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Weighted composition operators from Zygmund spaces to Bloch spaces on the unit ball

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Abstract. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$. Let φ be a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. The weighted composition operator uC_φ on $H(\mathbb{B})$ is defined by

$$uC_\varphi f(z) = u(z)f(\varphi(z)).$$

We investigate the boundedness and compactness of uC_φ induced by u and φ acting from Zygmund spaces to Bloch (or little Bloch) spaces in the unit ball.

1. Introduction. Let $H(\mathbb{B})$ be the class of all holomorphic functions on $\mathbb{B}$, where $\mathbb{B}$ is the unit ball in the n-dimensional complex space $\mathbb{C}^n$. Denote by $S(\mathbb{B})$ the collection of all holomorphic self-mappings of $\mathbb{B}$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$. Then we write

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}, \qquad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

For $f \in H(\mathbb{B})$, let ∇f and $\Re f$ denote the complex gradient and the radial derivative of $f \in H(\mathbb{B})$, i.e.

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right), \qquad \Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), \bar{z}\rangle.$$

Moreover, we write $\Re^m f = \Re(\Re^{m-1} f)$ for $f \in H(\mathbb{B})$ and $m \in \mathbb{N}$.

The *Bloch space* $\mathcal{B} = \mathcal{B}(\mathbb{B})$ is the space of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{B}} (1 - |z|^2)|\nabla f(z)| < \infty.$$

It is clear that $\mathcal{B}$ is a Banach space under the norm $|f(0)| + \|f\|_{\mathcal{B}}$. Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ for which $(1-|z|^2)|\nabla f(z)|$

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converges to zero as $|z| \to 1$. This space is called the *little Bloch space*. It follows from [19, Theorem 3.4] that $f \in \mathcal{B}$ if and only if

$$\sup_{z\in\mathbb{B}} (1-|z|^2)|\Re f(z)| < \infty.$$

Also $f \in \mathcal{B}_0$ if and only if $\lim_{|z|\to 1}(1-|z|^2)|\Re f(z)| = 0$. From [13, Theorem 2.1] we obtain

$$(1-|z|^2)|\Re f(z)| \asymp (1-|z|^2)|\nabla f(z)|,$$

thus

$$\|f\|_{\mathcal{B}} \asymp \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re f(z)|.$$

For $\alpha \in (0,1)$, $\mathcal{L}_{1-\alpha} = \mathcal{L}_{1-\alpha}(\mathbb{B})$ denotes the holomorphic $(1-\alpha)$-*Lipschitz space* which is the set of all $f \in H(\mathbb{B})$ such that for some $C > 0$, we have

$$|f(z)-f(w)| \le C|z-w|^{1-\alpha} \quad \text{for every } z, w \in \mathbb{B}.$$

Moreover, $\mathcal{L}_{1-\alpha}$ is endowed with a complete norm $\|\cdot\|_{\mathcal{L}_{1-\alpha}}$ given by

$$\|f\|_{\mathcal{L}_{1-\alpha}} = |f(0)| + \sup_{z\neq w:\, z,w\in\overline{\mathbb{B}}} \frac{|f(z)-f(w)|}{|z-w|^{1-\alpha}}. \tag{1.1}$$

In the above, $\mathbb{B}$ and $\overline{\mathbb{B}}$ are interchangeable, since functions in $\mathcal{L}_{1-\alpha}$ extend continuously to $\overline{\mathbb{B}}$. The supremum in (1.1) is called the Lipschitz constant for f. By [1, Theorem 3.5], it follows that the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B})$ equals $\mathcal{L}_{1-\alpha}(\mathbb{B})$ and

$$\|f\|_{\mathcal{B}^\alpha} \asymp \sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha|\Re f(z)| \asymp \|f\|_{\mathcal{L}_{1-\alpha}}.$$

Let $\mathcal{Z} = \mathcal{Z}(\mathbb{B})$ denote the space of all $f \in H(\mathbb{B})$ such that

$$\sup_{z\in\mathbb{B}} (1-|z|^2)|\Re^2 f(z)| < \infty.$$

It is also known that $f \in \mathcal{Z}$ if and only if $f \in A(\mathbb{B})$ and there exists a constant $C > 0$ such that

$$|f(\zeta+h)+f(\zeta-h)-2f(\zeta)| < C|h|$$

for all $\zeta \in \partial\mathbb{B}$ and $\zeta \pm h \in \partial\mathbb{B}$ (see e.g. [19, p. 261]). It is clear that $\mathcal{Z}$ is a Banach space with the norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re^2 f(z)|.$$

Let $\mathcal{Z}_0$ denote the closure in $\mathcal{Z}$ of the set of all polynomials. From [19, Theorem 7.18], we see that

$$f \in \mathcal{Z}_0 \iff \lim_{|z|\to 1} (1-|z|^2)|\Re^2 f(z)| = 0.$$

Equivalently (see e.g. [3]), $\mathcal{Z}$ consists of all $f \in H(\mathbb{B})$ whose first order partial derivatives are in the Bloch space, namely $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{B}}(1-|z|^2)\sum_{i,j=1}^{n}\left|\frac{\partial^2 f}{\partial z_i \partial z_j}(z)\right| < \infty.$$

Also, $f \in \mathcal{Z}_0$ if and only if

$$\lim_{|z|\to 1}(1-|z|^2)\sum_{i,j=1}^{n}\left|\frac{\partial^2 f}{\partial z_i \partial z_j}(z)\right| = 0. \tag{1.2}$$

Let $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$. The weighted composition operator uC_φ is defined by

$$uC_\varphi(f) = u(f\circ\varphi), \qquad f \in H(\mathbb{B}).$$

It is obvious that when $u = 1$, we have the composition operator C_φ given by

$$(C_\varphi f)(z) = f(\varphi(z)), \qquad f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

When $\varphi(z) = z$, we obtain the multiplication operator $M_u f(z) = u(z)f(z)$. Therefore the weighted composition operator can be regarded as a generalization of a multiplication operator and a composition operator.

Recently, there has been an increasing interest in describing the boundedness and compactness of weighted composition operators acting on different spaces of holomorphic functions in terms of the inducing functions u and mappings φ; see, for example, [2, 6–9, 14–18] and numerous references therein.

In 2008, Li and Stević [4] characterized the properties of

$$uC_\varphi : \mathcal{Z}\ (\text{or } \mathcal{Z}_0) \to \mathcal{B}\ (\text{or } \mathcal{B}_0)$$

on the unit disk.

In 2012, X. L. Zhu [20] gave some conditions for the boundedness and compactness of C_φ acting from $\mathcal{Z}$ to $\mathcal{B}$ (or $\mathcal{B}_0$) by using the notations

$$D\varphi(z) = \begin{pmatrix} \frac{\partial\varphi_1(z)}{\partial z_1} & \cdots & \frac{\partial\varphi_1(z)}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial\varphi_n(z)}{\partial z_1} & \cdots & \frac{\varphi_n(z)}{\partial z_n} \end{pmatrix},$$

$$|D\varphi(z)| = \left(\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|^2\right)^{1/2}.$$

Our main aim is to generalize the results of [4] and [20].

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then we write $a \asymp b$.

2. Some lemmas. In this section, we state several lemmas which will be used in the following proofs.

LEMMA 2.1 ([5, Lemma 1]). *Suppose that $f \in \mathcal{Z}$. Then:*

(a) *There is a positive constant C independent of f such that*

$$|\Re f(z)| \le C\|f\|_{\mathcal{Z}} \log \frac{e}{1-|z|^2}. \tag{2.1}$$

(b) *There is a positive constant C independent of f such that*

$$\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)| \le C\|f\|_{\mathcal{Z}}. \tag{2.2}$$

REMARK 2.2. (1) It is obvious that $\mathcal{Z}$ is a subset of the α-Bloch space $\mathcal{B}^\alpha$ consisting of all $f \in H(\mathbb{B})$ such that $\sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha|\Re f(z)| < \infty$, for $\alpha > 0$. Indeed, for any $f \in \mathcal{Z}$, (2.1) is true, and so

$$(1-|z|^2)^\alpha|\Re f(z)| \le C\|f\|_{\mathcal{Z}}(1-|z|^2)^\alpha \log \frac{e}{1-|z|^2} < \infty$$

for any $\alpha > 0$.

(2) Every bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in the α-Bloch space $\mathcal{B}^\alpha$ $(0<\alpha<1)$ which converges to zero uniformly on compact subsets of $\mathbb{B}$, converges to zero uniformly on $\overline{\mathbb{B}}$. We include the proof for the convenience of the readers.

For $\epsilon > 0$, let $r \in (1-\epsilon, 1)$ and $w \in r\overline{\mathbb{B}}$. Since $\mathcal{B}^\alpha = \mathcal{L}_{1-\alpha}$ and by (1.1), it follows that $|f_k(z) - f_k(w)| \le C|z-w|^{1-\alpha}$ for $z, w \in \overline{\mathbb{B}}$. Thus $|f_k(z)| \le |f_k(w)| + C|z-w|^{1-\alpha}$. Since $\sup_{w\in r\overline{\mathbb{B}}} |f_k(w)| \to 0$ and moreover, for any $z \in \overline{\mathbb{B}} \setminus r\mathbb{B}$, there exists $w_z \in \mathbb{B}$ satisfying $|z-w_z| < n\epsilon$, it follows that

$$\lim_{k\to\infty} \sup_{z\in\overline{\mathbb{B}}} |f_k(z)| \le \lim_{k\to\infty} \sup_{z\in\overline{\mathbb{B}}} |z-w_z|^{1-\alpha} \le (n\epsilon)^{1-\alpha}.$$

(3) If $0 < \alpha < 1$, we have $\mathcal{Z} \subset \mathcal{B}^\alpha$, and hence every bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{Z}$ which converges to zero uniformly on compact subsets of $\mathbb{B}$, converges to zero uniformly on $\overline{\mathbb{B}}$.

Using the above fact and a modification of [2, Proposition 3.11] yields the following lemma.

LEMMA 2.3. *Let $u \in H(\mathbb{B})$ and $\varphi = (\varphi_1, \dots, \varphi_n) \in S(\mathbb{B})$. Then $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact if and only if $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{Z}$ which converges to zero uniformly on $\overline{\mathbb{B}}$ as $k \to \infty$, the sequence $\|uC_\varphi f_k\|_{\mathcal{B}}$ converges to zero as $k \to \infty$.*

LEMMA 2.4 ([3, Lemma 2.3]). *Let $f \in \mathcal{Z}$ and*

$$Q_f(z) = \sup\left\{ \frac{|\langle \nabla f(z), \bar{u}\rangle|}{|u| + \log\frac{2}{1-|z|^2}|\langle u, z\rangle|} : 0 \ne u \in \mathbb{C}^n \right\}.$$

Then $\sup\{Q_f(z) : z \in \mathbb{B}\} \le C\|f\|_{\mathcal{Z}}$.

LEMMA 2.5 ([12, Lemma 4]). *A closed set K in $\mathcal{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)|\Re f(z)| = 0.$$

3. Main results. In this section we characterize the boundedness and compactness of the operator uC_φ acting from Zygmund spaces to Bloch (or little Bloch) spaces in the unit ball.

THEOREM 3.1. *Suppose $\varphi = (\varphi_1, \dots, \varphi_n) \in S(\mathbb{B})$ and $u \in H(\mathbb{B})$. Then the following statements are equivalent:*

(a) *$uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded.*

(b) *$uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is bounded.*

(c) (i) *$u \in \mathcal{B}$, that is,* $\sup_{z\in\mathbb{B}} (1-|z|^2)|\Re u(z)| < \infty$;

(ii) $\sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\Re\varphi(z)| < \infty$*;*

(iii) $\sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)| \log \frac{2}{1-|\varphi(z)|^2} |\langle \Re\varphi(z), \varphi(z)\rangle| < \infty$,

where $\Re\varphi(z) = (\Re\varphi_1(z), \dots, \Re\varphi_n(z))$.

Proof. (a)⇒(b). This follows from $\mathcal{Z}_0 \subset \mathcal{Z}$.

(b)⇒(c). Taking $f_0(z) = 1 \in \mathcal{Z}_0$, it follows that

$$\|uC_\varphi f_0\|_{\mathcal{B}} = \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re u(z)| < \infty,$$

that is, $u \in \mathcal{B}$. Then choosing $f_i = z_i \in \mathcal{Z}_0$, we get

$$\begin{aligned}\|uC_\varphi f_i\|_{\mathcal{B}} &= \sup_{z\in\mathbb{B}} (1-|z|^2)\big|\Re u(z)\varphi_i(z) + u(z)\Re\varphi_i(z)\big| \\ &\geq \sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)\Re\varphi_i(z)| - \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re u(z)\varphi_i(z)|.\end{aligned}$$

It follows that

$$\sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)\Re\varphi_i(z)| \leq \|uC_\varphi f_i\|_{\mathcal{B}} + \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re u(z)|\,|\varphi_i(z)| < \infty.$$

Thus

$$\sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\Re\varphi(z)| \leq \sum_{i=1}^{n} \sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\Re\varphi_i(z)| < \infty.$$

That is, (ii) of (c) holds.

Next, we choose

$$f_w(z) = (1-\langle z, \varphi(w)\rangle)\bigg(1 + \log\frac{2}{1-\langle z, \varphi(w)\rangle}\bigg). \tag{3.1}$$

It is easy to show that

$$\frac{\partial f_w(z)}{\partial z_i} = -\bigg(\log\frac{2}{1-\langle z, \varphi(w)\rangle}\bigg)\overline{\varphi_i(w)}$$

and

$$\frac{\partial^2 f_w(z)}{\partial z_i \partial z_j} = -\frac{\overline{\varphi_i(w)}\,\overline{\varphi_j(w)}}{1-\langle z,\varphi(w)\rangle}.$$

Then by (1.2), it follows that $f_w \in \mathcal{Z}_0$ and $\sup_{w\in\mathbb{B}} \|f_w\|_{\mathcal{Z}} < \infty$. By the boundedness of $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$, we have

$$\begin{aligned}\|uC_\varphi f_w\|_{\mathcal{B}} &\geq (1-|w|^2)|\Re(uf_w\circ\varphi)(w)| \\ &= (1-|w|^2)\big|\Re u(w) f_w(\varphi(w)) + u(w)\Re(f_w\circ\varphi)(w)\big| \\ &\geq (1-|w|^2)|u(w)|\,|\Re(f_w\circ\varphi)(w)| \\ &\quad - (1-|w|^2)|\Re u(w)|(1-|\varphi(w)|^2)\left(1+\log\frac{2}{1-|\varphi(w)|^2}\right).\end{aligned}$$

Since $\lim_{t\to 0} t\log(1/t) = 0$, we have $\sup_{t\in[0,1]} t\log(2/t) < \infty$. Thus

$$\sup_{w\in\mathbb{B}} (1-|\varphi(w)|^2)\left(1+\log\frac{2}{1-|\varphi(w)|^2}\right) < \infty.$$

From $u \in \mathcal{B}$, it follows that

$$\sup_{w\in\mathbb{B}} (1-|w|^2)|\Re u(w)|(1-|\varphi(w)|^2)\left(1+\log\frac{2}{1-|\varphi(w)|^2}\right) < \infty.$$

Therefore,

$$\sup_{w\in\mathbb{B}} (1-|w|^2)|u(w)|\,|\Re(f_w\circ\varphi)(w)| < \infty. \tag{3.2}$$

By an easy computation, we have

$$\begin{aligned}&\sup_{w\in\mathbb{B}} (1-|w|^2)|u(w)|\log\frac{2}{1-|\varphi(w)|^2}|\langle\Re\varphi(w),\varphi(w)\rangle| \\ &\qquad\qquad = \sup_{w\in\mathbb{B}} (1-|w|^2)|u(w)|\,|\Re(f_w\circ\varphi)(w)| < \infty.\end{aligned}$$

That is, (iii) of (c) is true.

(c)⇒(a). For any $f \in \mathcal{Z}$, using (c), Lemma 2.1 and Lemma 2.4, we deduce that

$$\begin{aligned}\|uC_\varphi f\|_{\mathcal{B}} &= \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re(uf\circ\varphi)(z)| \\ &\leq \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re u(z) f(\varphi(z))| + \sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\Re(f\circ\varphi)(z)| \\ &\leq \sup_{z\in\mathbb{B}} (1-|z|^2)|\Re u(z)|\,\|f\|_\infty \\ &\quad + \sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\langle\nabla f(\varphi(z)),\overline{\Re\varphi(z)}\rangle|\end{aligned}$$

$$\preceq \sup_{z\in\mathbb{B}}(1-|z|^2)|\Re u(z)|\,\|f\|_{\mathcal{Z}}$$
$$+\sup_{z\in\mathbb{B}}(1-|z|^2)|u(z)|\Big(|\Re\varphi(z)|+\log\frac{2}{1-|\varphi(z)|^2}|\langle\Re\varphi(z),\varphi(z)\rangle|\Big)\|f\|_{\mathcal{Z}}<\infty.$$

Also, $|uC_\varphi f(0)|=|u(0)|\,|f(\varphi(0))|<\infty$. Thus $uC_\varphi:\mathcal{Z}\to\mathcal{B}$ is bounded. ∎

THEOREM 3.2. *Suppose $\varphi=(\varphi_1,\dots,\varphi_n)\in S(\mathbb{B})$ and $u\in H(\mathbb{B})$. Then the following statements are equivalent:*

(a) *$uC_\varphi:\mathcal{Z}\to\mathcal{B}$ is compact.*
(b) *$uC_\varphi:\mathcal{Z}_0\to\mathcal{B}$ is compact.*

(c) (i) *$u\in\mathcal{B}$, that is, $\sup_{z\in\mathbb{B}}(1-|z|^2)|\Re u(z)|<\infty$;*
(ii) *$\lim_{|\varphi(z)|\to1}(1-|z|^2)|u(z)|\,|\Re\varphi(z)|=0$;*
(iii) *$\lim_{|\varphi(z)|\to1}(1-|z|^2)|u(z)|\log\frac{2}{1-|\varphi(z)|^2}|\langle\Re\varphi(z),\varphi(z)\rangle|=0$,*
where $\Re\varphi(z)=(\Re\varphi_1(z),\dots,\Re\varphi_n(z))$.

Proof. (a)⇒(b). This follows from $\mathcal{Z}_0\subset\mathcal{Z}$.

(b)⇒(c). Suppose that $uC_\varphi:\mathcal{Z}_0\to\mathcal{B}$ is compact, so in particular bounded. From Theorem 3.1, we find that (i) of (c) holds.

Choose $\{z_k\}_{k\in\mathbb{N}}\subset\mathbb{B}$ such that $|\varphi(z_k)|\to1$ as $k\to\infty$, where $z_k=(z_{1,k},\dots,z_{n,k})$. (If such a sequence does not exist, equalities (ii) and (iii) of (c) hold vacuously.) Define the function sequence

$$h_k(z)=(1-\langle z,\varphi(z_k)\rangle)\bigg(\Big(1+\log\frac{2}{1-\langle z,\varphi(z_k)\rangle}\Big)^2+1\bigg)\times\Big(\log\frac{2}{1-|\varphi(z_k)|^2}\Big)^{-1}.$$

It is obvious that h_k converges to zero uniformly on compact subsets of $\mathbb{B}$ as $k\to\infty$. Moreover,

$$\frac{\partial h_k(z)}{\partial z_i}(\varphi(z_k))=-\overline{\varphi_i(z_k)}\log\frac{2}{1-|\varphi(z_k)|^2},$$
$$\frac{\partial^2 h_k(z)}{\partial z_i\partial z_j}=\frac{-2\overline{\varphi_i(z_k)}\,\overline{\varphi_j(z_k)}}{1-\langle z,\varphi(z_k)\rangle}\Big(\log\frac{2}{1-\langle z,\varphi(z_k)\rangle}\Big)\Big(\log\frac{2}{1-|\varphi(z_k)|^2}\Big)^{-1},$$

so $h_k\in\mathcal{Z}_0$. Therefore,

$$\|uC_\varphi h_k\|_{\mathcal{B}}\ge(1-|z_k|^2)|\Re(uC_\varphi h_k)(z_k)|$$
$$\ge(1-|z_k|^2)|u(z_k)|\,|\Re(h_k\circ\varphi)(z_k)|-(1-|z_k|^2)|\Re u(z_k)|\,|h_k(\varphi(z_k))|.$$

Since

$$h_k(\varphi(z_k)) = 2(1-|\varphi(z_k)|^2)\Big(\log\frac{2}{1-|\varphi(z_k)|^2}\Big)^{-1} + 2(1-|\varphi(z_k)|^2) + (1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|^2} \to 0, \quad k\to\infty,$$

the compactness of $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ and (i) of (c) imply that

$$\lim_{k\to\infty}(1-|z_k|^2)|u(z_k)|\,|\Re(h_k\circ\varphi)(z_k)| = 0.$$

By an easy computation,

$$|\Re(h_k\circ\varphi)(z_k)| = \Big|\sum_{i=1}^{n}\frac{\partial h_k}{\partial z_i}(\varphi(z_k))\Re\varphi_i(z_k)\Big| = \log\frac{2}{1-|\varphi(z_k)|^2}|\langle\Re\varphi(z_k),\varphi(z_k)\rangle|.$$

Thus we have

$$\lim_{k\to\infty}(1-|z_k|^2)|u(z_k)|\log\frac{2}{1-|\varphi(z_k)|^2}|\langle\Re\varphi(z_k),\varphi(z_k)\rangle| = 0.$$

That is, (iii) of (c) holds.

Now we choose the test functions $f_k(z) = z_{j,k}^k/k$ for $j \in \{1,\dots,n\}$ and $k \in \mathbb{N}$. Obviously, the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{B}$ as $k\to\infty$. It follows that

$$\begin{aligned}\|uC_\varphi f_k\|_{\mathcal{B}} &\ge (1-|z_k|^2)|\Re(u\varphi_{j,k}^k/k)(z_k)| \\ &= (1-|z_k|^2)\big|\Re u(z_k)\varphi_{j,k}^k(z_k)/k + u(z_k)\varphi_{j,k}^{k-1}(z_k)\Re\varphi_{j,k}(z_k)\big| \\ &\ge (1-|z_k|^2)|u(z_k)\varphi_{j,k}^{k-1}(z_k)\Re\varphi_{j,k}(z_k)| \\ &\quad - (1-|z_k|^2)|\Re u(z_k)\varphi_{j,k}^k(z_k)/k|,\end{aligned}$$

from which we obtain

$$\lim_{k\to\infty}(1-|z_k|^2)|u(z_k)|\,|\Re\varphi_{j,k}(z_k)|\,|\varphi_{j,k}^{k-1}(z_k)| = 0.$$

By the elementary inequality $(a_1+\cdots+a_k)^p \le C(a_1^p+\cdots+a_k^p)$ for $a_1,\dots,a_k > 0$, where C is a positive constant, it follows that

$$\begin{aligned}&\lim_{k\to\infty}(1-|z_k|^2)|u(z_k)|\,|\Re\varphi_{j,k}(z_k)|\,|\varphi(z_k)|^{k-1} \\ &\quad\le \lim_{k\to\infty}(1-|z_k|^2)|u(z_k)|\,|\Re\varphi_{j,k}(z_k)|\Big(\sum_{j=1}^{n}|\varphi_{j,k}(z_k)|\Big)^{k-1} \\ &\quad\preceq \lim_{k\to\infty}(1-|z_k|^2)|u(z_k)|\,|\Re\varphi_{j,k}(z_k)|\sum_{j=1}^{n}|\varphi_{j,k}(z_k)|^{k-1} = 0.\end{aligned}$$

Since $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{B}$ with $|\varphi(z_k)| \to 1$ as $k \to \infty$, we have

$$\lim_{k\to\infty} (1-|z_k|^2)|u(z_k)|\,|\Re\varphi_{j,k}(z_k)| = 0.$$

Similarly,

$$\lim_{k\to\infty} (1-|z_k|^2)|u(z_k)|\,|\Re\varphi(z_k)| = 0.$$

That is, $\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u(z)|\,|\Re\varphi(z)| = 0$, thus (ii) of (c) holds.

(c)⇒(a). Suppose (c) holds. Then it obvious that (c) of Theorem 3.1 holds, so $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. We have

$$\sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\Re\varphi(z)| < \infty. \tag{3.3}$$

In order to prove that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact, according to Lemma 2.3, it suffices to show that if $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Z}$ converging to zero uniformly on $\overline{\mathbb{B}}$, then $\|uC_\varphi f_k\|_{\mathcal{B}}$ converges to zero as $k \to \infty$. So let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{Z}$ with $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Z}} \le L$, and suppose f_k converges to zero uniformly on $\overline{\mathbb{B}}$ as $k \to \infty$.

From (ii) and (iii) of (c), we find that for any $\epsilon > 0$, there is a $\delta \in (0,1)$ such that $\delta < |\varphi(z)| < 1$ implies

$$(1-|z|^2)|u(z)|\,|\Re\varphi(z)| < \epsilon/L, \tag{3.4}$$

$$(1-|z|^2)|u(z)| \log\frac{2}{1-|\varphi(z)|^2}\langle \Re\varphi(z), \varphi(z)\rangle < \epsilon/L. \tag{3.5}$$

From (3.3)–(3.5) and Lemma 2.4, we obtain

$$\begin{aligned}
\|uC_\varphi f_k\|_{\mathcal{B}} &= \sup_{z\in\mathbb{B}} (1-|z|^2)\big|\Re u(z) f_k(\varphi(z)) + u(z)\Re(f_k\circ\varphi(z))\big| \\
&\le \|u\|_{\mathcal{B}} \sup_{z\in\overline{\mathbb{B}}} |f_k(z)| + \sup_{z\in\mathbb{B}} (1-|z|^2)|u(z)|\,|\langle \nabla f_k(\varphi(z)), \Re\varphi(z)\rangle| \\
&\le \|u\|_{\mathcal{B}} \sup_{z\in\overline{\mathbb{B}}} |f_k(z)| + \sup_{|\varphi(z)|\le\delta} (1-|z|^2)|u(z)|\,|\langle \nabla f_k(\varphi(z)), \Re\varphi(z)\rangle| \\
&\quad + \sup_{|\varphi(z)|>\delta} (1-|z|^2)|u(z)|\,|\langle \nabla f_k(\varphi(z)), \Re\varphi(z)\rangle| \\
&\le \|u\|_{\mathcal{B}} \sup_{z\in\overline{\mathbb{B}}} |f_k(z)| + \sup_{|\varphi(z)|\le\delta} (1-|z|^2)|u(z)|\,|\Re\varphi(z)|\,|\nabla f_k(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|>\delta} (1-|z|^2)|u(z)| \\
&\qquad\qquad \times \left(|\Re\varphi(z)| + \log\frac{2}{1-|\varphi(z)|^2}|\langle \Re\varphi(z), \varphi(z)\rangle|\right)\|f_k\|_{\mathcal{Z}} \\
&\le 2\epsilon, \qquad k\to\infty.
\end{aligned}$$

On the other hand, $|u(0)f_k(\varphi(0))|$ converges to zero as $k \to \infty$. By Lemma 2.3, it follows that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is compact. ∎

THEOREM 3.3. *Suppose $\varphi = (\varphi_1, \ldots, \varphi_n) \in S(\mathbb{B})$ and $u \in H(\mathbb{B})$. Then $uC_\varphi : \mathcal{Z} \to \mathcal{B}_0$ is bounded if and only if $u \in \mathcal{B}_0$ and*

$$\lim_{|z|\to 1} (1-|z|^2)|u(z)|\,|\Re\varphi(z)| = 0, \tag{3.6}$$

$$\lim_{|z|\to 1} (1-|z|^2)|u(z)| \log \frac{2}{1-|\varphi(z)|^2} |\langle \Re\varphi(z), \varphi(z)\rangle| = 0, \tag{3.7}$$

where $\Re\varphi(z) = (\Re\varphi_1(z), \ldots, \Re\varphi_n(z))$.

Proof. Necessity. Taking $f(z) = 1$ and $f_j(z) = z_j$, we find that $u \in \mathcal{B}_0$ and

$$\lim_{|z|\to 1} (1-|z|^2)|u(z)|\,|\Re\varphi_j(z)| = 0.$$

It follows that

$$\lim_{|z|\to 1} (1-|z|^2)|u(z)|\,|\Re\varphi(z)| \le \lim_{|z|\to 1} \sum_{j=1}^{n} (1-|z|^2)|u(z)|\,|\Re\varphi_j(z)| = 0.$$

Similarly, taking $f_w(z)$ defined in (3.1), we deduce that $uC_\varphi f_w \in \mathcal{B}_0$. Thus, as $u \in \mathcal{B}_0$,

$$\begin{aligned}
&\lim_{|w|\to 1} (1-|w|^2)|\Re(uC_\varphi f_w)(w)| \\
&\quad = \lim_{|w|\to 1} (1-|w|^2)\big|\Re u(w) f_w(\varphi(w)) + u(w)\langle \nabla f_w(\varphi(w)), \overline{\Re\varphi(w)}\rangle\big| \\
&\quad \ge \lim_{|w|\to 1} (1-|w|^2)|u(w)| \log \frac{2}{1-|\varphi(w)|^2} |\langle \Re\varphi(w), \varphi(w)\rangle| \\
&\qquad - C \lim_{|w|\to 1} (1-|w|^2)|\Re u(w)|\,\|f\|_{\mathcal{Z}} \\
&\quad = \lim_{|w|\to 1} (1-|w|^2)|u(w)| \log \frac{2}{1-|\varphi(w)|^2} |\langle \Re\varphi(w), \varphi(w)\rangle|.
\end{aligned}$$

This shows that (3.7) holds.

Sufficiency. Since $u \in \mathcal{B}_0$, (3.6) and (3.7) hold, by Theorem 3.1 we know that $uC_\varphi : \mathcal{Z} \to \mathcal{B}$ is bounded. On the other hand, for any $f \in \mathcal{Z}$, we have

$$\begin{aligned}
&\lim_{|z|\to 1} (1-|z|^2)|\Re(uC_\varphi f)(z)| \\
&\qquad \preceq \lim_{|z|\to 1} (1-|z|^2)|\Re u(z)|\,\|f\|_{\mathcal{Z}} + \lim_{|z|\to 1} (1-|z|^2)|u(z)| \\
&\qquad\quad \times \left(|\Re\varphi(z)| + \log \frac{2}{1-|\varphi(z)|^2} |\langle \Re\varphi(z), \varphi(z)\rangle| \right) \|f\|_{\mathcal{Z}} \\
&\qquad = 0.
\end{aligned}$$

That is, $uC_\varphi f \in \mathcal{B}_0$ for any $f \in \mathcal{Z}$. Since $\mathcal{B}_0$ is a closed subset of $\mathcal{B}$, the map $uC_\varphi : \mathcal{Z} \to \mathcal{B}_0$ is bounded. ∎

THEOREM 3.4. *Suppose* $\varphi = (\varphi_1, \ldots, \varphi_n) \in S(\mathbb{B})$ *and* $u \in H(\mathbb{B})$. *Then* $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ *is bounded if and only if* $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ *is bounded and* $u \in \mathcal{B}_0$ *and*

$$\lim_{|z|\to 1} (1 - |z|^2)|u(z)\Re\varphi(z)| = 0, \tag{3.8}$$

where $|\Re\varphi(z)| = \sqrt{|\Re\varphi_1(z)|^2 + \cdots + |\Re\varphi_n(z)|^2}$.

Proof. Necessity. This is obvious by taking the test functions $f(z) = 1$ and $f_j(z) = z_j$, respectively.

Sufficiency. For any polynomial p, if $u \in \mathcal{B}_0$ and (3.8) holds, we have

$$\begin{aligned}
&(1 - |z|^2)|\Re(uC_\varphi p)(z)| \\
&\quad \le (1 - |z|^2)|\Re u(z)p(\varphi(z))| + (1 - |z|^2)|u(z)|\,|\langle \nabla p(\varphi(z)), \overline{\Re\varphi(z)}\rangle| \\
&\quad \le (1 - |z|^2)|\Re u(z)|\,\|p\|_\infty + (1 - |z|^2)|u(z)|\,|\Re\varphi(z)|\,\|\nabla p\|_\infty \\
&\quad \preceq (1 - |z|^2)|\Re u(z)| + (1 - |z|^2)|u(z)|\,|\Re\varphi(z)| \\
&\quad \to 0, \qquad |z| \to 1.
\end{aligned}$$

That is, $uC_\varphi p \in \mathcal{B}_0$ for any polynomial p. It is well known that the set of all polynomials is dense in $\mathcal{Z}_0$, so there exists a polynomial sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $\|f - p_n\|_{\mathcal{Z}}$ converges to zero as $n \to \infty$ for any $f \in \mathcal{Z}_0$. By the boundedness of $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$, we have

$$\|uC_\varphi f - uC_\varphi p_n\|_{\mathcal{B}} \le \|uC_\varphi\|_{\mathcal{Z}_0\to\mathcal{B}}\|f - p_n\|_{\mathcal{Z}} \le C\|f - p_n\|_{\mathcal{Z}} \to 0$$

as $n \to \infty$.

Since $uC_\varphi p_n \in \mathcal{B}_0$ and $\mathcal{B}_0$ is a closed subset of $\mathcal{B}$, we deduce that $uC_\varphi(\mathcal{Z}_0) \subset \mathcal{B}_0$. That is, $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is bounded. ∎

THEOREM 3.5. *Suppose* $\varphi = (\varphi_1, \ldots, \varphi_n) \in S(\mathbb{B})$ *and* $u \in H(\mathbb{B})$. *Then the following statements are equivalent:*

(a) $uC_\varphi : \mathcal{Z} \to \mathcal{B}_0$ *is compact;*
(b) $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ *is compact;*
(c) $u \in \mathcal{B}_0$ *and*

$$\lim_{|z|\to 1} (1 - |z|^2)|u(z)|\,|\Re\varphi(z)| = 0, \tag{3.9}$$

$$\lim_{|z|\to 1} (1 - |z|^2)|u(z)| \log \frac{2}{1 - |\varphi(z)|^2} |\langle \Re\varphi(z), \varphi(z)\rangle| = 0. \tag{3.10}$$

where $\Re\varphi(z) = (\Re\varphi_1(z), \ldots, \Re\varphi_n(z))$.

Proof. (a)⇒(b). This follows from $\mathcal{Z}_0 \subset \mathcal{Z}$.

(b)⇒(c). Suppose that $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}_0$ is compact, hence obviously bounded. Thus $u \in \mathcal{B}_0$ and (3.8) holds from Theorem 3.4. That is, (3.9) is true. Also, it is obvious that $uC_\varphi : \mathcal{Z}_0 \to \mathcal{B}$ is compact, so by Theorem 3.2, it follows that

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u(z)|\log\frac{2}{1-|\varphi(z)|^2}|\langle\Re\varphi(z),\varphi(z)\rangle| = 0. \tag{3.11}$$

Then there exists $r\in(0,1)$ such that

$$(1-|z|^2)|u(z)|\log\frac{2}{1-|\varphi(z)|^2}|\langle\Re\varphi(z),\varphi(z)\rangle| < \epsilon \tag{3.12}$$

whenever $r<|\varphi(z)|<1$.

On the other hand, by (3.9), there exists $\sigma\in(0,1)$ such that

$$(1-|z|^2)|u(z)|\,|\Re\varphi(z)| < \frac{\epsilon}{\log\frac{2}{1-r^2}} \tag{3.13}$$

whenever $\sigma<|z|<1$. Thus, when $\sigma<|z|<1$ and $r<|\varphi(z)|<1$, from (3.12) it follows that

$$(1-|z|^2)|u(z)|\log\frac{2}{1-|\varphi(z)|^2}|\langle\Re\varphi(z),\varphi(z)\rangle| < \epsilon. \tag{3.14}$$

On the other hand, when $|\varphi(z)|\le r$ and $\sigma<|z|<1$, by (3.13) we have

$$\begin{aligned}(1-|z|^2)|u(z)|\log\frac{2}{1-|\varphi(z)|^2}&|\langle\Re\varphi(z),\varphi(z)\rangle| \\ &\le (1-|z|^2)|u(z)|\,|\Re\varphi(z)|\log\frac{2}{1-r^2} < \epsilon.\end{aligned} \tag{3.15}$$

Combining (3.14) with (3.15) yields (3.10).

(c)⇒(a). From the conditions in (c), it follows by Theorem 3.1 that $uC_\varphi:\mathcal{Z}\to\mathcal{B}$ is bounded. Hence $uC_\varphi(\{f:\|f\|_{\mathcal{Z}}\le 1\})$ is bounded in $\mathcal{B}$. Moreover,

$$\begin{aligned}(1-|z|^2)&|\Re(uC_\varphi f)(z)| \\ &\preceq (1-|z|^2)|\Re u(z)|\,\|f\|_{\mathcal{Z}} + (1-|z|^2)|u(z)|\,|\Re\varphi(z)|\,\|f\|_{\mathcal{Z}} \\ &\quad + (1-|z|^2)|u(z)|\log\frac{2}{1-|\varphi(z)|^2}|\langle\Re\varphi(z),\varphi(z)\rangle|\,\|f\|_{\mathcal{Z}}.\end{aligned}$$

It is clear that $uC_\varphi(\{f:\|f\|_{\mathcal{Z}}\le 1\})$ is bounded in $\mathcal{B}_0$ by conditions in (c). On the other hand, by taking the supremum in the above inequality over the unit ball in $\mathcal{Z}$, then letting $|z|\to 1$ and using the conditions in (c), it follows that

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathcal{Z}}\le 1}(1-|z|^2)|\Re(uC_\varphi f)(z)| = 0.$$

From the above equality and Lemma 2.5, we conclude that $uC_\varphi:\mathcal{Z}\to\mathcal{B}_0$ is compact. ∎

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