vii).

(c) The hypersurface x is a relative sphere with $S = H\,\mathrm{id}$ if and only if the hypersurface x_t is a relative sphere with $S_t = \frac{H}{1-tH}\,\mathrm{id}$.

4.1. Parallel deformation of quadrics in Blaschke's geometry. This and the following subsection compare parallel deformations of hyperquadrics in different hypersurface theories. From both subsections we get the proof of Theorem 1.2.

Proposition 4.3. *Let x be a non-degenerate centered hyperquadric with Blaschke structure in unimodular $\mathbb{R}^{n+1}$. Then any x_t is a centered hyperquadric.*

Proof. Every non-degenerate centered hyperquadric is a proper affine sphere, thus $\nabla S \equiv 0$. Now 3.4(s) states that $K(t) \equiv 0$ if and only if $K \equiv 0$. ∎

4.2. Parallel deformation of quadrics in Euclidean geometry. We list some trivial observations.

Observation.

(i) $p \in M$ is umbilical for x if and only if p is umbilical for x_t.
(ii) If x is spherical then x_t is spherical.
(iii) In Euclidean geometry, S is always diagonalizable with eigenbasis $\{e_i\}$, thus L_t and S_t have the same eigenbasis.
(iv) Assume that $Se_i = k_i e_i$ and $I(e_i, e_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$ at $p \in M$. Then at $p \in M$:

$$\begin{aligned} \mathit{II}(e_i, e_j) &= k_i \delta_{ij}, & \mathit{III}(e_i, e_j) &= k_i k_j \delta_{ij}, \\ L_t(e_i) &= (1 - tk_i) e_i, & I(t)(e_i, e_j) &= (1 - tk_i)(1 - tk_j)\delta_{ij}, \\ \mathit{II}(t)(e_i, e_j) &= k_i(1 - tk_j)\delta_{ij}, & \mathit{III}(t)(e_i, e_j) &= \mathit{III}(e_i, e_j). \end{aligned}$$

(v) $\mathit{II}(t) = \mathit{II} - t \cdot \mathit{III}$.
(vi) $\det L_t \cdot H_n(t) = H_n$.
(vii) x is non-degenerate if and only if x_t is non-degenerate.
(viii) $\nabla(\mathit{III}(t))\mathit{II}(t) = \nabla(\mathit{III})\mathit{II}$.

Lemma 4.4. *Let x be a non-degenerate, non-spherical quadric. Then* $H_n \neq \mathrm{const}$.

Proof. Assume that $H_n = \mathrm{const} = c$. Then $c \neq 0$ as x is non-degenerate. From the quadric characterization it follows that $\nabla(\mathit{III})\mathit{II} \equiv 0$ on M. As III, considered as a Riemannian metric, has constant curvature 1, this metric is irreducible, thus $\mathit{II} = \lambda \cdot \mathit{III}$ with $\lambda^n \cdot H_n = 1$, and so x is spherical; a contradiction. ∎

Notation. Let Z be a $(0,3)$-tensor on M with local components Z_{ijk}; then $\mathfrak{z}(Z)$ denotes the *totally symmetrized* $(0,3)$-tensor with local components

$$\mathfrak{z}(Z)_{ijk} := Z_{ijk} + Z_{jki} + Z_{kij}.$$

Proposition 4.5. *Let x be part of a Euclidean non-degenerate centered quadric without umbilics; then there is no quadric in the parallel family x_t.*

Proof. Both x and x_t satisfy the quadric equation (11); we set $G := |H_n|$ and $G(t) := |H_n(t)|$, and obey (18) and $G(t) \cdot |\det L_t| = G$. Then

$$\begin{aligned}
0 &= \nabla(III(t))II(t) + \frac{1}{n+2}\mathfrak{z}(II(t)\otimes d\ln G(t))\\
&= \left[\nabla(III)II + \frac{1}{n+2}\mathfrak{z}(II\otimes d\ln G)\right] - \frac{1}{n+2}\mathfrak{z}(II\otimes d\ln\det|L_t|)\\
&\quad - \frac{1}{n+2}\cdot t\cdot \mathfrak{z}(III\otimes d\ln G(t)).
\end{aligned}$$

As x is a quadric, we have $[\ldots] = 0$. We evaluate the last equation at an arbitrary point $p\in M$ and choose a local frame $\{e_i\}$ such that, at p,

$$II_{ij} = k_i\delta_{ij} \quad \text{and} \quad III_{ij} = k_ik_j\delta_{ij}.$$

Then, for fixed indices k and $i=j$,

$$II_{ii}e_k(\ln\det|L_t|) + t\cdot III_{ii}e_k(\ln G(t)) = k_i\{e_k(\ln\det|L_t|) + tk_i\cdot e_k(\ln G(t))\}.$$

As x is non-degenerate we have $k_i\neq 0$ for any $i=1,\ldots,n$, thus

$$e_k(\ln\det|L_t|) + tk_i\cdot e_k(\ln G(t)) = 0.$$

This equation is true for any pair (i,k); recall that $G(t)\neq \text{const}$; thus we finally get $k_1=\cdots=k_n$ at p; this is a contradiction, as p cannot be umbilical. Therefore x_t cannot be part of a quadric.

4.3. A characterization of proper relative spheres. One can characterize relative spheres in terms of preservation of intrinsic invariants.

PROPOSITION 4.6. *Let $x: M\to\mathbb{R}^{n+1}$ be a relative hypersurface and* $\operatorname{rk} S > 1$. *Consider a parallel deformation x_t and let x_t be an immersion. Then x is a proper relative sphere if and only if the curvature tensors of the induced connections coincide: $R(t)=R$.*

Proof. We need only prove that $R(t)=R$ implies that x is a proper relative sphere. This assumption, the definition of L_t and 3.4(g) imply that

$$R(u,v)S = SR(u,v) \qquad \text{for any } u,v.$$

Using now the assumption $\operatorname{rk} S>1$ and the Gauß and Ricci equations, one easily sees that S must be a multiple of the identity. ■

5. Hypersurfaces with parallel shape operator. Relative hypersurfaces with parallel shape operator were investigated in [1]. There the author proved:

LEMMA 5.1. *Let (x,Y,y) be a relative hypersurface; then we have the following equivalences:*

(i) $\nabla S\equiv 0$,
(ii) $\nabla^* S\equiv 0$,
(iii) $\nabla(h)S\equiv 0$.

In particular the author gave a complete classification in dimensions $n = 2,3$ of such hypersurfaces without any restriction on $\operatorname{rk} S$. If (x, Y, y) is a Blaschke hypersurface and $\operatorname{rk} S = n \geq 3$ then $\nabla S \equiv 0$ implies that (x, Y, y) is an affine sphere (see [7, Corollary 3.11]).

If $\operatorname{rk} S = n$ then the list in Subsection 3.4 and [7, Section 3] have the following consequences:

Corollary 5.2.

(a) *If, for a parallel family $x_t = x + ty$, the relation $\nabla S \equiv 0$ holds for $x = x_0$ then, for any admissible t, the following conditions are satisfied:*

(i) $\nabla = \nabla(t)$.
(ii) $K = K(t)$.
(iii) $\nabla(h) = \nabla(h(t))$.
(iv) $\nabla(t) S_t \equiv 0$.

Vice versa, if for some admissible t one of conditions (i)–(iv) *is satisfied then all these conditions hold for any admissible t, and additionally $\nabla S \equiv 0$.*

(b) *If $\nabla S \equiv 0$ then, for any admissible t,* $\det L_t = \mathrm{const}$. *Namely,* (iii) *implies that the Riemannian volume forms coincide* (*modulo a non-zero constant*), *and then* 3.4(b) *gives the assertion. This has a remarkable consequence: As the polynomial $P(t) := \det L_t$ has only finitely many roots, we have* $\det L_t \neq 0$ *for almost all t. Thus for hypersurfaces satisfying $\nabla S \equiv 0$ only finitely many t are non-admissible.*

(c) *We have $\nabla S \equiv 0 \Leftrightarrow \nabla L_t \equiv 0$ for any* (*admissible*) *t. Then it follows from* [7, *Corollary* 3.6] *that x and x_t are affine homothetic if $n > 2$.*

(d) *Recall Lemma* 3.8. *If x is a Blaschke hypersurface with $\nabla S \equiv 0$ then* $\det L_t = \mathrm{const} \neq 0$, *and x_t is again a Blaschke hypersurface.*

Proof. (i) The relation $\nabla S \equiv 0$ and the definitions of $\nabla(t)$ and L_t give

$$L_t \nabla(t)_u v = \nabla_u v - t \cdot \nabla_u (Sv) = \nabla_u v - t \cdot S \nabla_u v = L_t \nabla_u v.$$

(ii) follows from 3.4(s).

(iii) follows from $\nabla(h) = \nabla - K$ (see [9, Section 4.4.3]).

(iv) We have

$$\begin{aligned}(\nabla(t)_u S_t) v &= \nabla(t)_u (S_t v) - S_t(\nabla(t)_u v) = L_t^{-1}(\nabla_u L_t L_t^{-1} S) v - L_t^{-1} S(\nabla_u v) \\ &= L_t^{-1}(\nabla_u S) v. \ \blacksquare\end{aligned}$$

6. Weingarten hypersurfaces. In the literature, a *Weingarten hypersurface* is defined by some differentiable relation between its curvature functions, e.g. there exists a differentiable function $\mathfrak{W}_i$, where $i \in \{H, P, k, R\}$, such that

- $\mathfrak{W}_H(H_1, \ldots, H_n) = 0$, or
- $\mathfrak{W}_P(P_1, \ldots, P_n) = 0$, or
- $\mathfrak{W}_k(k_1, \ldots, k_n) = 0$, or
- $\mathfrak{W}_R(R_1, \ldots, R_n) = 0$.

The subindices for $\mathfrak{W}$ mark the type of the relation. In case such a relation is linear, the term *linear Weingarten hypersurface* is used. In the Introduction we pointed out that such linearity relations are not necessarily equivalent.

We will specify below when the assumption $\operatorname{rk} S = n$ is needed.

6.1. Linear Weingarten hypersurfaces—part I. For this subsection we assume that the relative shape operator S, and thus S, S_t and also L_t have a (joint) eigenbasis. As already stated, in Euclidean hypersurface theory S, S_t and also L_t always have a (joint) eigenbasis.

Definition 6.1.

(i) We say that a hypersurface x is *polynomial Weingarten* if there exists a polynomial relation of one of the foregoing four types.

(ii) For the definition of an *H-linear Weingarten* hypersurface see the Introduction.

(iii) We say that a hypersurface x is *P-linear Weingarten* if there are real numbers $b_0, \ldots, b_n$ such that $\sum_{i=1}^n (b_i)^2 \neq 0$ and
$$P_n b_n + \cdots + P_1 b_1 + b_0 = 0$$
at each point of M.

(iv) We say that a hypersurface x is *k-linear Weingarten* if there are real numbers $c_0, \ldots, c_n$ such that $\sum_{i=1}^n (c_i)^2 \neq 0$ and
$$k_n c_n + \cdots + k_1 c_1 + c_0 = 0$$
at each point of M.

(v) We say that a hypersurface x is *R-linear Weingarten* if there are real numbers $d_0, \ldots, d_n$ such that $\sum_{i=1}^n (d_i)^2 \neq 0$ and
$$R_n d_n + \cdots + R_1 d_1 + d_0 = 0$$
at each point of M.

Remark 6.2. In (iii) and (v) of the foregoing definition we have to assume that $\operatorname{rk} S = n$. If $\operatorname{rk} S = n$ then the hypersurface x is H-linear Weingarten if and only if it is P-linear Weingarten; thus it is sufficient to investigate one of the two classes in this case.

Indeed, from $\binom{n}{k} = \binom{n}{n-k}$ we have $H_{n-k} = H_n \cdot P_k$ and $P_{n-k} = P_n \cdot H_k$.

Then $0 = \sum a_i H_i = \sum a_i \frac{P_{n-i}}{P_n}$ gives $0 = \sum a_i P_{n-i} = \sum b_j P_j$ where $b_j = a_{n-j}$. The rest is analogous.

REMARK 6.3. (i) In the Introduction we formulated a theorem on the deformation of H-linear Weingarten hypersurfaces. The proof is a straightforward but long computation.

Note that
$$\frac{1}{n!}H_0(t)W^{(n)}(t) = a_0.$$

(ii) In particular, for $n = 2$ the H-linear relation from the Introduction reads
$$0 = a_0 + [2a_0t + a_1]H_1(t) + [a_0t^2 + a_1t + a_2]H_2(t).$$

(iii) For $n = 3$ we have
$$\begin{aligned}0 = a_0 &+ [3ta_0 + a_1]H_1(t) + [3a_0t^2 + 2a_1t + a_2]H_2(t)\\ &+ [a_0t^3 + a_1t^2 + a_2t + a_3]H_3(t).\end{aligned}$$

(iv) If x satisfies a polynomial relation $\mathfrak{W}_H = 0$ then any x_t is again H-polynomial. Analogously, if $\operatorname{rk} S = n$ and x satisfies a polynomial relation $\mathfrak{W}_P = 0$ then x_t is again P-polynomial.

THEOREM 6.4. *Consider a linear Weingarten hypersurface x.*

(P) *Let x be P-linear satisfying the relation*
$$\sum_{k=0}^{n} b_kP_k = 0;$$
then x_t satisfies the relation
$$\sum_{j=0}^{n} P_j(t)\sum_{l=j}^{n} b_lt^{l-j} = 0.$$

(k) *Let x be k-linear; then x_t is H-polynomial.*

(R) *Let x be R-linear satisfying the relation*
$$0 = \sum_i a_iR_i + a_0;$$
then x_t is R-linear satisfying
$$0 = \sum_i a_iR_i(t) + a_0^*,$$
where $a_0^* := a_0 + t\sum_i a_i$.

Proof. (P) Use the expression (12) for P_k. ∎

PROPOSITION 6.5. *Consider a one-parameter family $\{x_t\}$ parallel to x.*

(i) *If $H_k = \text{const}$ for some $k \in \{1, \ldots, n\}$ then the family is H-linear Weingarten.*
(ii) *If $P_k = \text{const}$ for some $k \in \{1, \ldots, n\}$ then the family is P-linear Weingarten.*
(iii) *If $\omega(h_t) = \omega(h)$ for some t then $\{x_t\}$ is $H(t)$-linear Weingarten.*

Proof. The assumption in (iii) implies $\det L_t = 1$, thus $\prod(1 - tk_i) = 1$.

REMARK 6.6. A modified version of (iii) is: If $\det L_t = \mathrm{const} \neq 0$ then the family $\{x_t\}$ is $H(t)$-linear.

COROLLARY 6.7. *Let (x, Y, y) be a Blaschke hypersurface and let $x_\tau = x + \tau y$, for fixed $\tau \neq 0$, be in the parallel family a Blaschke hypersurface again. Then (x, Y, y) is H-linear Weingarten.*

Proof. Apply Lemma 3.8 and Proposition 6.5.

6.2. Linear Weingarten hypersurfaces—part II

6.2.1. *Algebraic results on polynomials*

LEMMA 6.8. *Let $a, k_1, \ldots, k_n$ be real numbers such that $k_1 \neq 0$; in this subsection, as in (10), we denote their normalized elementary symmetric functions by H_l for $l = 1, \ldots, n$. If we have the equality of polynomials*

$$t^n + a_1 t^{n-1} + \cdots + a_n = \left(t - \frac{1}{k_1}\right)^{n-1}(t - a)$$

then

$$nk_1^{n-1}(1 + a_1H_1 + \cdots + a_nH_n) = (1 - ak_1)(k_1 - k_2)\cdots(k_1 - k_n). \tag{19}$$

Proof. We have

$$\begin{aligned}
&\left(t - \frac{1}{k_1}\right)^{n-1}(t - a)\\
&= \sum_{l=0}^{n-l} t^{n-l}(-1)^l\left(\frac{1}{k_1}\right)^l\binom{n-1}{l} - \sum_{l=0}^{n-1} t^{n-1-l}(-1)^l\cdot a\left(\frac{1}{k_1}\right)^l\binom{n-1}{l}\\
&= t^n + \sum_{l=1}^{n-1} t^{n-l}(-1)^l\left(\frac{1}{k_1}\right)^l\binom{n-1}{l}\\
&\quad - a\sum_{l=1}^{n-1} t^{n-l}(-1)^{l-1}\left(\frac{1}{k_1}\right)^{l-1}\binom{n-1}{l-1} + a(-1)^n\left(\frac{1}{k_1}\right)^{n-1}\\
&= t^n\\
&\quad + \sum_{l=1}^{n-1} t^{n-l}(-1)^l\left(\frac{1}{k_1}\right)^l\frac{1}{n}\binom{n}{l}[(n-l) + k_1la] + a(-1)^n\left(\frac{1}{k_1}\right)^{n-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
a_l &= (-1)^l\left(\frac{1}{k_1}\right)^l\frac{1}{n}\binom{n}{l}[(n-l) + k_1la] \quad \text{for } l = 1, \ldots, n-1,\\
a_n &= a(-1)^n\left(\frac{1}{k_1}\right)^{n-1}.
\end{aligned}$$

We introduce the following notations:

$$\sum_{1\mapsto n} := \sum_{1\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l} \quad \text{and} \quad \sum_{2\mapsto n} := \sum_{2\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l}.$$

Now for $l=1,\dots,n-1$ we have

$$\begin{aligned}
nk_1^{n-1}a_lH_l &= (-1)^l k_1^{n-l-1}[(n-l)+k_1la]\sum_{1\mapsto n}\\
&= (-1)^l k_1^{n-l-1}(n-l)\sum_{1\mapsto n} + (-1)^l k_1^{n-l}la\sum_{1\mapsto n}\\
&= (-1)^l k_1^{n-l-1}(n-l)\Big(\sum_{2\le i_1<\cdots<i_{l-1}\le n} k_1k_{i_1}\cdots k_{i_{l-1}} + \sum_{2\mapsto n}\Big)\\
&\quad + (-1)^l k_1^{n-l}la\Big(\sum_{2\le i_1<\cdots<i_{l-1}\le n} k_1k_{i_1}\cdots k_{i_{l-1}} + \sum_{2\mapsto n}\Big)\\
&= (-1)^l k_1^{n-l}(n-l)\sum_{2\le i_1<\cdots<i_{l-1}\le n} k_{i_1}\cdots k_{i_{l-1}}\\
&\quad + (-1)^l k_1^{n-l-1}(n-l)\sum_{2\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l}\\
&\quad + (-1)^l k_1^{n-l+1}la\sum_{2\le i_1<\cdots<i_{l-1}\le n} k_{i_1}\cdots k_{i_{l-1}} + (-1)^l k_1^{n-l}la\sum_{2\mapsto n}.
\end{aligned}$$

Using these computations we get

$$\begin{aligned}
&nk_1^{n-1} + \sum_{l=1}^{n-1}(-1)^l k_1^{n-l-1}(n-l)\sum_{1\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l}\\
&\quad = k_1^{n-1}\\
&\qquad + \sum_{l=1}^{n-2}[(-1)^l(n-l)+(-1)^{l+1}(n-l-1)]k_1^{n-l-1}\sum_{2\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l}\\
&\qquad + (-1)^{n-1}k_2\cdots k_n\\
&\quad = k_1^{n-1} + \sum_{l=1}^{n-1}(-1)^l k_1^{n-l-1}\sum_{2\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{l=1}^{n-1}(-1)^l k_1^{n-l-1}(n-l)k_1la\sum_{1\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l} + nk_1^{n-1}a_nH_n\\
&\qquad = (-1)k_1^na\\
&\qquad\quad + \sum_{l=1}^{n-2} ak_1^{n-l}[(-1)^l l + (-1)^{l+1}(l+1)]\sum_{2\le i_1<\cdots<i_l\le n} k_{i_1}\cdots k_{i_l}
\end{aligned}$$

$$+ (-1)^{n-1}(n-1)ak_1 \cdots k_n + (-1)^n nak_1 \cdots k_n$$
$$= (-1)k_1^n a + \sum_{l=1}^{n-1} (-1)^{l+1} k_1^{n-l} a \sum_{2 \le i_1 < \cdots < i_l \le n} k_{i_1} \cdots k_{i_l}.$$

Now one can easily deduce the equality (19). ■

LEMMA 6.9. *Let $k_1, \ldots, k_n$, for $n \geq 2$, be real numbers such that $k_1 \neq 0$. If we have the equality of polynomials*

$$t^{n-1} + a_2 t^{n-2} + \cdots + a_n = \left(t - \frac{1}{k_1}\right)^{n-1}$$

then

$$nk_1^{n-2}(H_1 + a_2 H_2 + \cdots + a_n H_n) = (k_1 - k_2) \cdots (k_1 - k_n).$$

Proof. We have

$$\left(t - \frac{1}{k_1}\right)^{n-1} = \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \left(\frac{1}{k_1}\right)^1 t^{n-1-l}$$
$$= t^{n-1} + \sum_{l=2}^{n} \binom{n-1}{l-1} (-1)^{l-1} \left(\frac{1}{k_1}\right) t^{n-l}.$$

Thus for $l = 2, \ldots, n$ we have

$$a_l = \binom{n-1}{l-1} (-1)^{l-1} \left(\frac{1}{k_1}\right)^{l-1};$$

it follows that

$$nk_1^{n-2} a_l H_l = (-1)^{l-1} l k_1^{n-l-1} \sum_{1 \le k_{i_1} < \cdots < k_{i_l} \le n} k_{i_1} \cdots k_{i_l}$$

for $l = 2, \ldots, n$. Consequently,

$$nk_1^{n-2}(H_1 + a_2 H_2 + \cdots + a_n H_n)$$
$$= k_1^{n-1}$$
$$+ \sum_{i=2}^{n} k_1^{n-2} k_i + \sum_{l=2}^{n-1} (-1)^{l-1} l k_1^{n-l} \sum_{2 \le i_1 < \cdots < i_{l-1} \le n} k_{i_1} \cdots k_{i_{l-1}}$$
$$+ \sum_{l=2}^{n-1} (-1)^{l-1} l k_1^{n-l-1} \sum_{2 \le i_1 < \cdots < i_l \le n} k_{i_1} \cdots k_{i_l} + (-1)^{n-1} nk_2 \cdots k_n$$
$$= k_1^{n-1} + \sum_{l=2}^{n-1} (-1)^{l-1} [l - (l-1)] k_1^{n-l} \sum_{2 \le i_1 < \cdots < i_{l-1} \le n} k_{i_1} \cdots k_{i_{l-1}}$$
$$+ (-1)^{n-2}(n-1) k_2 \cdots k_n + (-1)^{n-1} nk_2 \cdots k_n$$

$$
\begin{aligned}
&= k_1^{n-1} \\
&\quad + \sum_{l=2}^{n-1} (-1)^{l-1} k_1^{n-l} \sum_{2 \le i_1 < \cdots < i_{l-1} \le n} k_{i_1} \cdots k_{i_{l-1}} + (-1)^{n-1} k_2 \cdots k_n \\
&= (k_1 - k_2) \cdots (k_1 - k_n). \ \blacksquare
\end{aligned}
$$

6.2.2. *Geometric results on hypersurfaces.* From Theorem 1.4 we obtain:

THEOREM 6.10. *If a relative hypersurface x with mutually distinct principal curvatures at each point of M is H-linear Weingarten satisfying the equation* (2) *with $a_0 \neq 0$, and if the polynomial* (3) *has a root t_0 of multiplicity $n-1$, then x_{t_0} is an immersion with constant non-zero mean curvature.*

Proof. By (4) it suffices to prove that x_{t_0} is an immersion. Assume that it is not. Then $\det L_{t_0} = 0$ at some point $p \in M$, so

$$(1 - tk_1(p)) \cdots (1 - tk_n(p)) = 0.$$

We can assume that $1 - tk_1(p) = 0$. It follows that $k_1(p) \neq 0$ and $t_0 = 1/k_1(p)$. Since $a_0 \neq 0$, further we can assume that $a_0 = 1$. The polynomial associated with x is now of the form $(t - 1/k_1(p))^{n-1}(t - a)$, where $ak_1(p) \neq 1$. Using now Lemma 6.8 and the fact that the principal curvatures $k_1(p), \ldots, k_n(p)$ are mutually distinct we get a contradiction. ∎

Using Lemma 6.9, in the same manner as Theorem 6.10, one can prove the following theorem.

THEOREM 6.11. *If a relative immersion $x : M \to \mathbb{R}^{n+1}$ has mutually distinct principal curvatures at each point of M and satisfies the equation* (2) *with $a_0 = 0$, $a_1 \neq 0$, and if the polynomial* (3) *has a root t_0 of multiplicity $n-1$, then x_{t_0} is an immersion and $H_1(t_0) = 0$ on M.*

6.3. H-linear Weingarten hypersurfaces in dimension $n = 2$. In this subsection *we do not assume that S is diagonalizable.* In particular, the results can be applied to surfaces in a pseudo-Euclidean space $\mathbb{R}^3$. It is clear that if a surface $x : M \to \mathbb{R}^3$ is pseudo-Euclidean and x_t is an immersion then x_t is pseudo-Euclidean of the same index as x. As before, $H_1 = \frac{1}{2} \operatorname{tr} S$ and $H_2 = \det S$.

Assume that $x : M \to \mathbb{R}^3$ is a relative linear Weingarten surface satisfying the condition

$$a_2 H_2 + a_1 H_1 + a_0 = 0, \tag{20}$$

where not all a_0, a_1, a_2 are zero. If $a_0 \neq 0$, we have the trinomial

$$W(t) = a_0 t^2 + a_1 t + a_2$$

and $\varDelta := a_1^2 - 4a_0 a_2$. A mapping x_t is an immersion if and only if $\det L_t \neq 0$.

Since in the 2-dimensional case

$$\det L_t = H_2 t^2 - 2H_1 t + 1,$$

we see that x_t is an immersion if and only if

$$H_2 t^2 - 2H_1 t + 1 \neq 0$$

at each point of M. By a straightforward computation we obtain

PROPOSITION 6.12. *Let x be a relative H-linear Weingarten surface satisfying the equality* (20)*. If x_t is an immersion then x_t is also H-linear Weingarten, that is, the following relation holds on M:*

$$0 = a_0 + [2a_0 t + a_1]H_1(t) + [a_0 t^2 + a_1 t + a_2]H_2(t). \tag{21}$$

PROPOSITION 6.13. *Assume that $x : M \to \mathbb{R}^3$ is a relative H-linear Weingarten surface satisfying* (20) *and such that $H_1^2 - H_2 \neq 0$ at each point of M. If in its associated polynomial $a_0 \neq 0$, $\Delta > 0$, and if t_1, t_2 are the roots of the polynomial, then x_{t_1}, x_{t_2} are immersions of constant mean curvature satisfying $H_1(t_1) = 1/(t_2 - t_1)$ and $H_1(t_2) = 1/(t_1 - t_2)$.*

Proof. We have $a_1 = -a_0(t_1 + t_2)$ and $a_2 = a_0 t_1 t_2$. Observe first that x_{t_1} is an immersion. Indeed, assume it is not. Then at some point p of M we have $\det L_{t_1} = 0$, i.e.

$$H_2 t_1^2 - 2H_1 t_1 + 1 = 0. \tag{22}$$

We also have

$$0 = H_2 a_2 + H_1 a_1 + a_0 = H_2 t_1 t_2 a_0 - H_1(t_1 + t_2)a_0 + a_0.$$

By subtracting the two equalities we obtain

$$H_2 t_1(t_1 - t_2) - H_1(t_1 - t_2) = 0,$$

hence

$$H_2 t_1 = H_1. \tag{23}$$

Inserting this into the equality (22), we get $H_1 t_1 = 1$. It follows that $t_1 \neq 0$ and $H_1 = 1/t_1$. Using now (23) we obtain $H_2 = 1/t_1^2$, which contradicts the assumption that $H_1^2 - H_2 \neq 0$ at each point of M. Now we use (21) to get $H_1(t) = 1/(t_2 - t_1)$. ■

PROPOSITION 6.14. *Assume that $x : M \to \mathbb{R}^3$ is a relative H-linear Weingarten surface satisfying* (20) *and such that $H_2 \neq 0$ at each point of M. If in the associated polynomial $a_0 \neq 0$ and $\Delta \neq 0$ then there is $t \in \mathbb{R}$ such that x_t is an immersion with constant curvature $H_2(t) = 4a_0^2/\Delta$.*

Proof. Take t such that $2a_0 t + a_1 = 0$, i.e. $t = -a_1/(2a_0)$. Then

$$\det L_t = \left(-\frac{a_1}{2a_0}\right)^2 H_2 - 2\left(-\frac{a_1}{2a_0}\right)H_1 + 1 = \frac{1}{4a_0^2}[a_1^2 H_2 + 4a_1 a_0 H_1 + 4a_0^2].$$

Since $a_0 + a_1 H_1 + a_2 H_2 = 0$, we have

$$4a_1 a_0 H_1 + 4a_0^2 = 4a_0(a_1 H_1 + a_0) = -4a_0 a_2 H_2.$$

Hence

$$\det L_t = \frac{1}{4a_0^2}[a_1^2 H_2 - 4a_0 a_2 H_2] = \frac{1}{4a_0^2} H_2 \Delta \neq 0$$

at each point of M. Thus x_t is an immersion and using formula (21) one sees that x_t has constant curvature $H_2(t) = 4a_0^2/\Delta$. ∎

PROPOSITION 6.15. *Assume that $x : M \to \mathbb{R}^3$ is a relative H-linear Weingarten surface satisfying* (20) *such that $H_2 \neq 0$ and $H_1^2 - H_2 \neq 0$ at each point of M. If in the associated polynomial $a_0 = 0$ then there is $t \in \mathbb{R}$ such that x_t is an immersion and $H_1(t) = 0$ on M.*

Proof. Now the equation

$$a_1 H_1 + a_2 H_2 = 0 \tag{24}$$

is satisfied. Since $H_2 \neq 0$, we have $a_1 \neq 0$. Take $t = -a_2/a_1$. By (21) it is sufficient to prove that x_t is an immersion. Suppose it is not. Then $\det L_t = 0$ at some point p, i.e.

$$0 = 1 + 2H_1 \frac{a_2}{a_1} + H_2 \left(\frac{a_2}{a_1}\right)^2$$

at p. If we insert $a_2 H_2 = -a_1 H_1$ into this formula, we get $a_1 = -H_1 a_2$, which implies that $a_2 \neq 0$ and $H_1 = -a_1/a_2$. Using again (24) we get a contradiction to the assumption $H_1^2 - H_2 \neq 0$. ∎

Proof of Proposition 1.5. Take $t = -a_2/a_1$ as in the proof of the foregoing proposition. It suffices to observe that x_t is an immersion. We have $H_1 = -a_2 H_2/a_1$. Then $H_2 \leq 0$ implies that

$$\det L_t = 1 - 2\frac{a_2}{a_1}\frac{a_2 H_2}{a_1} + \left(\frac{a_2}{a_1}\right)^2 H_2 > 0. \ ∎$$

PROPOSITION 6.16.

(a) *If a relative immersion $x : M \to \mathbb{R}^3$ has positive constant curvature H_2 and $H_1^2 - H_2 \neq 0$ at each point of M then there is t such that x_t is an immersion of constant mean curvature $H_1(t)$.*

(b) *If for x the curvature H_2 is non-zero at each point of M and H_1 is a non-zero constant then there is t such that x_t has positive constant curvature $H_2(t)$.*

Proof. (a) If x has constant non-zero curvature H_2, we can set $a_2 = 1/H_2$, $a_0 = -1$, $a_1 = 0$. Since $H_2 > 0$, we have $\Delta > 0$ and we can use Proposition 6.13.

(b) If H_1 is constant then we set $a_2 = 0$, $a_1 = -1$, $a_0 = H_1$. Take $t = 1/(2H_1)$. We now have

$$\det L_t = 1 - 2\frac{1}{2H_1}H_1 + \frac{1}{4H_1^2}H_2 = \frac{1}{4}\frac{H_2}{H_1^2} \neq 0.$$

From (21) we obtain

$$0 = H_1 + \left[H_1\frac{1}{4H_1^2} - \frac{1}{2H_1}\right]H_2(t) = H_1 - \frac{1}{4H_1}H_2(t).$$

Thus $H_2(t) = 4H_1^2$. ∎

6.4. The case of Blaschke hypersurfaces. In this subsection we do not assume, in general, that S is diagonalizable. We assume that the ambient space $\mathbb{R}^{n+1}$ is equipped with a fixed determinant. Hence the affine normal vector field is unique up to sign. Assume that $x : M \to \mathbb{R}^{n+1}$ is a Blaschke hypersurface with affine normal field y and affine normal bundle $\mathcal{N} = \mathbb{R}y$. Let $x_t = x + ty$ be a one-parameter deformation.

Denote

$$\begin{aligned} c(t) &= \det L_t = \det(\mathrm{id} - tS) \\ &= 1 + (-t)\binom{n}{1}H_1 + \cdots + (-t)^i\binom{n}{i}H_i + \cdots + (-t)^n H_n. \end{aligned}$$

We already know that $\mathcal{N}$ is the affine normal for an immersion x_t if and only if $c(t)$ is a non-zero constant. Assume that $c(t)$ is a non-zero constant. The bundle $\mathcal{N}$ is the affine normal bundle for x_t but y is not necessarily the affine normal for x_t. The affine normal $\tilde{y}_t$ to x_t is equal to $\Phi(t)y$ where

$$\Phi(t) = \varepsilon|c(t)|^{-1/(n+2)}$$

and ε is the sign of $c(t)$ (see [7, (32)]). Denote by $\tilde{S}_t$ the affine shape operator for x_t. Then

$$\tilde{S}_t = \Phi(t)S_t = \Phi(t)L_t^{-1}S.$$

Assume that S is diagonalizable. Then $\tilde{S}_t$ is diagonalizable as well. It is worth emphasizing that if x is locally strongly convex then x_t need not be, but the shape operator for x_t is diagonalizable (of course if x_t is an immersion).

As before we shall denote by k_i, H_i the curvature quantities determined by y. The eigenvalues of $\tilde{S}_t$, i.e. the affine principal curvature functions $\tilde{k}_i(t)$ for x_t, are equal to $\Phi(t)k_i(t)$. Hence the affine $\tilde{H}$-curvatures for x_t are given by

$$\tilde{H}_k(t) = \Phi(t)^k H_k(t).$$

The same formula can be proved if S is not diagonalizable. In that case H_i, $\tilde{H}_i$ are normalized coefficients of the characteristic polynomials of the corresponding shape operators. By the above considerations we have

PROPOSITION 6.17.

(a) *If there is* t_0 *such that* x_{t_0} *is an immersion and* $\mathcal{N}$ *is the affine normal bundle for* x_{t_0} *then* x *is* H*-linear Weingarten. The* H*-curvatures satisfy the equality*

$$0 = (1-c(t_0))+(-t_0)\binom{n}{1}H_1 + \cdots + (-t_0)^i\binom{n}{i}H_i + \cdots + (-t_0)^n H_n. \tag{25}$$

(b) *If moreover* S *is diagonalizable then* x_{t_0} *is also* H*-linear Weingarten and* $\tilde{H}$*-linear Weingarten.*

(c) *If* $n = 2$ (*and* S *is not necessarily diagonalizable*) *then*

$$\begin{aligned}
&t_0^2 H_2(t_0) + 2t_0 H_1(t_0) + \frac{c(t_0)-1}{c(t_0)} = 0,\\
&t_0^2 H_2 - 2t_0 H_1 + (1-c(t_0)) = 0,\\
&t_0^2 c(t_0)^{3/2}\tilde{H}_2(t_0) + 2\varepsilon c(t_0)^{5/4}\tilde{H}_1(t_0) + c(t_0) - 1 = 0.
\end{aligned}$$

In particular, if $c(t_0) = 1$ *then*

$$\tilde{H}_2(t_0)H_1 = -\tilde{H}_1(t_0)H_2.$$

If S *is diagonalizable and non-singular at every point of* M *then*

$$P_1 = -\tilde{P}_1(t_0).$$

Proof. To prove the last sentence it suffices to observe that $H_2(t_0) = \frac{1}{c(t_0)}H_2$. Now we see that the inequality $H_2 \neq 0$ implies $\tilde{H}_2(t_0) \neq 0$. ∎

Observe that, in general, the H-curvatures satisfying (25) are not constant. In such a case, if $n = 2$ then t_0 is the only parameter (except for $t = 0$) for which $\mathcal{N}$ is its affine normal bundle. Namely, if x_t is an immersion and $\mathcal{N}$ is its affine normal bundle then $t^2H_2 - 2tH_1$ and $t_0^2H_2 - 2t_0H_1$ are constants. Hence $2dH_1 = tdH_2$ and $2dH_1 = t_0dH_2$. This is not possible unless H_1 and H_2 are constant.

Assume now that all the curvature functions $H_1, \ldots, H_n$ are constant. Then $c(t)$ is constant for all t. Hence if $c(t)$ is not zero then the bundle $\mathcal{N}$ is the affine normal bundle for x_t. If S is diagonalizable then using induction from n to 1 and the formula

$$\binom{n}{k}H_k(t) = \frac{1}{c(t)}\sum_{j=0}^{n-k}\binom{n}{n-j}(-t)^{k-j}H_{n-j}$$

we see that all curvatures $H_1(t), \ldots, H_n(t)$ are constant. It follows that $\tilde{H}_1(t), \ldots, \tilde{H}_n(t)$ are constant as well.

We can also use the considerations of Subsection 6.4. For instance, we have

PROPOSITION 6.18. *Let $x : M \to \mathbb{R}^3$ be a Blaschke surface with constant curvatures $H_1, H_2 \neq 0$ and $H_1^2 - H_2 \neq 0$. There is t such that x_t is a minimal Blaschke immersion.*

Proof. We set $a_0 = 0$, $a_1 = 1$, $a_2 = -H_1/H_2$. If we take $t_0 = H_1/H_2$ then we have $c(t_0) = -H_1^2/H_2 + 1 \neq 0$. Now we can use (21). ∎

Acknowledgments. B. Opozda is supported by the AvH-Stiftung and the grant NN 201 545738; U. Simon is partially supported by DFG 158/8-1.

References

[1] Th. Binder, *On relative hypersurfaces with parallel shape operator*, Results Math. 46 (2004), 227–236.

[2] J. A. Gálvez, A. Martinez and F. Milán, *Linear Weingarten surfaces in* $\mathbb{R}^3$, Monatsh. Math. 138 (2003), 133–144.

[3] H. Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. 4 (1951), 232–249.

[4] W. Kühnel and M. Steller, *On closed Weingarten surfaces*, Monatsh. Math. 146 (2005), 113–126.

[5] A. M. Li, U. Simon and G. Zhao, *Global Affine Differential Geometry of Hypersurfaces*, de Gruyter, Berlin, 1993.

[6] R. López, *On linear Weingarten surfaces*, Int. J. Math. 19 (2008), 439–448.

[7] B. Opozda, *Parallel submanifolds*, Results Math. 56 (2009), 231–244.

[8] A. Schwenk-Schellschmidt and U. Simon, *Codazzi-equivalent affine connections*, Results Math. 56 (2009), 211–229.

[9] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science Univ. Tokyo, 1991.

[10] U. Simon, *Affine hypersurface theory revisited: gauge invariant structures*, Russian Math. (Izv. VUZ) 48 (2004), 48–73.

[11] U. Simon, A. Schwenk-Schellschmidt and L. Vrancken, *Codazzi-equivalent Riemannian metrics*, Asian J. Math. 14 (2010), 291–302.

[12] K. Voss, *Über geschlossene Weingartensche Flächen*, Math. Ann. 138 (1959), 42–54.

Barbara Opozda
Wydział Matematyki i Informatyki UJ
Łojasiewicza 6
30-048 Kraków, Poland
E-mail: Barbara.Opozda@im.uj.edu.pl

Udo Simon
Mathematisches Institut MA 8-3
TU Berlin
Straße des 17. Juni 136
D-10623 Berlin, Germany
E-mail: simon@math.tu-berlin.de

Received 15.3.2013
and in final form 14.6.2013 (3062)