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Mixed 3-Sasakian structures and curvature

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Abstract. We deal with two classes of mixed metric 3-structures, namely the mixed 3-Sasakian structures and the mixed metric 3-contact structures. First, we study some properties of the curvature of mixed 3-Sasakian structures. Then we prove the identity between the class of mixed 3-Sasakian structures and the class of mixed metric 3-contact structures.

1. Introduction. The geometry of 3-Sasakian manifolds has been a well-known topic, since their introduction, independently, by Udrişte [22] and Kuo [19]. It was studied, in a first stage, by Ishihara, Kashiwada, Konishi, Kuo, Tachibana, Tanno, Yu and other geometers of the Japanese school, and then from different viewpoints by Boyer, Galicki and Mann; in particular, we mention the remarkable survey [4], to which we refer the reader for more details about such structures, as well as for historical remarks. On the other hand, studies of analogous odd-dimensional geometries related to the algebra of paraquaternionic numbers have begun very recently (see, for example, [1], [2], [8], [11] and [12]).

In analogy with an early result of Kashiwada [15] for Sasakian 3-structures, a first result we shall present in this paper is for manifolds endowed with mixed 3-Sasakian structures, which are also considered in [8], where they are called split three Sasakian structures. We give a direct proof that they are Einstein, which is analogous to the well-known fact that a paraquaternionic Kähler manifold is Einstein (cf. [10]). To this end, we shall need some formulas for the curvature tensor of a manifold with parasasakian structure and of a manifold with indefinite Sasakian structure. Some results recently proved in [23] will also be recovered.

The second result is concerned with the identity between the class of mixed metric 3-contact structures and the class of mixed 3-Sasakian struc-

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tures (see Kashiwada [17] for the case of 3-contact metric manifolds). It is based on an extension of Kashiwada's generalization of a lemma of Hitchin (cf. [16]) to the almost hyper parahermitian case.

The content of the paper is now briefly described.

In Section 2 we give some fundamental definitions and facts about paracontact metric structures (cf. [9], [23]), which together with the notion of indefinite almost contact metric structure ([5]) are at the root of the notion of mixed metric 3-structure. We also recall a few definitions concerning almost hyper parahermitian structures. In Section 3, after introducing the notion of $[r]$-Sasakian structure, $r = \pm 1$, to mean an indefinite Sasakian structure for $r = +1$, and a parasasakian structure for $r = -1$, we consider some preliminary issues, needed to state, in Section 4, the result concerning the mixed 3-Sasakian manifolds. Finally, Section 5 is devoted to proving that a mixed metric 3-contact structure is in fact a mixed 3-Sasakian structure.

All manifolds and tensor fields are assumed to be smooth.

2. Preliminaries. We recall a few definitions about paracomplex and hyper paracomplex structures. For more details we refer the reader to [7] and [13].

Definition 2.1. An *almost product structure* on a manifold M is a $(1,1)$-type tensor field $F \neq \pm I$ satisfying $F^2 = I$; the pair (M, F) is then said to be an *almost product manifold*.

On an almost product manifold (M, F) we have $TM = T^+M \oplus T^-M$, where T^+M and T^-M are the eigensubbundles associated to the eigenvalues $+1$ and -1 of F. (M, F) is called an *almost paracomplex manifold* if $\mathrm{rank}(T^+M) = \mathrm{rank}(T^-M)$. Finally, an almost product (resp. almost paracomplex) manifold (M, F) is called a *product* (resp. *paracomplex*) *manifold* if $N_F = 0$, N_F being the Nijenhuis tensor field of the structure F. Any (almost) paracomplex manifold has even dimension.

An (almost) paracomplex manifold (M, F) is called (*almost*) *parahermitian* if there exists a metric tensor g compatible with F, i.e. such that $g(FX, Y) + g(X, FY) = 0$ for any $X, Y \in \Gamma(TM)$. Such a metric is necessarily semi-Riemannian, with neutral signature.

Definition 2.2. An *almost hyper parahermitian structure* on a manifold M is a triple (J_1, J_2, J_3) of $(1,1)$-type tensor fields, together with a semi-Riemannian metric g satisfying:

(i) $(J_a)^2 = -\tau_a I$ for any $a \in \{1, 2, 3\}$,
(ii) $J_a J_b = \tau_c J_c = -J_b J_a$ for any cyclic permutation (a, b, c) of $(1, 2, 3)$,
(iii) $g(J_a X, Y) + g(X, J_a Y) = 0$ for any $a \in \{1, 2, 3\}$ and $X, Y \in \Gamma(TM)$,

where $\tau_1 = -1$, $\tau_2 = -1$ and $\tau_3 = +1$. Then (M, J_1, J_2, J_3, g) will be said to be an *almost hyper parahermitian manifold*.

Such a manifold has dimension divisible by four and the metric has neutral signature. An almost hyper parahermitian structure on a manifold M will be called *hyper parahermitian* if for any $a \in \{1, 2, 3\}$, the Nijenhuis tensor field N_a vanishes, that is, each structure J_a is integrable. Then M will be called a *hyper parahermitian manifold*. An almost hyper parahermitian manifold is hyper parahermitian if and only if at least two of the Nijenhuis tensor fields vanish (cf. [13]).

Definition 2.3. Let M be a manifold. An *almost paracontact structure* on M is a triple (φ, ξ, η), where $\varphi \in \mathfrak{T}^1_1(M)$, $\xi \in \Gamma(TM)$ and $\eta \in \bigwedge^1(M)$, satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$. Then M is said to be an *almost paracontact manifold*, denoted by (M, φ, ξ, η). An almost paracontact structure (φ, ξ, η) will be called *normal* if $N_\varphi = 2d\eta \otimes \xi$, N_φ being the Nijenhuis tensor field of φ.

Almost paracontact structures were originally introduced by I. Satō in [20] and [21], where he also studied the properties of manifolds endowed with such structures and with a Riemannian metric satisfying suitable compatibility conditions. Moreover, one may find similar definitions in [14] and [23], where the further condition that the restriction $\varphi|_{\mathrm{Im}(\varphi)}$ is an almost paracomplex structure on the distribution $\mathrm{Im}(\varphi)$ is required. The notion of normality for an almost paracontact structure is defined, as in the classical almost contact case (cf. [3]), through the integrability of the almost product structure F canonically induced on the manifold $M \times \mathbb{R}$, defined by $F\big(X, f\frac{d}{dt}\big) := \big(\varphi X + f\xi, \eta(X)\frac{d}{dt}\big)$ (cf. [14], [23]).

Other properties of almost paracontact manifolds (M, φ, ξ, η), which are immediate consequences of the above definition, are $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\ker(\varphi) = \mathrm{Span}(\xi)$, $\ker(\eta) = \mathrm{Im}(\varphi)$ and $TM = \mathrm{Im}(\varphi) \oplus \mathrm{Span}(\xi)$.

Endowing an almost paracontact manifold with a metric tensor field and considering a suitable compatibility condition, we obtain the notion of almost paracontact metric manifold.

Definition 2.4 ([23]). Let (M, φ, ξ, η) be an almost paracontact manifold and g a metric tensor field on M, that is, a symmetric, nondegenerate $(0, 2)$-type tensor field on M. Then g is said to be *compatible* with the structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = -g(X, Y) + \varepsilon\eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$, with $\varepsilon = \pm 1$ according as ξ is spacelike or timelike. Then (φ, ξ, η, g) is said to be an *almost paracontact metric structure*. We shall call the structure *positive* or *negative* according as $\varepsilon = +1$ or $\varepsilon = -1$.

Then $(M, \varphi, \xi, \eta, g)$ will be called an *almost paracontact metric manifold.* Such a structure (φ, ξ, η, g) will be called *normal* if $N_\varphi = 2d\eta \otimes \xi$.

In [9], the author refers to the same kind of structure called almost paracontact hyperbolic metric structure.

As a consequence of the above definition, for an almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$, the pair (F, g), where $F := \varphi|_{\mathrm{Im}(\varphi)}$, is an almost parahermitian structure on the distribution $\mathrm{Im}(\varphi)$. Hence $\mathrm{rank}(\mathrm{Im}(\varphi)) = 2m$ and $\dim(M) = 2m + 1$. Furthermore, the signature of g on $\mathrm{Im}(\varphi)$ is (m, m), where we put first the minus signs, and the signature of g on TM is $(m, m+1)$ or $(m+1, m)$ according as ξ is spacelike (the structure is positive) or timelike (the structure is negative). It follows that g is a Lorentzian metric only if $m = 1$ and $\dim(M) = 3$.

We know that TM is the orthogonal direct sum of $\mathrm{Im}(\varphi)$ and $\mathrm{Span}(\xi)$, and finally that $\eta(X) = \varepsilon g(X, \xi)$ and $g(\varphi X, Y) + g(X, \varphi Y) = 0$ for any $X, Y \in \Gamma(TM)$.

Particular classes of almost paracontact metric structures are defined as follows.

Definition 2.5 ([9], [23]). Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. Then it is said to be a

(i) *paracontact metric manifold* if $d\eta = \Phi$;
(ii) *parasasakian manifold* if $d\eta = \Phi$ and the structure is normal;
(iii) *para-K-contact manifold* if $d\eta = \Phi$ and ξ is a Killing vector field,

where $\Phi(X, Y) := g(X, \varphi Y)$ is the fundamental 2-form associated with the almost paracontact metric structure.

Furthermore, we recall the following result.

Proposition 2.6. *Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. Then it is a parasasakian manifold if and only if*

$$(\nabla_X \varphi)(Y) = -g(X, Y)\xi + \varepsilon\eta(Y)X$$

for any $X, Y \in \Gamma(TM)$, where $\varepsilon = g(\xi, \xi) = \pm 1$.

We assume the following definition of mixed (metric) 3-structure, which is introduced in [11] and [12], although in a different form.

Definition 2.7. Let M be a manifold. A *mixed 3-structure* on M is a triple of structures $(\varphi_a, \xi_a, \eta^a)$, $a \in \{1, 2, 3\}$, which are almost paracontact structures for $a = 1, 2$ and an almost contact structure for $a = 3$, satisfying

$$\varphi_a \varphi_b - \tau_a \eta^b \otimes \xi_a = \tau_c \varphi_c = -\varphi_b \varphi_a + \tau_b \eta^a \otimes \xi_b, \tag{1}$$

$$\eta^a \circ \varphi_b = \tau_c \eta^c = -\eta^b \circ \varphi_a, \tag{2}$$

$$\varphi_a(\xi_b) = \tau_b \xi_c, \quad \varphi_b(\xi_a) = -\tau_a \xi_c, \tag{3}$$

for any cyclic permutation (a, b, c) of $(1, 2, 3)$, with $\tau_1 = \tau_2 = -1 = -\tau_3$. A *mixed metric* 3*-structure* on M is a triple of structures $(\varphi_a, \xi_a, \eta^a, g)$, $a \in \{1, 2, 3\}$, which are almost paracontact metric structures for $a = 1, 2$, and an almost contact metric structure for $a = 3$, satisfying (1)–(3).

From now on, a mixed 3-structure and a mixed metric 3-structure on a manifold M will be denoted simply by $(\varphi_a, \xi_a, \eta^a)$ and $(\varphi_a, \xi_a, \eta^a, g)$, with the condition $a \in \{1, 2, 3\}$ understood.

Remark 2.8. Equivalently, a mixed metric 3-structure on a manifold M is given by a mixed 3-structure $(\varphi_a, \xi_a, \eta^a)$, together with a metric tensor g satisfying the following compatibility condition:

$$g(\varphi_a X, \varphi_a Y) = \tau_a(g(X, Y) - \varepsilon_a \eta^a(X)\eta^a(Y)) \tag{4}$$

for any $a \in \{1, 2, 3\}$, and any $X, Y \in \Gamma(TM)$, where $\varepsilon_a = g(\xi_a, \xi_a) = \pm 1$.

Remark 2.9. We point out that the above definition of mixed 3-structure, without the metric compatibility, is equivalent to the definition given in [11], and very recently in [12], providing that one substitutes the structures $(\varphi_1, \xi_1, \eta^1)$, $(\varphi_2, \xi_2, \eta^2)$ and $(\varphi_3, \xi_3, \eta^3)$ of [11] and [12] with $(\varphi_3, \xi_3, \eta^3)$, $(\varphi_1, \xi_1, \eta^1)$ and $(\varphi_2, \xi_2, \eta^2)$, respectively, and then the vector fields ξ_1 and ξ_2 with their negatives.

We remark that the conditions (1)–(4) are compatible. We first observe that from (3), one has $\eta^a(\xi_c) = 0$ whenever $a \neq c$, and by the definition of almost (para)contact metric structures, one gets

$$\eta^a(\xi_c) = \delta^a_c \tag{5}$$

for any $a, c \in \{1, 2, 3\}$. Moreover, since each structure $(\varphi_a, \xi_a, \eta^a, g)$ is almost (para)contact metric, one has

$$\eta^a(X) = \varepsilon_a g(X, \xi_a) \tag{6}$$

for any $X \in \Gamma(TM)$, and any $a \in \{1, 2, 3\}$. From (3) one has $\varphi_2(\xi_3) = \xi_1 = \varphi_3(\xi_2)$, and using (4) and (5) we find, on one hand,

$$g(\xi_1, \xi_1) = g(\varphi_2(\xi_3), \varphi_2(\xi_3)) = -g(\xi_3, \xi_3),$$

and on the other hand,

$$g(\xi_1, \xi_1) = g(\varphi_3(\xi_2), \varphi_3(\xi_2)) = g(\xi_2, \xi_2).$$

Thus, $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3$. Analogously, starting from $\varphi_1(\xi_2) = -\xi_3 = -\varphi_2(\xi_1)$ and from $\varphi_3(\xi_1) = -\xi_2 = \varphi_1(\xi_3)$, we obtain the same restrictions on the values of ε_1, ε_2 and ε_3. Let us now verify that (4) makes sense for arbitrary choices of ξ_1, ξ_2 and ξ_3. Fixing a mixed metric 3-structure $(\varphi_a, \xi_a, \eta^a, g)$ with $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (+1, +1, -1)$, for $a = 1$ the condition (4) becomes

$$g(\varphi_1 X, \varphi_1 Y) = -g(X, Y) + \eta^1(X)\eta^1(Y),$$

and using (3), (5) and (6), putting $(X,Y)=(\xi_1,\xi_1)$, we have $0=-g(\xi_1,\xi_1)+\eta^1(\xi_1)\eta^1(\xi_1)=-\varepsilon_1+1=0$. If the mixed metric 3-structure is such that $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(-1,-1,+1)$, it is easy to check that we get the same identity. Analogously, one verifies the consistency for the other choices of (ξ_b,ξ_c), choosing $a=2$ and $a=3$ in (4).

Let us check that (1) and (4) are compatible for any $X,Y\in\Gamma(TM)$. If we fix $(a,b,c)=(1,2,3)$, then (1) becomes

$$\varphi_1\varphi_2+\eta^2\otimes\xi_1=\varphi_3=-\varphi_2\varphi_1-\eta^1\otimes\xi_2. \tag{7}$$

From (4), on one hand we have

$$g(\varphi_3X,\varphi_3Y)=g(X,Y)-\varepsilon_3\eta^3(X)\eta^3(Y), \tag{8}$$

and on the other hand, by (2), (4) and (7),

$$\begin{aligned}g(\varphi_3X,\varphi_3Y)&=g(\varphi_1\varphi_2X+\eta^2(X)\xi_1,\varphi_1\varphi_2Y+\eta^2(Y)\xi_1)\\&=g(\varphi_1\varphi_2X,\varphi_1\varphi_2Y)+\varepsilon_1\eta^2(X)\eta^2(Y)\\&=-g(\varphi_2X,\varphi_2Y)+\varepsilon_1\eta^1(\varphi_2X)\eta^1(\varphi_2Y)+\varepsilon_1\eta^2(X)\eta^2(Y)\\&=g(X,Y)-\varepsilon_2\eta^2(X)\eta^2(Y)\\&\quad+\varepsilon_1\eta^1(\varphi_2X)\eta^1(\varphi_2Y)+\varepsilon_1\eta^2(X)\eta^2(Y),\end{aligned}$$

from which, using $\varepsilon_1=\varepsilon_2=-\varepsilon_3$, (8) follows. Again, by (2), (4) and (7), we have

$$\begin{aligned}g(\varphi_3X,\varphi_3Y)&=g(\varphi_2\varphi_1X+\eta^1(X)\xi_2,\varphi_2\varphi_1Y+\eta^1(Y)\xi_2)\\&=g(\varphi_2\varphi_1X,\varphi_2\varphi_1Y)+\varepsilon_2\eta^1(X)\eta^1(Y)\\&=-g(\varphi_1X,\varphi_1Y)+\varepsilon_2\eta^2(\varphi_1X)\eta^2(\varphi_1Y)+\varepsilon_2\eta^1(X)\eta^1(Y)\\&=g(X,Y)-\varepsilon_1\eta^1(X)\eta^1(Y)\\&\quad+\varepsilon_2\eta^2(\varphi_1X)\eta^2(\varphi_1Y)+\varepsilon_2\eta^1(X)\eta^1(Y),\end{aligned}$$

from which, using $\varepsilon_1=\varepsilon_2=-\varepsilon_3$, (8) follows again. Analogously, one verifies that the consistency also holds starting from the other two cyclic permutations of (a,b,c).

Let M be a manifold endowed with a mixed 3-structure (φ_a,ξ_a,η^a). Considering the two distributions $\mathcal{H}:=\bigcap_{a=1}^3\ker(\eta^a)$ and $\mathcal{V}:=\mathrm{Span}(\xi_1,\xi_2,\xi_3)$, one has the decomposition $TM=\mathcal{H}\oplus\mathcal{V}$. It follows that $(\varphi_1|_{\mathcal{H}},\varphi_2|_{\mathcal{H}},\varphi_3|_{\mathcal{H}})$ is an almost hyper paracomplex structure on the distribution $\mathcal{H}$. Hence $\mathrm{rank}(\mathcal{H})=2n$ and $\dim(M)=2n+3$. Furthermore, if we have a mixed metric 3-structure $(\varphi_a,\xi_a,\eta^a,g)$ on M, then $(\varphi_a|_{\mathcal{H}},g)$, $a\in\{1,2,3\}$, becomes an almost hyper parahermitian structure on the distribution $\mathcal{H}$. Hence $\mathrm{rank}(\mathcal{H})=4m$ and $\dim(M)=4m+3$. As an obvious consequence we have the following result.

PROPOSITION 2.10. *Let M be a manifold with $\dim(M)=2n+3$, endowed with a mixed 3-structure (φ_a,ξ_a,η^a). If $n\neq 2m$, then there is no metric* *tensor field g on M compatible with the mixed 3-structure, and M cannot have any mixed metric 3-structure.*

The compatibility condition (4) between a metric tensor g and a mixed 3-structure $(\varphi_a, \xi_a, \eta^a)$ on a $(4m+3)$-dimensional manifold M, together with (3), has some consequences on the signature of the metric g too. Since $g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -g(\xi_3, \xi_3)$, the vector fields ξ_1 and ξ_2 related to the almost paracontact metric structures are either both spacelike or both timelike. We may therefore distinguish between *positive* and *negative* mixed metric 3-structures according as ξ_1 and ξ_2 are both spacelike ($\varepsilon_1 = \varepsilon_2 = +1$) or both timelike ($\varepsilon_1 = \varepsilon_2 = -1$). This forces the causal character of the third vector field ξ_3. Since the signature of g on $\mathcal{H}$ is necessarily neutral $(2m, 2m)$, we have only the following two possibilities:

- (i) the signature of g on TM is $(2m+1, 2m+2)$ if the mixed metric 3-structure is positive $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (+1, +1, -1)$;
- (ii) the signature of g on TM is $(2m+2, 2m+1)$ if the mixed metric 3-structure is negative $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, +1)$.

We point out that any metric g which is compatible with a mixed 3-structure, in the sense of (4), can never be Lorentzian and that the definition of mixed metric 3-structure given in [12] is equivalent to that of a negative mixed metric 3-structure.

Example 2.11 ([6]). Let M^{4m+3} be any orientable nondegenerate hypersurface of an almost hyper parahermitian manifold $(\bar{M}^{4m+4}, J_a, G)_{a=1,2,3}$. If $N \in \Gamma(TM^{\perp})$ is a unit normal vector field such that $G(N, N) = s = \pm 1$, put $\xi_a := -\tau_a J_a N$ for any $a \in \{1, 2, 3\}$, and define three $(1,1)$-type tensor fields φ_a and three 1-forms η^a on M such that $J_a X = \varphi_a X + \eta^a(X) N$, for any $X \in \Gamma(TM)$ and any $a \in \{1, 2, 3\}$. Then, denoting by g the metric induced on M from G, it is easy to check that $(\varphi_a, \xi_a, \eta^a, g)$ is a mixed metric 3-structure on M with sign $\sigma = -s$.

Finally, we adopt the following definition of mixed 3-Sasakian structure on a manifold, which is already given in [8], although in a different form and called split three Sasakian structure.

Definition 2.12. Let M be a manifold with a mixed metric 3-structure $(\varphi_a, \xi_a, \eta^a, g)$. This structure will be said to be a *mixed 3-Sasakian structure* if $(\varphi_1, \xi_1, \eta^1, g)$ and $(\varphi_2, \xi_2, \eta^2, g)$ are both parasasakian structures, and $(\varphi_3, \xi_3, \eta^3, g)$ is an indefinite Sasakian structure. Then $(M, \varphi_a, \xi_a, \eta^a, g)$ will be called *mixed 3-Sasakian manifold.*

Remark 2.13. The previous definition is equivalent to the notion of split three Sasakian structure given in [8], providing that one replaces the structures (Φ_1, ξ^1), (Φ_2, ξ^2) and (Φ_3, ξ^3) of [8] with $(\varphi_3, \xi_3, \eta^3)$, $(\varphi_2, \xi_2, \eta^2)$

and $(\varphi_1, \xi_1, \eta^1)$, and the vector field ξ_3 with its negative, taking the vector fields ξ_1, ξ_2 and ξ_3 with $g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -1$ and $g(\xi_3, \xi_3) = 1$, that is, $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, -1, +1)$.

Remark 2.14. By Proposition 2.6, a mixed metric 3-structure $(\varphi_a, \xi_a, \eta^a, g)$ on a manifold M is mixed 3-Sasakian if and only if

$$(\nabla_X \varphi_a)(Y) = \tau_a(g(X, Y)\xi_a - \varepsilon_a \eta^a(Y)X) \tag{9}$$

for any $X, Y \in \Gamma(TM)$ and any $a \in \{1, 2, 3\}$, with $\tau_1 = \tau_2 = -1 = -\tau_3$.

We remark that Definition 2.12 is not equivalent to that given in [12]. More precisely, referring to [12], the condition $(\nabla_X \varphi_2)(Y) = g(\varphi_2 X, \varphi_2 Y)\xi_2 + \eta^2(Y)(\varphi_2)^2(X)$ in Definition 4.3, using the compatibility condition (29) there, may be rewritten in the form $(\nabla_X \varphi_2)(Y) = -g(X, Y)\xi_2 + \eta^2(Y)X$, which corresponds to

$$(\nabla_X \varphi_1)(Y) = g(X, Y)\xi_1 + \eta^1(Y)X \tag{10}$$

in our notation. Since the definition of mixed metric 3-structure given in [12] is equivalent to that of negative mixed metric 3-structure, writing the condition (9) for $\tau_a = \tau_1 = -1$ and $\varepsilon_a = \varepsilon_1 = -1$, we get $(\nabla_X \varphi_1)(Y) = -g(X, Y)\xi_1 - \eta^1(Y)$, which is clearly the negative of (10). One obtains an analogous result considering the condition on $(\nabla_X \varphi_3)(Y)$ of [12].

3. On the curvature of $[r]$-Sasakian structures. In this section, we prove some useful formulas concerning the curvature of both parasasakian structures and indefinite Sasakian structures. To treat both cases simultaneously, we introduce the synthetic notation of $[r]$-*Sasakian structure* on a manifold M, considering a system (φ, ξ, η, g) where $\varphi \in \mathfrak{T}^1_1(M)$, $\xi \in \Gamma(TM)$, $\eta \in \bigwedge^1(M)$ and $g \in \mathfrak{T}^0_2(M)$ is a metric tensor field, such that $g(\xi, \xi) = \varepsilon = \pm 1$, $\varphi^2 = r(-I + \eta \otimes \xi)$, $\eta(\xi) = 1$ and

$$g(\varphi X, \varphi Y) = r(g(X, Y) - \varepsilon \eta(X)\eta(Y)), \tag{11}$$

$$(\nabla_X \varphi)(Y) = r(g(X, Y)\xi - \varepsilon \eta(Y)X). \tag{12}$$

Thus, we obtain an indefinite Sasakian structure for $r = +1$ and a parasasakian structure for $r = -1$. From (12) it follows that $\nabla_X \xi = -\varepsilon \varphi(X)$ for any $X \in \Gamma(TM)$.

Following [18], the curvature tensor field $R \in \mathfrak{T}^1_3(M)$ of the Levi-Civita connection ∇, the Riemannian curvature tensor field $R \in \mathfrak{T}^0_4(M)$, and the Ricci curvature tensor field $\rho \in \mathfrak{T}^0_2(M)$ are defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$
$$R(X, Y, Z, W) := g(R(Z, W)Y, X) = -g(R(X, Y)W, Z),$$
$$\rho(X, Y) := \mathrm{tr}_g\{Z \mapsto R(Z, X)Y\} = \sum_{i=1}^{m} \varepsilon_i g(R(E_i, X)Y, E_i),$$

where $(E_i)_{1\leq i\leq m}$ is a local orthonormal frame, $\varepsilon_i = g(E_i, E_i)$ and $m = \dim(M)$.

LEMMA 3.1. *Let M be a manifold endowed with an $[r]$-Sasakian structure (φ,ξ,η,g). Then, for any $X,Y,Z,W\in\Gamma(TM)$,*

$$g(R(X,Y)Z,\varphi W)+g(R(X,Y)\varphi Z,W)=-r\varepsilon P(X,Y,Z,W), \tag{13}$$

where $P\in\mathfrak{T}^0_4(M)$ is the tensor field defined by

$$\begin{aligned}P(X,Y,Z,W):=\,&d\eta(X,Z)g(Y,W)-d\eta(X,W)g(Y,Z)\\&-d\eta(Y,Z)g(X,W)+d\eta(Y,W)g(X,Z).\end{aligned}$$

Proof. Denoting by Φ the fundamental 2-form defined by $\Phi(X,Y):=g(X,\varphi Y)$, let us consider the derivation R_{XY} of the tensor algebra $\mathfrak{T}(M)$, canonically induced from the $(1,1)$-tensor field $R(X,Y):=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$. For any $X,Y,Z,W\in\Gamma(TM)$, we have

$$\begin{aligned}(R_{XY}\Phi)(Z,W)&=R_{XY}(g(Z,\varphi W))-\Phi(R_{XY}Z,W)-\Phi(Z,R_{XY}W)\\&=-g(R_{XY}Z,\varphi W)-g(R_{XY}\varphi Z,W)\\&=-g(R(X,Y)Z,\varphi W)-g(R(X,Y)\varphi Z,W)\end{aligned} \tag{14}$$

Let us compute again the term $(R_{XY}\Phi)(Z,W)$, using (12). One has

$$\begin{aligned}(\nabla_X\nabla_Y\Phi)(Z,W)&=X(\nabla_Y\Phi(Z,W))-\nabla_Y\Phi(\nabla_XZ,W)-\nabla_Y\Phi(Z,\nabla_XW)\\&=X(g(Z,(\nabla_Y\varphi)(W)))-g(\nabla_XZ,(\nabla_Y\varphi)(W))\\&\quad+g((\nabla_Y\varphi)(Z),\nabla_XW)\\&=r\varepsilon(X(\eta(Z)g(Y,W))-X(\eta(W)g(Z,Y))\\&\quad-\eta(\nabla_XZ)g(Y,W)+\eta(W)g(\nabla_XZ,Y)\\&\quad+\eta(\nabla_XW)g(Y,Z)-\eta(Z)g(Y,\nabla_XW)).\end{aligned}$$

Switching X and Y, we have

$$\begin{aligned}(\nabla_Y\nabla_X\Phi)(Z,W)=r\varepsilon(&Y(\eta(Z)g(X,W))-Y(\eta(W)g(Z,X))\\&-\eta(\nabla_YZ)g(X,W)+\eta(W)g(\nabla_YZ,X)\\&+\eta(\nabla_YW)g(X,Z)-\eta(Z)g(X,\nabla_YW)).\end{aligned}$$

Finally,

$$\begin{aligned}(\nabla_{[X,Y]}\Phi)(Z,W)&=g(Z,(\nabla_{[X,Y]}\varphi)(W))\\&=r\varepsilon(\eta(Z)g([X,Y],W)-\eta(W)g(Z,[X,Y])).\end{aligned}$$

It follows that

$$\begin{aligned}(R_{XY}\Phi)(Z,W)=r\varepsilon(&(\nabla_X\eta)(Z)g(Y,W)-(\nabla_X\eta)(W)g(Z,Y)\\&-(\nabla_Y\eta)(Z)g(X,W)+(\nabla_Y\eta)(W)g(Z,X)).\end{aligned}$$

Since $\nabla_X \xi = -\varepsilon\varphi(X)$ and $\Phi = d\eta$, $(\nabla_X \eta)(Y) = d\eta(X,Y)$, we have

$$(15)\qquad \begin{aligned}(R_{XY}\Phi)(Z,W) &= r\varepsilon(d\eta(X,Z)g(Y,W) - d\eta(X,W)g(Z,Y)\\ &\quad - d\eta(Y,Z)g(X,W) + d\eta(Y,W)g(Z,X))\\ &= r\varepsilon P(X,Y,Z,W).\end{aligned}$$

Now, (14) and (15) imply (13). ∎

It is easy to prove the following lemma.

LEMMA 3.2. *Let M be a manifold endowed with an almost (para)contact metric structure (φ,ξ,η,g). Then, for any $X_1, X_2, X_3, X_4 \in \Gamma(TM)$,*

(i) $P(X_1,X_2,X_3,X_4) = -P(X_2,X_1,X_3,X_4)$;
(ii) $P(X_1,X_2,X_3,X_4) = -P(X_1,X_2,X_4,X_3)$;
(iii) $P(X_1,X_2,X_3,X_4) = -P(X_3,X_4,X_1,X_2)$;
(iv) $P(X_1,X_2,X_3,X_4) = P(X_4,X_3,X_2,X_1)$.

PROPOSITION 3.3. *Let M^{2n+1} be a manifold with an $[r]$-Sasakian structure (φ,ξ,η,g). Then*

$$(16)\qquad \rho(X,\xi) = 2nr\eta(X)$$

for any $X \in \Gamma(TM)$.

Proof. We choose a local orthonormal frame $(E_i)_{1\le i\le 2n+1}$ on M. Putting $\alpha_i := g(E_i,E_i)$, using (11), (13) and the definition of P, since $I = -r\varphi^2 + \eta\otimes\xi$ and $d\eta(X,Y) = \Phi(X,Y) = g(X,\varphi Y)$, one has, for any $X \in \Gamma(TM)$,

$$\begin{aligned}\rho(X,\xi) &= \sum_{i=1}^{2n+1} \alpha_i R(X,E_i,\xi,E_i) = -r\sum_{i=1}^{2n+1} \alpha_i R(X,E_i,\xi,\varphi^2 E_i)\\ &= -r\Big(\sum_{i=1}^{2n+1} \alpha_i g(R(X,E_i)\varphi(\xi),\varphi E_i) + \varepsilon r\sum_{i=1}^{2n+1} \alpha_i P(X,E_i,\xi,\varphi E_i)\Big)\\ &= -\varepsilon\sum_{i=1}^{2n+1} \alpha_i (g(\varphi X,\varphi E_i)g(\xi,E_i) - g(\varphi E_i,\varphi E_i)g(X,\xi))\\ &= -r\varepsilon\sum_{i=1}^{2n+1} \alpha_i (g(X,E_i)g(\xi,E_i) - g(E_i,E_i)g(X,\xi))\\ &= -r\varepsilon\Big\{g(X,\xi) - \sum_{i=1}^{2n+1} \alpha_i^2 g(X,\xi)\Big\} = r2n\eta(X).\ ∎\end{aligned}$$

4. Mixed 3-Sasakian structures and Ricci curvature. As stated in [8], a split three Sasakian manifold is Einstein. We give here a direct proof and examine some consequences.

THEOREM 4.1. *Any mixed* 3*-Sasakian manifold* $(M^{4n+3}, \varphi_a, \xi_a, \eta^a, g)$ *is Einstein. More precisely, for any* $X, Y \in \Gamma(TM)$*, one has*

$$\rho(X,Y) = -\sigma(4n+2)g(X,Y),$$

where $\sigma = \pm 1$*, according as the* 3*-structure is positive or negative.*

Proof. Let us put, for any $X, Y \in \Gamma(TM)$,

$$Q(X,Y) := \rho(X, \varphi_3 Y) - \rho(Y, \varphi_3 X) + 2\sigma(4n+1)g(X, \varphi_3 Y). \tag{17}$$

We are going to prove that

$$Q(X,Y) = \sum_{i=1}^{4n+3} \varepsilon_i g(R(X,Y)e_i, \varphi_3(e_i)), \tag{18}$$

where $(e_i)_{1 \leq i \leq 4n+3}$ is an arbitrary orthonormal local frame on M, and $\varepsilon_i := g(e_i, e_i)$. Since the structure $(\varphi_3, \xi_3, \eta^3, g)$ is indefinite Sasakian, from (13), with $r = 1$ and $\varepsilon = g(\xi_3, \xi_3) = \mp 1 = -\sigma$ according as the 3-structure is positive or negative, we have

$$g(R(X,Y)Z, \varphi_3 W) = -g(R(X,Y)\varphi_3 Z, W) + \sigma P_3(X,Y,Z,W) \tag{19}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Using Bianchi's First Identity, (19) and Lemma 3.2, the right hand side of (18) becomes

$$\begin{aligned}
(20)\quad & \sum_{i=1}^{4n+3} \varepsilon_i g(R(X,Y)e_i, \varphi_3(e_i)) \\
&= -\sum_{i=1}^{4n+3} \varepsilon_i \{g(R(Y,e_i)X, \varphi_3(e_i)) + g(R(e_i,X)Y, \varphi_3(e_i))\} \\
&= \sum_{i=1}^{4n+3} \varepsilon_i \{g(R(Y,e_i)\varphi_3 X, e_i) - \sigma P_3(Y, e_i, X, e_i) \\
&\quad + g(R(e_i,X)\varphi_3 Y, e_i) - \sigma P_3(e_i, X, Y, e_i)\} \\
&= -\rho(Y, \varphi_3 X) + \rho(X, \varphi_3 Y) - 2\sigma \sum_{i=1}^{4n+3} \varepsilon_i P_3(Y, e_i, X, e_i).
\end{aligned}$$

Computing the last term, by the definition of P_3, one has

$$\begin{aligned}
(21)\quad \sum_{i=1}^{4n+3} \varepsilon_i P_3(Y, e_i, X, e_i) &= \sum_{i=1}^{4n+3} \varepsilon_i \{d\eta^3(Y,X)g(e_i,e_i) - d\eta^3(Y,e_i)g(e_i,X) \\
&\quad - d\eta^3(e_i,X)g(Y,e_i) - d\eta^3(e_i,e_i)g(Y,X)\} \\
&= (4n+3)g(\varphi_3 X, Y) + g(X, \varphi_3 Y) - g(\varphi_3 X, Y)
\end{aligned}$$

$$= (4n+3)g(\varphi_3 X, Y) - 2g(\varphi_3 X, Y)$$
$$= -(4n+1)g(X, \varphi_3 Y).$$

From (20) and (21), we obtain (17).

Now, let us choose a local orthonormal frame adapted to the 3-structure

$$(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i, \xi_1, \xi_2, \xi_3)_{1 \le i \le n}.$$

For any $i \in \{1, \dots, n\}$, we put $e_i := E_i$, $e_{n+i} := \varphi_1 E_i$, $e_{2n+i} := \varphi_2 E_i$ and $e_{3n+i} := \varphi_3 E_i$, and

$$\alpha_i := g(E_i, E_i) = -g(\varphi_1 E_i, \varphi_1 E_i) = -g(\varphi_2 E_i, \varphi_2 E_i) = g(\varphi_3 E_i, \varphi_3 E_i);$$

for any $a \in \{1, 2, 3\}$, we put also $e_{4n+a} := \xi_a$, and $\alpha_{4n+a} := g(\xi_a, \xi_a) = \varepsilon_a$. We get

$$\begin{aligned} Q(X,Y) = \sum_{i=1}^{n} \alpha_i \{ & g(R(X,Y)E_i, \varphi_3 E_i) - g(R(X,Y)\varphi_1 E_i, \varphi_3 \varphi_1 E_i) \\ & - g(R(X,Y)\varphi_2 E_i, \varphi_3 \varphi_2 E_i) + g(R(X,Y)\varphi_3 E_i, \varphi_3^2 E_i)\} \\ & + \varepsilon_1 g(R(X,Y)\xi_1, \varphi_3 \xi_1) + \varepsilon_2 g(R(X,Y)\xi_2, \varphi_3 \xi_2) \\ = \sum_{i=1}^{n} \alpha_i \{ & g(R(X,Y)E_i, \varphi_3 E_i) + g(R(X,Y)\varphi_1 E_i, \varphi_1 \varphi_3 E_i) \\ & + g(R(X,Y)\varphi_2 E_i, \varphi_2 \varphi_3 E_i) + g(R(X,Y)E_i, \varphi_3 E_i)\} \\ & + \varepsilon_1 g(R(X,Y)\xi_1, \varphi_1 \xi_3) + \varepsilon_2 g(R(X,Y)\xi_2, \varphi_2 \xi_3). \end{aligned}$$

Since the structures $(\varphi_1, \xi_1, \eta^1, g)$ and $(\varphi_2, \xi_2, \eta^2, g)$ are both parasasakian, using (13) with $r = -1$, one has

$$\begin{aligned} Q(X,Y) = \sum_{i=1}^{n} \alpha_i \{ & g(R(X,Y)E_i, \varphi_3 E_i) - g(R(X,Y)\varphi_1^2 E_i, \varphi_3 E_i) \\ & + \varepsilon_1 P_1(X, Y, \varphi_1 E_i, \varphi_3 E_i) - g(R(X,Y)\varphi_2^2 E_i, \varphi_3 E_i) \\ & + \varepsilon_2 P_2(X, Y, \varphi_2 E_i, \varphi_3 E_i) + g(R(X,Y)E_i, \varphi_3 E_i)\} \\ & + P_1(X, Y, \xi_1, \xi_3) + P_2(X, Y, \xi_2, \xi_3) \\ = \sum_{i=1}^{n} \alpha_i \{ & \varepsilon_1 P_1(X, Y, \varphi_1 E_i, \varphi_3 E_i) + \varepsilon_2 P_2(X, Y, \varphi_2 E_i, \varphi_3 E_i)\} \\ & + P_1(X, Y, \xi_1, \xi_3) + P_2(X, Y, \xi_2, \xi_3). \end{aligned}$$

Recalling the definition of the tensor field P, since $d\eta^1 = \Phi_1$, $d\eta^2 = \Phi_2$,

$\varepsilon_1 = \varepsilon_2 = \sigma = -\varepsilon_3$ and $\sigma\varepsilon_1 = \sigma\varepsilon_2 = 1$, using (1), (3) and (4), one has

$$(22)\quad Q(X,Y) = -2\sigma\Big\{\sum_{i=1}^{n}\alpha_i((g(X,E_i)g(\varphi_3Y,E_i)-g(X,\varphi_2E_i)g(\varphi_3Y,\varphi_2E_i) + g(\varphi_3Y,\varphi_3E_i)g(X,\varphi_3E_i) - g(\varphi_3Y,\varphi_1E_i)g(X,\varphi_1E_i)) + \varepsilon_1 g(X,\xi_1)g(\varphi_3Y,\xi_1) + \varepsilon_2 g(X,\xi_2)g(\varphi_3Y,\xi_2)\Big\} = -2\sigma g(X,\varphi_3Y).$$

From (17) and (22), it follows that

$$(23)\qquad \rho(X,\varphi_3Y) - \rho(\varphi_3X,Y) = -2\sigma(4n+2)g(X,\varphi_3Y).$$

Since the structure $(\varphi_3,\xi_3,\eta^3,g)$ is indefinite Sasakian, one has $\rho(X,\varphi_3Y) = -\rho(\varphi_3X,Y)$ for any X,Y orthogonal to ξ_3 (cf. [3] for the Riemannian case). From (23) it follows that $\rho(X,\varphi_3Y) = -\sigma(4n+2)g(X,\varphi_3Y)$ for any X,Y orthogonal to ξ_3. Replacing Y with φ_3Y, since Y is orthogonal to ξ_3, one has

$$(24)\qquad \rho(X,Y) = -\sigma(4n+2)g(X,Y), \qquad X,Y \perp \xi_3.$$

Using (16), we have

$$(25)\qquad \rho(X,\xi_3) = -\sigma(4n+2)g(X,\xi_3), \qquad X \in \Gamma(TM),$$

hence, putting $X = \xi_3$,

$$(26)\qquad \rho(\xi_3,\xi_3) = -\sigma(4n+2)g(\xi_3,\xi_3).$$

Finally, if $X,Y \in \Gamma(TM)$, writing $X = X_0 + \lambda\xi_3$ and $Y = Y_0 + \mu\xi_3$ with X_0,Y_0 orthogonal to ξ_3, and $\lambda,\mu \in \mathfrak{F}(M)$, using (24)–(26), one gets $\rho(X,Y) = -\sigma(4n+2)g(X,Y)$ for any $X,Y \in \Gamma(TM)$, concluding the proof. ∎

As an obvious consequence of the above result, we have

PROPOSITION 4.2. *Any mixed* 3*-Sasakian manifold* $(M^{4n+3},\varphi_a,\xi_a,\eta^a,g)$ *has constant scalar curvature*

$$\mathrm{Sc} = -\sigma(4n+2)(4n+3),$$

therefore negative or positive according as the 3*-structure is positive or negative.*

PROPOSITION 4.3. *Let* $(M^{4n+3},\varphi_a,\xi_a,\eta^a,g)$ *be a mixed* 3*-Sasakian manifold. Then* M *has* (*pointwise*) *constant sectional curvature* k *if and only if* $k = \mp 1$ *according as the* 3*-structure is positive or negative.*

Proof. Since the 3-structure $(\varphi_a,\xi_a,\eta^a,g)$ is mixed 3-Sasakian, (13) holds for any $a \in \{1,2,3\}$. Using the constant $\sigma = \pm 1$ according as the 3-structure is positive or negative, and recalling that $\tau_a\varepsilon_a = -\sigma$, we have, for any

$a \in \{1,2,3\}$ and $X,Y,Z,W \in \Gamma(TM)$,

$$g(R(X,Y)Z,\varphi_a W) + g(R(X,Y)\varphi_a Z, W) = \sigma P_a(X,Y,Z,W).$$

Supposing that M has pointwise constant sectional curvature $k \in \mathfrak{F}(M)$, i.e. $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}$, we have

$$\begin{aligned}\sigma P_a(X,Y,Z,W) &= g(R(X,Y)Z,\varphi_a W) + g(R(X,Y)\varphi_a Z,W)\\ &= k\{g(Y,Z)g(X,\varphi_a W) - g(X,Z)g(Y,\varphi_a W)\\ &\quad + g(Y,\varphi_a Z)g(X,W) - g(X,\varphi_a Z)g(Y,W)\}\\ &= k\{d\eta^a(X,W)g(Y,Z) - d\eta^a(Y,W)g(X,Z)\\ &\quad + d\eta^a(Y,Z)g(X,W) - d\eta^a(X,Z)g(Y,W)\}\\ &= -\,kP_a(X,Y,Z,W),\end{aligned}$$

hence, for any $a \in \{1,2,3\}$ and any $X,Y,Z,W \in \Gamma(TM)$, it follows that $(k+\sigma)P_a(X,Y,Z,W) = 0$, and so $k = -\sigma = \mp 1$ according as the 3-structure is positive or negative. Namely, choosing a vector field Y orthogonal to ξ_1,ξ_2,ξ_3 such that $g(Y,Y) \neq 0$, by the definition of P_a given in Lemma 3.1, we get $P_a(\xi_a,Y,\xi_a,\varphi_a Y) = -\varepsilon_a g(Y,Y) \neq 0$. ■

5. Mixed metric 3-contact and mixed 3-Sasakian structures. In this section we shall be concerned with some properties of particular classes of mixed metric 3-structures, namely the class of mixed metric 3-contact structures, which reflect analogous properties of classical metric 3-structures (see [3] for more details).

Definition 5.1. Let M be a manifold with a mixed metric 3-structure $(\varphi_a,\xi_a,\eta^a,g)$. The structure is said to be a *mixed metric* 3-*contact structure* if $d\eta^a = \Phi_a$ for each $a \in \{1,2,3\}$, where Φ_a is the fundamental 2-form defined by $\Phi_a(X,Y) := g(X,\varphi_a Y)$. Then $(M,\varphi_a,\xi_a,\eta^a,g)$ will be called a *mixed metric* 3-*contact manifold*.

Our intent here is to prove that any mixed metric 3-contact manifold is in fact a mixed 3-Sasakian manifold.

Let M be a manifold with a mixed metric 3-structure $(\varphi_a,\xi_a,\eta^a,g)$. Setting $\tilde{M} = M \times \mathbb{R}$, and denoting by t the coordinate on $\mathbb{R}$, define three $(1,1)$-type tensor fields J_a, $a = 1,2,3$, by putting, for any $\tilde{X} = \big(X, f\frac{d}{dt}\big) \in \Gamma(T\tilde{M})$, with $X \in \Gamma(TM)$ and $f \in \mathfrak{F}(\tilde{M})$,

$$J_a(\tilde{X}) = J_a\Big(X, f\,\frac{d}{dt}\Big) := \Big(\varphi_a X - \tau_a f \xi_a, \eta^a(X)\,\frac{d}{dt}\Big),$$

where $\tau_1 = \tau_2 = -1 = -\tau_3$. Furthermore, define the $(0,2)$-type tensor field G, by putting, for any $\tilde{X} = \big(X, f\frac{d}{dt}\big)$ and $\tilde{Y} = \big(Y, h\frac{d}{dt}\big)$ in $\Gamma(T\tilde{M})$, with

$X, Y \in \Gamma(TM)$ and $f, h \in \mathfrak{F}(\tilde{M})$,

$$G(\tilde{X}, \tilde{Y}) := g(X, Y) - \sigma f h,$$

where $\sigma = \pm 1$ according as the 3-structure is positive or negative.

Proposition 5.2. *$(\tilde{M}, J_a, G)_{a=1,2,3}$ is an almost hyper parahermitian manifold.*

Proof. Let $a \in \{1,2,3\}$ and $\tilde{X} \in \Gamma(T\tilde{M})$ with $\tilde{X} = \left(X, f\,\frac{d}{dt}\right)$. Since by definition $\varphi_a^2 = -\tau_a(I - \eta^a \otimes \xi_a)$, we have

$$(J_a)^2(\tilde{X}) = \left((\varphi_a)^2 X - \tau_a \eta^a(X)\xi_a, -\tau_a f\,\frac{d}{dt}\right) = -\tau_a \tilde{X},$$

hence $(J_a)^2 = -\tau_a I$. Let now (a,b,c) be a cyclic permutation of $(1,2,3)$. Using (1)–(3), one has, for any $\tilde{X} \in \Gamma(T\tilde{M})$ with $\tilde{X} = \left(X, f\,\frac{d}{dt}\right)$,

$$\begin{aligned} J_a J_b(\tilde{X}) &= \left(\varphi_a\varphi_b X - \tau_b f \varphi_a \xi_b - \tau_a \eta^b(X)\xi_a, (\eta^a(\varphi_b X) - \tau_b f \eta^a \xi_b)\,\frac{d}{dt}\right) \\ &= \left(\tau_c \varphi_c X - f\xi_c, \tau_c \eta^c(X)\,\frac{d}{dt}\right) = \tau_c J_c(\tilde{X}), \end{aligned}$$

hence $J_a J_b = \tau_c J_c$. Analogously, $J_b J_a = -\tau_c J_c$, and this proves that $(J_a)_{a=1,2,3}$ is an almost hyper paracomplex structure on $\tilde{M}$. Let now $a \in \{1,2,3\}$, $\tilde{X} = \left(X, f\,\frac{d}{dt}\right)$ and $\tilde{Y} = \left(Y, h\,\frac{d}{dt}\right)$. Since, by (4), $g(\varphi_a X, Y) = -g(X, \varphi_a Y)$, using the identity $\tau_a \varepsilon_a = -\sigma$, by standard calculations we have $G(\tilde{X}, J_a\tilde{Y}) = -G(J_a(\tilde{X}), \tilde{Y})$, and by Definition 2.2 it follows that $(\tilde{M}, J_a, G)$, $a \in \{1,2,3\}$, is an almost hyper parahermitian manifold. ∎

Remark 5.3. It is clear that the tensor fields J_a constructed on $\tilde{M}$ are almost product structures for $a = 1, 2$, and an almost complex structure for $a = 3$. The three structures $(\varphi_a, \xi_a, \eta^a, g)$ are normal if and only if the manifold $(\tilde{M}, J_a, G)$, $a \in \{1,2,3\}$, is hyper parahermitian.

Thus, we may state:

Proposition 5.4. *Let M be a manifold endowed with a mixed 3-structure $(\varphi_a, \xi_a, \eta^a)$. Then the structures are normal if and only if at least two of them are normal.*

We shall see in a moment that the manifold $(\tilde{M}, J_a, G)$, $a \in \{1,2,3\}$, is indeed hyper parahermitian if the 3-structure is a mixed metric 3-contact structure. To this end, let us prove the following preliminary results.

Lemma 5.5. *Let M be a manifold endowed with a mixed metric 3-contact structure. Denoting, for any $a \in \{1,2,3\}$, by Ω_a the fundamental 2-form associated with the structure (J_a, G) defined by $\Omega_a(\tilde{X}, \tilde{Y}) := G(\tilde{X}, J_a\tilde{Y})$, we have*

$$d\Omega_a = 2\sigma dt \wedge \Omega_a$$

for any $a \in \{1,2,3\}$, *where* $\sigma = \pm 1$ *according as the* 3*-structure is positive or negative.*

Proof. Fixing $a \in \{1,2,3\}$, let us compute $d\Omega_a$ using the formula

$$3d\Omega_a(\tilde X,\tilde Y,\tilde Z) = \underset{(\tilde X,\tilde Y,\tilde Z)}{\mathfrak{S}} \{\tilde X(\Omega_a(\tilde Y,\tilde Z)) - \Omega_a([\tilde X,\tilde Y],\tilde Z)\}, \tag{27}$$

for any $\tilde X,\tilde Y,\tilde Z \in \Gamma(T\tilde M)$. Putting $\tilde X = \big(X, f\,\frac{d}{dt}\big)$, $\tilde Y = \big(Y, h\,\frac{d}{dt}\big)$ and $\tilde Z = \big(Z, k\,\frac{d}{dt}\big)$ and using $\tau_a\varepsilon_a = -\sigma$, we have

$$\Omega_a(\tilde Y,\tilde Z) = \Phi_a(Y,Z) + \sigma(k\eta^a(Y) - h\eta^a(Z)). \tag{28}$$

Furthermore, $[\tilde X,\tilde Y] = \big([X,Y], \big(X(h) - Y(f) + f\,\frac{dh}{dt} - h\,\frac{df}{dt}\big)\frac{d}{dt}\big)$ and

$$\begin{aligned}\Omega_a([\tilde X,\tilde Y],\tilde Z) &= \Phi_a([X,Y],Z)\\ &\quad + \sigma\bigg\{k\eta^a[X,Y] - \bigg(X(h) - Y(f) + f\,\frac{dh}{dt} - h\,\frac{df}{dt}\bigg)\eta^a(Z)\bigg\}.\end{aligned}$$

Finally, from (28),

$$\begin{aligned}\tilde X(\Omega_a(\tilde Y,\tilde Z)) &= X(\Phi_a(Y,Z) + \sigma(k\eta^a(Y) - h\eta^a(Z)))\\ &\quad + f\,\frac{d}{dt}(\Phi_a(Y,Z) + \sigma(k\eta^a(Y) - h\eta^a(Z)))\\ &= X(\Phi_a(Y,Z)) + \sigma(X(k)\eta^a(Y) + kX(\eta^a(Y))\\ &\quad - X(h)\eta^a(Z) - hX(\eta^a(Z))) + \sigma\bigg(f\,\frac{dk}{dt}\,\eta^a(Y) - f\,\frac{dh}{dt}\,\eta^a(Z)\bigg).\end{aligned}$$

From (27), using the above identities and $d\Phi_a = 0$, one gets

$$3d\Omega_a(\tilde X,\tilde Y,\tilde Z) = 2\sigma(\Phi_a(X,Y)k + \Phi_a(Y,Z)f + \Phi_a(Z,X)h).$$

Finally, using (28), it follows that

$$\begin{aligned}3d\Omega_a(\tilde X,\tilde Y,\tilde Z) &= 2\sigma\big(f\Omega_a(\tilde Y,\tilde Z) - \sigma(fk\eta^a(Y) - fh\eta^a(Z))\\ &\quad + h\Omega_a(\tilde Z,\tilde X) - \sigma(hf\eta^a(Z) - hk\eta^a(X))\\ &\quad + k\Omega_a(\tilde X,\tilde Y) - \sigma(kh\eta^a(X) - kf\eta^a(Y))\big)\\ &= 2\sigma(f\Omega_a(\tilde Y,\tilde Z) + h\Omega_a(\tilde Z,\tilde X) + k\Omega_a(\tilde X,\tilde Y))\\ &= 6\sigma(dt \wedge \Omega_a)(\tilde X,\tilde Y,\tilde Z),\end{aligned}$$

hence $d\Omega_a = 2\sigma dt \wedge \Omega_a$. ∎

LEMMA 5.6. *Let* (M, J_a, g), $a \in \{1,2,3\}$, *be an almost hyper parahermitian manifold such that, denoting by* Ω_a *the fundamental* 2*-form associated with* J_a, *there exists a* 1*-form* ω *satisfying* $d\Omega_a = k\omega \wedge \Omega_a$ *for any* $a \in \{1,2,3\}$ *with* $k \in \mathfrak{F}(M)$. *Then each structure* J_a *is integrable and the manifold is hyper parahermitian.*

Proof. Let us prove that $N_1 = 0$. It is well known that

$$N_1(X,Y) = (\nabla_{J_1X}J_1)(Y) - (\nabla_{J_1Y}J_1)(X) - J_1(\nabla_XJ_1)(Y) + J_1(\nabla_YJ_1)(X),$$

hence, using (i) and (ii) of Definition 2.2, we get

$$(29)\qquad \begin{aligned} J_2N_1(X,Y) &= -J_2(\nabla_{J_1Y}J_1)(X) - J_3(\nabla_Y J_1)(X) \\ &\quad + J_2(\nabla_{J_1X}J_1)(Y) + J_3(\nabla_X J_1)(Y). \end{aligned}$$

Then, for any $Z \in \Gamma(TM)$, using (iii) of Definition 2.2, by standard calculations, one has

$$\begin{aligned} g(-J_2(\nabla_{J_1Y}J_1)(X),Z) &= -g(J_2\nabla_{J_1Y}(J_1X),Z) - g(J_3\nabla_{J_1Y}X,Z) \\ &= -g(X,(\nabla_{J_1Y}J_3)(Z)) - g(J_1X,(\nabla_{J_1Y}J_2)(Z)) \\ &= (\nabla_{J_1Y}\Omega_3)(Z,X) + (\nabla_{J_1Y}\Omega_2)(Z,J_1X). \end{aligned}$$

Switching X and Y one has

$$g(J_2(\nabla_{J_1X}J_1)(Y),Z) = (\nabla_{J_1X}\Omega_3)(Y,Z) + (\nabla_{J_1X}\Omega_2)(J_1Y,Z).$$

Analogously, one obtains

$$g(-J_3(\nabla_Y J_1)(X),Z) = (\nabla_Y\Omega_2)(Z,X) + (\nabla_Y\Omega_3)(Z,J_1X)$$

and switching X and Y one gets

$$g(J_3(\nabla_X J_1)(Y),Z) = (\nabla_X\Omega_2)(Y,Z) + (\nabla_X\Omega_3)(J_1Y,Z).$$

Since $3d\Omega(X,Y,Z) = \underset{(X,Y,Z)}{\mathfrak{S}} (\nabla_X\Omega)(Y,Z)$, from (29) we have

$$\begin{aligned} g(J_2N_1(X,Y),Z) &= 3d\Omega_2(X,Y,Z) + 3d\Omega_3(X,J_1Y,Z) \\ &\quad + 3d\Omega_3(J_1X,Y,Z) + 3d\Omega_2(J_1X,J_1Y,Z). \end{aligned}$$

As $d\Omega_a = k\omega \wedge \Omega_a$, we get

$$\begin{aligned} g(J_2N_1(X,Y),Z) = k\{&\omega(X)\Omega_2(Y,Z) + \omega(Y)\Omega_2(Z,X) \\ &+ \omega(Z)\Omega_2(X,Y) + \omega(X)\Omega_3(J_1Y,Z) \\ &+ \omega(J_1Y)\Omega_3(Z,X) + \omega(Z)\Omega_3(X,J_1Y) \\ &+ \omega(J_1X)\Omega_3(Y,Z) + \omega(Y)\Omega_3(Z,J_1X) \\ &+ \omega(Z)\Omega_3(J_1X,Y) + \omega(J_1X)\Omega_2(J_1Y,Z) \\ &+ \omega(J_1Y)\Omega_2(Z,J_1X) + \omega(Z)\Omega_2(J_1X,J_1Y)\}. \end{aligned}$$

It is easy to check that $\Omega_3(J_1Y,Z) = -\Omega_2(Y,Z)$, $\Omega_3(Y,J_1Z) = -\Omega_2(Y,Z)$, $\Omega_2(Z,J_1X) = -\Omega_3(Z,X)$, $\Omega_2(J_1Z,X) = -\Omega_3(Z,X)$ and $\Omega_2(J_1X,J_1Y) = \Omega_2(X,Y)$. Therefore, $g(J_2N_1(X,Y),Z) = 0$, hence $N_1 = 0$. In an analogous way, one proves that $N_2 = 0$ and $N_3 = 0$. ∎

As an obvious consequence of Lemmas 5.5 and 5.6, one obtains the following result.

THEOREM 5.7. *Any mixed metric* 3*-contact structure on a manifold is mixed* 3*-Sasakian.*

Thus, Theorems 4.1–4.3 may be formulated for mixed metric 3-contact manifolds.

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