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Flat tensor product surfaces of pseudo-Euclidean curves

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Abstract. We determine the flat tensor product surfaces of two curves in pseudo-Euclidean spaces of arbitrary dimensions.

1. Introduction. In [2] B. Y. Chen defined the tensor product of two immersions of a Riemannian manifold and started its study. In [4] the authors examined the tensor products of two immersions of, in general, different manifolds. I. Mihai and L. Verstraelen [15] gave an overview of the origins of the study of tensor products of submanifolds.

As a particular case, the tensor product of two curves results in a tensor product surface. Taking into account some curvature conditions and characterizations, many authors studied this topic.

In [11], minimal, totally real, complex, slant and pseudo-umbilical tensor product surfaces of two Euclidean planar curves are studied (see also [10] and [9]). Classification theorems for minimal, totally real and pseudo-minimal tensor product surfaces of two Lorentzian planar curves are given in [14]. Minimal and pseudo-minimal tensor product surfaces of a Lorentzian planar curve and a Euclidean planar curve are studied in [13].

Recently, in [1] and [6], the authors generalized previous results on minimal tensor product surfaces; classification theorems for minimal tensor product surfaces of two curves in Euclidean and pseudo-Euclidean spaces, respectively, of arbitrary dimensions were proved.

When the ambient space is a sphere, tensor products are used in [8] to construct examples of Willmore surfaces. It is proved that a surface in a unit sphere which is a tensor product immersion of two curves is flat.

To relate this topic to another interesting topic in differential geometry, submanifolds of finite type, we mention that flat tensor product surfaces of two curves of finite type on a unit sphere are surfaces of finite type [3].

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Besides the minimality condition, an interesting curvature condition for a surface is the vanishing of the Gaussian curvature, that is, one wants to determine under which conditions the surface is flat. In [12] a classification of flat tensor product surfaces of two Euclidean planar curves in 4-dimensional Euclidean space is proved.

In [5], the authors classified flat tensor product surfaces of two pseudo-Euclidean planar curves and also gave examples of flat tensor product surfaces in higher dimensional pseudo-Euclidean spaces.

In the present article we determine the flat tensor product surfaces in pseudo-Euclidean spaces of arbitrary dimensions.

2. Preliminaries. In this section we shall fix the notations and recall some known formulae which we shall use.

The Gaussian curvature of a pseudo-Riemannian surface parameterized by $f(s,t)$ is given by the formula of Brioschi (see [16], [5]):

$$K(s,t) = \frac{1}{(EG-F^2)^2} \times \left\{ \begin{vmatrix} -\frac{1}{2}E_{tt} + F_{st} - \frac{1}{2}G_{ss} & \frac{1}{2}E_s & -\frac{1}{2}E_t + F_s \\ F_t - \frac{1}{2}G_s & E & F \\ \frac{1}{2}G_t & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_t & \frac{1}{2}G_s \\ \frac{1}{2}E_t & E & F \\ \frac{1}{2}G_s & F & G \end{vmatrix} \right\},$$

where $E(s,t)$, $F(s,t)$ and $G(s,t)$ are the components of the induced metric on the surface and the subscript s or t denotes differentiation with respect to s or t, respectively.

Definition 2.1. A surface is called *flat* if its Gaussian curvature vanishes identically.

Let $\mathbb{E}^m_\mu$ denote the m-dimensional pseudo-Euclidean space of index μ with the standard flat metric g_1. Analogously, denote the metric on $\mathbb{E}^n_\nu$ by g_2. Let G_1 and G_2 denote the metric matrices of $\mathbb{E}^m_\mu$ and $\mathbb{E}^n_\nu$ respectively. Consider the elements of $\mathbb{E}^m_\mu$ and $\mathbb{E}^n_\nu$ as column vectors and identify $\mathbb{E}^{mn}$ with the space $\mathcal{M}$ of real valued $m \times n$ matrices. Define a metric g in $\mathcal{M}$ by

$$g(A,B) = \mathrm{trace}(G_1 A G_2\, {}^tB)$$

for $A, B \in \mathcal{M}$, where ${}^tB$ denotes the transpose of B. Then $(\mathcal{M}, g)$ is isometric to the pseudo-Euclidean space $\mathbb{E}^{mn}_\rho$ of index $\rho = \mu(n-\nu) + \nu(m-\mu)$.

Definition 2.2. The *tensor product* is defined as

$$\otimes : \mathbb{E}^m_\mu \times \mathbb{E}^n_\nu \to \mathcal{M};\ (X,Y) \mapsto X \otimes Y = X\, {}^tY.$$

We recall the following lemma [5]:

LEMMA 2.3. *If $W, X \in \mathbb{E}^m_\mu$ and $Y, Z \in \mathbb{E}^n_\nu$, then*
$$g(W \otimes Y, X \otimes Z) = g_1(W, X) g_2(Y, Z).$$

Finally, assume that the tensor product
$$\begin{aligned} f(s,t) &:= c_1(s) \otimes c_2(t) = c_1(s)\,{}^t c_2(t) \\ &= (c_1^1(s)c_2^1(t), \ldots, c_1^1(s)c_2^n(t), c_1^2(s)c_2^1(t), \ldots, c_1^m(s)c_2^n(t)) \end{aligned}$$
of two pseudo-Euclidean curves
$$\begin{aligned} c_1 &: \mathbb{R} \to \mathbb{E}^m_\mu, \quad s \mapsto c_1(s) = (c_1^1(s), \ldots, c_1^m(s)), \\ c_2 &: \mathbb{R} \to \mathbb{E}^n_\nu, \quad t \mapsto c_2(t) = (c_2^1(t), \ldots, c_2^n(t)) \end{aligned}$$
defines an immersion of $\mathbb{R}^2$ into $\mathcal{M}$. Then
$$f_s(s,t) = \frac{\partial f}{\partial s}(s,t) = \dot{c}_1(s) \otimes c_2(t)$$
and
$$f_t(s,t) = \frac{\partial f}{\partial t}(s,t) = c_1(s) \otimes \dot{c}_2(t),$$
where the dot denotes ordinary differentiation.

By applying the previous lemma, the components of the induced metric on the surface $f(s,t) = c_1(s) \otimes c_2(t)$ are ([5])
$$\begin{aligned} E(s,t) &= g(f_s, f_s) = g_1(\dot{c}_1, \dot{c}_1) g_2(c_2, c_2), \\ F(s,t) &= g(f_s, f_t) = g_1(c_1, \dot{c}_1) g_2(c_2, \dot{c}_2), \\ G(s,t) &= g(f_t, f_t) = g_1(c_1, c_1) g_2(\dot{c}_2, \dot{c}_2). \end{aligned}$$

The position vectors of c_1 and c_2 cannot be null since $EG - F^2$ must be non-zero in order for the surface to be non-degenerate.

3. Flat tensor product surfaces. In [5] the authors studied flat tensor product surfaces of two pseudo-Euclidean planar curves.

We shall use the following formula expressing the flatness of a tensor product surface, arising from the formula of Brioschi [5, (1) and (2)]):

$$\begin{aligned} (3.1)\quad & g_1(\dot{c}_1, \dot{c}_1)\{g_1(c_1, \dot{c}_1)^2 - g_1(c_1, c_1) g_1(\dot{c}_1, \dot{c}_1)\} \\ &\qquad \cdot g_2(\dot{c}_2, \dot{c}_2)\{g_2(c_2, \dot{c}_2)^2 - g_2(c_2, c_2) g_2(\dot{c}_2, \dot{c}_2)\} \\ &= g_1(c_1, c_1)\{g_1(\dot{c}_1, \dot{c}_1) g_1(c_1, \ddot{c}_1) - g_1(c_1, \dot{c}_1) g_1(\dot{c}_1, \ddot{c}_1)\} \\ &\qquad \cdot g_2(c_2, c_2)\{g_2(\dot{c}_2, \dot{c}_2) g_2(c_2, \ddot{c}_2) - g_2(c_2, \dot{c}_2) g_2(\dot{c}_2, \ddot{c}_2)\}. \end{aligned}$$

We recall the following remark from [5]:

REMARK ([5]). From equation (3.1), two special cases of flat tensor product surfaces follow directly.

(i) Assume one of the curves c_1 or c_2 is a straight line through the origin. Then the tensor product surface $f(s,t) = c_1(s) \otimes c_2(t)$ has zero Gaussian curvature, and therefore is flat.

(ii) Assume one of the curves c_1 or c_2 is a null curve. For instance, if $g_1(\dot{c}_1, \dot{c}_1) = 0$, then also $g_1(\dot{c}_1, \ddot{c}_1) = 0$, so that equation (3.1) is satisfied. Hence, the tensor product surface $f(s,t) = c_1(s) \otimes c_2(t)$ is flat if c_1 or c_2 is a null curve.

We shall now consider the general case for any m and n.

The formula (3.1) can be written as

$$\begin{aligned}(3.2)\quad &\frac{g_1(\dot{c}_1, \dot{c}_1)\{g_1(c_1, \dot{c}_1)^2 - g_1(c_1, c_1)g_1(\dot{c}_1, \dot{c}_1)\}}{g_1(c_1, c_1)\{g_1(\dot{c}_1, \dot{c}_1)g_1(c_1, \ddot{c}_1) - g_1(c_1, \dot{c}_1)g_1(\dot{c}_1, \ddot{c}_1)\}} \\ &\qquad = \frac{g_2(c_2, c_2)\{g_2(\dot{c}_2, \dot{c}_2)g_2(c_2, \ddot{c}_2) - g_2(c_2, \dot{c}_2)g_2(\dot{c}_2, \ddot{c}_2)\}}{g_2(\dot{c}_2, \dot{c}_2)\{g_2(c_2, \dot{c}_2)^2 - g_2(c_2, c_2)g_2(\dot{c}_2, \dot{c}_2)\}}.\end{aligned}$$

The left hand side depends only on c_1 and the right hand side depends only on c_2. It follows that both sides are constant, say equal to k, i.e.,

$$(3.3)\qquad \frac{g_1(\dot{c}_1, \dot{c}_1)\{g_1(c_1, \dot{c}_1)^2 - g_1(c_1, c_1)g_1(\dot{c}_1, \dot{c}_1)\}}{g_1(c_1, c_1)\{g_1(\dot{c}_1, \dot{c}_1)g_1(c_1, \ddot{c}_1) - g_1(c_1, \dot{c}_1)g_1(\dot{c}_1, \ddot{c}_1)\}} = k,$$

$$(3.4)\qquad \frac{g_2(c_2, c_2)\{g_2(\dot{c}_2, \dot{c}_2)g_2(c_2, \ddot{c}_2) - g_2(c_2, \dot{c}_2)g_2(\dot{c}_2, \ddot{c}_2)\}}{g_2(\dot{c}_2, \dot{c}_2)\{g_2(c_2, \dot{c}_2)^2 - g_2(c_2, c_2)g_2(\dot{c}_2, \dot{c}_2)\}} = k.$$

We shall solve the above equation according to the causal character of c_1 and c_2, respectively.

I. (i) c_1 is space-like, i.e. $g_1(\dot{c}_1, \dot{c}_1) = 1$.

We denote $g_1(c_1, c_1) = r_1$. Then $g_1(c_1, \dot{c}_1) = \dot{r}_1/2$ and $g_1(c_1, \dot{c}_1) = \ddot{r}_1/2$–$1$.

Substituting in (3.3), we get

$$\dot{r}_1^2/4 - r_1 = kr_1(\ddot{r}_1/2 - 1),$$

or equivalently

$$(3.5)\qquad 2kr_1\ddot{r}_1 - \dot{r}_1^2 - 4(k-1) = 0.$$

To solve (3.5), we define

$$p_1(y) = \int_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{1/k}}}, \qquad c \in \mathbb{R}.$$

Then the inverse function of $p_1(y) = x$ is a solution of (3.5).

(ii) c_1 is time-like, i.e. $g_1(c_1, c_1) = -1$.

We denote again $g_1(c_1, c_1) = r_1$. Then (3.3) becomes

$$-\dot{r}_1^2/4 + r_1 = kr_1(\ddot{r}_1/2 - 1),$$

or, equivalently,

$$(3.6)\qquad 2kr_1\ddot{r}_1 + \dot{r}_1^2 - 4(k+1) = 0.$$

The solution of (3.6) is the inverse function of $p_2(y) = x$, where

$$p_2(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{-1/k}}}, \qquad c \in \mathbb{R}.$$

Analogously, according to the causal character of c_2, we have the following cases.

II. (i) c_2 is space-like, i.e. $g_2(\dot{c}_2, \dot{c}_2) = 1$.

We denote $g_2(c_2, c_2) = r_2$. Then $g_2(c_2, \dot{c}_2) = \dot{r}_2/2$ and $g_2(c_2, \ddot{c}_2) = \ddot{r}_2/2 - 1$. Then (3.4) becomes

$$\dot{r}_2^2/4 - r_2 = \tfrac{1}{k} r_2(\ddot{r}_2/2 - 1),$$

or, equivalently,

$$\tfrac{2}{k} r_2 \ddot{r}_2 - \dot{r}_2^2 - 4(1/k - 1) = 0. \tag{3.7}$$

The solution of (3.7) is the inverse function of $p_3(y) = x$, where

$$p_3(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{k}}}, \qquad c \in \mathbb{R}.$$

(ii) c_2 is time-like, i.e. $g_2(\dot{c}_2, \dot{c}_2) = -1$.

We denote again $g_2(c_2, c_2) = r_2$. Then (3.4) becomes

$$-\dot{r}_2^2/4 + r_2 = \tfrac{1}{k} r_2(\ddot{r}_2/2 - 1),$$

or, equivalently,

$$\tfrac{2}{k} r_2 \ddot{r}_2 + \dot{r}_2^2 - 4(1/k + 1) = 0. \tag{3.8}$$

The solution of (3.8) is the inverse function of $p_4(y) = x$, where

$$p_4(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{-k}}}, \qquad c \in \mathbb{R}.$$

Summarising, we proved the following classification theorem:

THEOREM 3.1. *Let $c_1 : \mathbb{R} \to \mathbb{E}^m_\mu$ and $c_2 : \mathbb{R} \to \mathbb{E}^n_\nu$ be two pseudo-Euclidean curves. Then their tensor product surface $c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{E}^{mn}_\rho$, where $\rho = \mu(n-\nu) + \nu(m-\mu)$, is flat if and only if one of conditions* (i)–(iii) *below holds:*

(i) *c_1 or c_2 is a straight line through the origin.*
(ii) *c_1 or c_2 is a null curve.*
(iii) *Either c_1 is a space-like curve and $r_1 = g_1(c_1, c_1)$ is the inverse function of*

$$p_1(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{1/k}}}, \qquad c \in \mathbb{R},$$

or c_1 *is a time-like curve and* $r_1 = g_1(c_1, c_1)$ *is the inverse function of*

$$p_2(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{-1/k}}}, \qquad c \in \mathbb{R};$$

and either c_2 *is a space-like curve and* $r_2 = g_2(c_2, c_2)$ *is the inverse function of*

$$p_3(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{k}}}, \qquad c \in \mathbb{R},$$

or c_2 *is a time-like curve and* $r_2 = g_2(c_2, c_2)$ *is the inverse function of*

$$p_4(y) = \int\limits_{y_0}^{y} \frac{dy}{\sqrt{4y + cy^{-k}}}, \qquad c \in \mathbb{R}.$$

Remarks. 1. The flatness condition depends only on the functions $r_1(s)$ and $r_2(t)$ and does not depend on all components of the curves $c_1(s)$ and $c_2(t)$.

2. For $k = \pm 1$ the functions p_1, p_2, p_3 and p_4 can be elementarily computed.

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