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# A characterization of elliptic operators

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**Abstract.** We give a characterization of constant coefficients elliptic operators in terms of estimates of their iterations on smooth functions.

Let $P$ be a linear differential operator of order $m \in \mathbb{N}$ with real analytic coefficients on an open set $\Omega \subset \mathbb{R}^n$. Such an operator can be written in the form

$$(Pu)(x) = \sum_{\alpha \in \mathbb{N}_0^n,\, |\alpha| \le m} a_\alpha(x)(D^\alpha u)(x), \quad u \in C^m(\Omega),\, x \in \Omega,$$

where $a_\alpha$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n \le m$ belong to the set $\mathcal{A}(\Omega)$ of real analytic, complex-valued functions on $\Omega$. Recall that the operator $P$ is called *elliptic* on $\Omega$ if its principal symbol

$$\sigma_m(x,\xi) := \sum_{\alpha \in \mathbb{N}_0^n,\, |\alpha| = m} a_\alpha(x)\xi^\alpha$$

does not vanish on $\Omega \times \mathbb{R}^n \backslash \{0\}$.

In 1962 M. S. Narasimhan and Kotake proved that if $P$ is elliptic on $\Omega$, $u \in C^\infty(\Omega)$ and for any compact set $K \Subset \Omega$ one can find a constant $C < \infty$ such that

$$\|P^N u\|_{L^2(K)} \le C^{N+1}(Nm)! \quad \text{for } N \in \mathbb{N}_0,$$

then $u \in \mathcal{A}(\Omega)$ (see [NK, Theorem 1]). Clearly, the same conclusion holds if in the above estimates the $L^2$-norm is replaced by the sup norm.

The aim of the present note is to prove that ellipticity of a constant coefficients operator $P$ can be characterized in terms of estimates of its iterations $P^N$ on smooth functions. Namely we prove

Theorem 1. *Let $P$ be a constant coefficients differential operator of order $m \in \mathbb{N}$ and $\Omega$ an open subset of $\mathbb{R}^n$. Fix $u \in C^\infty(\Omega)$ and consider the*

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*following property of the iterates:*

(I) *for any compact set* $K \Subset \Omega$ *there exists a constant* $C = C(u,K) < \infty$ *such that*

$$\sup_{x\in K} |P^N u(x)| \le C^{N+1}(Nm)! \quad \text{for all } N \in \mathbb{N}_0.$$

*If for any* $u \in C^\infty(\Omega)$ *the property* (I) *implies that* $u \in \mathcal{A}(\Omega)$, *then* $P$ *is elliptic.*

Note that the ellipticity of a constant coefficients operator $P$ does not depend on $\Omega$. Before the proof of Theorem 1 recall that a differential operator $P$ is called *analytic-hypoelliptic* on $\Omega$ if for any $u \in C^\infty(\Omega)$,

$$Pu \in \mathcal{A}(\Omega) \quad \text{implies} \quad u \in \mathcal{A}(\Omega).$$

The idea of the proof of Theorem 1 is the following. Fix $u \in C^\infty(\Omega)$ and assume that $Pu \in \mathcal{A}(\Omega)$. We shall show that the property (I) holds. By assumption this implies that $u \in \mathcal{A}(\Omega)$. Thus $P$ is analytic-hypoelliptic and by [R, Theorem 2.2.11], $P$ is elliptic.

*Proof of Theorem 1.* Let $u \in C^\infty(\Omega)$ be such that $Pu \in \mathcal{A}(\Omega)$. Fix a compact set $K \Subset \Omega$. Our aim is to show the estimate in (I) for any $N \in \mathbb{N}_0$. Clearly it holds for $N = 0$ with any $C \ge \sup_K |u|$. Let $N = 1$. Since $Pu$ is real analytic it is well known [N, Prop. 1.1.14] that there exists a constant $M = M(u,K) < \infty$ such that

$$\sup_{x\in K} |D^\alpha(Pu)(x)| \le M^{|\alpha|+1}\alpha! \quad \text{for any } \alpha \in \mathbb{N}_0^n. \tag{1}$$

So in particular for $\alpha = 0$ we get $\sup_K |Pu| \le M \le C^2 m!$ if $C \ge (M/m!)^{1/2}$. For $N = 2$ using (1) and $\alpha! \le |\alpha|!$ we estimate

$$\begin{aligned}\sup_K |P^2 u| &\le \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\le m} |a_\alpha| \sup_K |D^\alpha(Pu)| \overset{(1)}{\le} \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\le m} |a_\alpha| M^{|\alpha|+1}\alpha! \\ &\le \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\le m} |a_\alpha| M^{|\alpha|+1} m! \le C^3(2m)!\end{aligned}$$

if $C \ge (AM^{m+1}m!/(2m)!)^{1/3}$ with $A = \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\le m} |a_\alpha|$.

Finally, fix a general $N \in \mathbb{N}$, $N \ge 2$. In the sums below we use $\beta^i \in \mathbb{N}_0^n$ for $i = 1,\ldots,N-1$. Since $P$ is a constant coefficients operator we have

$$\begin{aligned}P^N u &= \sum_{|\beta^1|\le m} a_{\beta^1} D^{\beta^1}(P^{N-1}u) \\ &= \sum_{|\beta^1|\le m}\sum_{|\beta^2|\le m}\cdots\sum_{|\beta^{N-1}|\le m} a_{\beta^1}a_{\beta^2}\cdots a_{\beta^{N-1}} D^{\beta^1+\beta^2+\cdots+\beta^{N-1}}(Pu).\end{aligned}$$

Hence

$$\sup_K |P^N u| \le \sum_{\substack{|\beta^i|\le m\\ i=1,\dots,N-1}} |a_{\beta^1}|\cdots|a_{\beta^{N-1}}|\cdot \sup_K |D^{\beta^1+\dots+\beta^{N-1}}(Pu)|$$
$$\stackrel{(1)}{\le} \sum_{\substack{|\beta^i|\le m\\ i=1,\dots,N-1}} |a_{\beta^1}|\cdots|a_{\beta^{N-1}}| M^{|\beta^1+\dots+\beta^{N-1}|+1}\cdot(\beta^1+\dots+\beta^{N-1})!$$
$$\le A^{N-1}M^{(N-1)m+1}((N-1)m)!$$
$$\le C^{N+1}(Nm)!$$

if $C \ge \big[A^{N-1}M^{(N-1)m+1}\big((N-1)m\big)!/(Nm)!\big]^{1/(N+1)}$. So if $C \ge \max(1, A, \sup_K |u|)\cdot \max(1, M^{m+1})$, then the estimate in (I) holds, which by assumption implies that $u \in \mathcal{A}(\Omega)$. The application of [R, Theorem 2.2.11] finishes the proof. ∎

Since elliptic operators with constant coefficients are analytic-hypoelliptic [R, Theorem 2.2.8], by Theorem 1 and [NK, Theorem 1] we get

COROLLARY 2. *Let $P$ be a constant coefficient differential operator of order $m \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *$P$ is elliptic.*
(ii) *$P$ is analytic-hypoelliptic.*
(iii) *For every open set $\Omega \subset \mathbb{R}^n$, if $u \in C^\infty(\Omega)$ and* (I) *is satisfied, then $u \in \mathcal{A}(\Omega)$.*

Finally, let us remark that the above equivalence does not hold for operators with variable coefficients. Indeed, let $P = \frac{\partial}{\partial x_1} + i x_1^2 \frac{\partial}{\partial x_2}$. Then $P$ is an analytic-hypoelliptic operator [R, Theorem 2.3.5] as well as (iii) is satisfied [BM, Theorem 1.1], but $P$ is not elliptic.

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