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Asymptotic behavior of solutions to a class of differential variational inequalities

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Abstract. We address some questions concerning a class of differential variational inequalities with finite delays. The existence of exponential decay solutions and a global attractor for the associated multivalued semiflow is proved.

1. Introduction. We consider the following differential variational inequality (DVI):

$$x'(t) = Ax(t) + h(x(t)) + B(x(t), x_t)u(t), \quad t \in J = [0, T], \tag{1.1}$$

$$\langle v - u(t), F(x(t)) + G(u(t))\rangle \geq 0, \quad \forall v \in K, \text{ for a.e. } t \in J, \tag{1.2}$$

$$x(s) = \varphi(s), \quad s \in [-\tau, 0], \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in K$ with K being a closed convex subset in $\mathbb{R}^m$, x_t denotes the history of the state function up to time t; A, B, F, G and h are given maps which will be specified in the next section.

The notion of differential variational inequality can be traced back to Aubin and Cellina [2] in 1984. In a later work of Avgerinos and Papageorgiou [3], this concept was revisited and expanded. However, DVIs were first systematically studied by Pang and Stewart [15]. As mentioned in that paper, DVIs are useful for representing models involving both dynamics and constraints in the form of inequalities which arise in many problems in reality, for example, mechanical impact problems, electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics and related problems such as dynamic traffic networks. In case $K = \mathbb{R}^m$, system (1.1)–(1.3) becomes a differential algebraic equation with the unknown $y = (x, u)$, while if K is a cone, it is a differential complementarity problem. Some existence results for DVIs can be found in [9, 10, 13] and the references therein.

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In the theory of differential equations, problem (1.1)–(1.3) is called a differential system with unilateral constraints. As a matter of fact, it can be seen as a control problem subject to constraints. In this work, we will look for conditions ensuring the existence of solutions to this problem, and study their behavior. Specifically, after giving a short proof for the solvability of (1.1)–(1.3) on bounded intervals, we show that this system has a compact set of exponentially decaying solutions. In addition, the multivalued semiflow generated by (1.1)–(1.3) admits a compact global attractor in $C([-\tau,0];\mathbb{R}^n)$. Up to our knowledge, no attempt has been made to study the behavior of solutions to (1.1)–(1.3) in the literature. So this is a motivation of our work.

The rest of this paper is organized as follows. In the next section, we recall the notion of measure of noncompactness (MNC), and construct a regular MNC to determine the compactness in $BC(0,\infty;\mathbb{R}^n)$. On the other hand, the theory of global attractors for multivalued semiflows introduced in [14] will be taken into account. Section 3 contains the existence result for (1.1)–(1.3) on compact intervals. In Section 4, we prove the existence of exponentially decaying solutions to our problem, and Section 5 is devoted to showing that the multivalued semiflow associated with (1.1)–(1.3) has a compact global attractor in $C([-\tau,0];\mathbb{R}^n)$.

2. Preliminaries

2.1. Measure of noncompactness. Let E be a Banach space. Denote

$$\mathcal{P}(E)=\{B\subset E: B\neq\emptyset\},\qquad \mathcal{B}(E)=\{B\in\mathcal{P}(E): B \text{ is bounded}\}.$$

We recall the definition of measure of noncompactness introduced in [1].

Definition 2.1. A function $\beta:\mathcal{B}(E)\to\mathbb{R}^+$ is called a *measure of noncompactness* (MNC) on E if

$$\beta(\overline{\mathrm{co}}\,\Omega)=\beta(\Omega)\qquad\text{for every } \Omega\in\mathcal{B}(E),$$

where $\overline{\mathrm{co}}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called

- *monotone* if Ω_0, $\Omega_1\in\mathcal{B}(E), \Omega_0\subset\Omega_1$ implies $\beta(\Omega_0)\subset\beta(\Omega_1)$;
- *nonsingular* if $\beta(\{a\}\cup\Omega)=\beta(\Omega)$ for any $a\in E$, $\Omega\in\mathcal{B}(E)$;
- *invariant with respect to unions with compact sets* if $\beta(K\cup\Omega)=\beta(\Omega)$ for every relatively compact set $K\subset E$ and $\Omega\in\mathcal{B}(E)$;
- *algebraically semiadditive* if $\beta(\Omega_0+\Omega_1)\le\beta(\Omega_0)+\beta(\Omega_1)$ for any $\Omega_0,\Omega_1\in\mathcal{B}(E)$;
- *regular* if $\beta(\Omega)=0$ is equivalent to the relative compactness of Ω.

An important example of MNC is the Hausdorff MNC $\chi(\cdot)$, which is defined as follows: for $\Omega \in \mathcal{B}(E)$,

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \text{ has a finite } \epsilon\text{-net}\}.$$

In particular, it is known that the Hausdorff MNC on $C([0,T];\mathbb{R}^n)$, the space of continuous functions on $[0,T]$ taking values in $\mathbb{R}^n$, is given by (see [1])

$$\chi_T(D) = \frac{1}{2}\lim_{\delta\to 0}\sup_{x\in D}\max_{t,s\in[0,T],\,|t-s|<\delta}\|x(t)-x(s)\|. \tag{2.1}$$

The measure $\chi_T(D)$ of a subset D can be seen as the modulus of equicontinuity of a subset in $C([0,T];\mathbb{R}^n)$.

Consider the space $BC(0,\infty;\mathbb{R}^n)$ of bounded continuous functions on $[0,\infty)$ taking values in $\mathbb{R}^n$. Denote by π_T the restriction operator on $BC(0,\infty;\mathbb{R}^n)$, that is, $\pi_T(x)$ is the restriction of x on $[0,T]$. Then

$$\chi_\infty(D) = \sup_{T>0}\chi_T(\pi_T(D)), \qquad D \subset BC(0,\infty;\mathbb{R}^n), \tag{2.2}$$

is an MNC. One can check that χ_∞ has all the properties given in Definition 2.1, but regularity. Indeed, we will prove this claim by choosing the sequence $\{f_k\} \subset BC(0,\infty;\mathbb{R})$ as follows:

$$f_k(t) = \begin{cases} 0, & t \notin [k,k+1], \\ 2t-2k, & t \in [k,k+1/2], \\ -2t+2k+2, & t \in [k+1/2,k+1]. \end{cases}$$

Then it is obvious that $\{\pi_T(f_k)\}$ is compact (converging to 0 in $C([0,T];\mathbb{R})$) for any $T>0$. However,

$$\sup_{t\ge 0}|f_k(t)-f_l(t)| = 1 \qquad \text{for } k \ne l,$$

and so $\{f_k\}$ is not a Cauchy sequence in $BC(0,\infty;\mathbb{R})$. This tells us that $\chi_T(\pi_T(\{f_k\}))) = 0$ for any $T>0$, and hence $\chi_\infty(\{f_k\}) = 0$, but $\{f_k\}$ is not relatively compact.

We now construct a regular MNC on $BC(0,\infty;\mathbb{R}^n)$. Recall the following MNCs on $BC(0,\infty;\mathbb{R}^n)$ (see [4]):

$$d_T(D) = \sup_{x\in D}\sup_{t\ge T}\|x(t)\|, \qquad d_\infty(D) = \lim_{T\to\infty} d_T(D).$$

Define

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{2.3}$$

By a simple check, χ^* is an MNC on $BC(0,\infty;\mathbb{R}^n)$.

Lemma 2.2. *The MNC χ^* defined by* (2.3) *is regular.*

Proof. Let $D \subset BC(0,\infty;\mathbb{R}^n)$ be a bounded set such that $\chi^*(D) = 0$. We will show that D is relatively compact. Let $PBC(0,\infty;\mathbb{R}^n)$ be the space of piecewise continuous and bounded functions on $\mathbb{R}^+$, taking values in $\mathbb{R}^n$. This is a Banach space with the norm

$$\|x\|_{PBC} = \sup_{t\geq 0} \|x(t)\|,$$

and contains $BC(0,\infty;\mathbb{R}^n)$ as a closed subspace.

For $\epsilon > 0$, since $d_\infty(D) = 0$, one can choose $T > 0$ such that $\sup_{t\geq T} \|x(t)\| < \epsilon/2$ for all $x \in D$. This means that

$$\|x - \pi_T(x)\|_{PBC} < \epsilon/2, \qquad \forall x \in D,$$

here $\pi_T(x)$ is understood as a function in $PBC(0,\infty;\mathbb{R}^n)$ in the following manner:

$$\pi_T(x)(t) = \begin{cases} x(t), & t \in [0,T], \\ 0, & t > T. \end{cases}$$

Now since D is bounded and $\chi_T(D) = 0$, by the Arzelà–Ascoli theorem $\pi_T(D)$ is a relatively compact set in $C([0,T];\mathbb{R}^n)$, so we can write

$$\pi_T(D) \subset \bigcup_{i=1}^{N} B_T(x_i, \epsilon/2), \tag{2.4}$$

where $x_i \in C([0,T];\mathbb{R}^n)$, $i = 1,\ldots,N$, and $B_T(x,r)$ stands for the ball in $C([0,T];\mathbb{R}^n)$ centered at x with radius r. Set

$$\hat{x}_i(t) = \begin{cases} x_i(t), & t \in [0,T], \\ 0, & t > T; \end{cases}$$

then $\{\hat{x}_i\}_{i=1}^N \subset PBC(0,\infty;\mathbb{R}^n)$. We assert that

$$D \subset \bigcup_{i=1}^{N} B_\infty(\hat{x}_i, \epsilon),$$

where $B_\infty(x,r)$ is the ball in $PBC(0,\infty;\mathbb{R}^n)$ with center x and radius r. Indeed, if $x \in D$ then by (2.4) there is $k \in \{1,\ldots,N\}$ such that

$$\|\pi_T(x) - x_k\|_C < \epsilon/2,$$

where $\|\cdot\|_C$ is the norm in $C([0,T];\mathbb{R}^n)$. This implies

$$\|\pi_T(x) - \hat{x}_k\|_{PBC} < \epsilon/2.$$

Then

$$\|x - \hat{x}_k\|_{PBC} \leq \|x - \pi_T(x)\|_{PBC} + \|\pi_T(x) - \hat{x}_k\|_{PBC} \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $x \in B_\infty(\hat{x}_k, \epsilon)$. We have $D \subset \bigcup_{i=1}^N B_\infty(\hat{x}_i, \epsilon)$, and hence D is relatively compact in $PBC(0,\infty;\mathbb{R}^n)$. In order to show that D is also relatively compact in $BC(0,\infty;\mathbb{R}^n)$, we observe that if $\{x_n\} \subset D$ then one can find a function $x \in PBC(0,\infty;\mathbb{R}^n)$ such that

$$\lim_{n\to\infty} \|x_n - x\|_{PBC} = \lim_{n\to\infty} \sup_{t\geq 0} \|x_n(t) - x(t)\| = 0,$$

up to a subsequence. This means that $\{x_n\}$ converges to x uniformly on $\mathbb{R}^+$. As $\{x_n\}$ are continuous functions, we have $x \in BC(0, \infty; \mathbb{R}^n)$. ∎

We also make use of some notions and facts of set-valued analysis. Let Y be a metric space.

DEFINITION 2.3. A multivalued map (multimap) $\mathcal{F} : Y \to \mathcal{P}(E)$ is said to be

- *upper semicontinuous* (u.s.c.) if $\mathcal{F}^{-1}(V) = \{y \in V : \mathcal{F}(y) \cap V \neq \emptyset\}$ is a closed subset of Y for every closed set $V \subset E$;
- *weakly upper semicontinuous* (weakly u.s.c.) if $\mathcal{F}^{-1}(V)$ is a closed subset of Y for all weakly closed sets $V \subset E$;
- *closed* if its graph $\Gamma_{\mathcal{F}} = \{(y, z) : z \in \mathcal{F}(y)\}$ is a closed subset of $Y \times E$;
- *compact* if $\mathcal{F}(B)$ is relatively compact in E for any bounded set $B \subset Y$;
- *quasicompact* if its restriction to any compact subset $A \subset Y$ is compact.

LEMMA 2.4 ([12, Theorem 1.1.12]). *Let $G : Y \to \mathcal{P}(E)$ be a closed quasicompact multimap with compact values. Then G is u.s.c.*

LEMMA 2.5 ([5, Proposition 2]). *Let X be a Banach space and Ω be a nonempty subset of another Banach space. Assume that $\mathcal{G} : \Omega \to \mathcal{P}(X)$ is a multimap with weakly compact, convex values. Then $\mathcal{G}$ is weakly u.s.c. iff $\{x_n\} \subset \Omega$, $x_n \to x_0$ and $y_n \in \mathcal{G}(x_n)$ implies $y_n \rightharpoonup y_0 \in \mathcal{G}(x_0)$, up to a subsequence.*

We will use the following fixed point principle, which is a special case of [12, Corollary 3.3.1].

THEOREM 2.6. *Let $\mathcal{M}$ be a bounded convex and closed subset of a Banach space E, and let $\mathcal{F} : \mathcal{M} \to \mathcal{P}(\mathcal{M})$ be a compact, u.s.c. multimap with compact convex values. Then* $\mathrm{Fix}(\mathcal{F}) := \{x \in E : x \in \mathcal{F}(x)\}$ *is a nonempty compact set.*

2.2. Multivalued semiflows and their attractors. We summarize some definitions and results regarding the theory of global attractors of multivalued semiflows (m-semiflows) given in [14]. Let Γ be a nontrivial subgroup of the additive group of real numbers $\mathbb{R}$ and $\Gamma_+ = \Gamma \cap [0, \infty)$.

DEFINITION 2.7. A mapping $G : \Gamma_+ \times E \to \mathcal{P}(E)$ is called an *m-semiflow* if:

(1) $G(0, w) = \{w\}$ for all $w \in E$.
(2) $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$ for all $t_1, t_2 \in \Gamma_+$, $x \in E$,

where $G(t, B) = \bigcup_{x \in B} G(t, x)$ and $B \subset E$.

The m-semiflow is called *strict* if $G(t_1 + t_2, w) = G(t_1, G(t_2, w))$ for all $w \in E$ and $t_1, t_2 \in \Gamma_+$, and *eventually bounded* if for each bounded set

$B \subset E$, there is a number $T(B) > 0$ such that $\gamma^+_{T(B)}(B)$ is bounded, where $\gamma^+_{T(B)}(B) = \bigcup_{t \geq T(B)} G(t, B)$.

DEFINITION 2.8. A set $\mathcal{A}$ is called a *global attractor* of the m-semiflow G if

(1) $\mathcal{A}$ is negatively semi-invariant, i.e. $\mathcal{A} \subset G(t, \mathcal{A})$ for all $t \in \mathit{\Gamma}_+$;
(2) $\mathcal{A}$ attracts any $B \in \mathcal{B}(E)$, i.e. $\operatorname{dist}(G(t, B), \mathcal{A}) \to 0$ as $t \to \infty$, for all bounded sets $B \subset E$, where $\operatorname{dist}(\cdot, \cdot)$ is the Hausdorff semidistance of two subsets in E:

$$\operatorname{dist}(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} \|x - y\|.$$

DEFINITION 2.9. The m-semiflow G is called *pointwise dissipative* if there is a bounded set B_0 attracting any point $x \in E$, i.e. there exists $K > 0$ such that for $w \in E$ and $u(t) \in G(t, w)$, one has $\|u(t)\|_E \leq K$ for $t \geq t_0(\|w\|_E)$.

DEFINITION 2.10. The m-semiflow G is called *asymptotically upper semicompact* if for each $B \in \mathcal{B}(E)$ such that $\gamma^+_{T(B)} \in \mathcal{B}(E)$ for some $T(B) \in \mathit{\Gamma}_+$, any sequence $\xi_n \in G(t_n, B)$ with $t_n \to \infty$ is precompact in E.

DEFINITION 2.11. A bounded set $B_1 \subset E$ which has the property that, for any bounded set $B \subset E$ there exists $\tau = \tau(B) \geq 0$ such that $\gamma^+_\tau(B) \subset B_1$, is called an *absorbing set* for the m-semiflow G.

It is obvious that if the m-semiflow G has an absorbing set, then it is pointwise dissipative and eventually bounded.

The following theorem gives a sufficient condition for the existence of a global attractor for the m-semiflow G.

THEOREM 2.12 ([14]). *Assume that the m-semiflow G has the following properties:*

(1) *$G(t, \cdot)$ is u.s.c. and has closed values for each $t \in \mathit{\Gamma}_+$;*
(2) *G is pointwise dissipative;*
(3) *G is asymptotically upper semicompact.*

If G is eventually bounded then it has a compact global attractor $\mathcal{A}$ in E. Moreover, if G is a strict m-semiflow then $\mathcal{A}$ is invariant, that is, $\mathcal{A} = G(t, \mathcal{A})$ for any $t \in \mathit{\Gamma}_+$.

3. Existence result on compact intervals. Set

$$J = [0, T], \quad C_T = C([0, T]; \mathbb{R}^n), \quad C_\tau = C([-\tau, 0]; \mathbb{R}^n), \quad \mathcal{C} = C([-\tau, T]; \mathbb{R}^n).$$

In what follows, we use the following assumptions:

(H1) *A is a linear operator on $\mathbb{R}^n$.*

(H2) $B : \mathbb{R}^n \times C_\tau \to \mathbb{R}^{n\times m}$ *is a continuous map and there exist positive constants* η_B, ζ_B *such that*

$$\|B(v,w)\| \le \eta_B(\|v\| + \|w\|_{C_\tau}) + \zeta_B$$

for all $v \in \mathbb{R}^n$ *and* $w \in C_\tau$.

(H3) *The function* $F : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous and there is a positive number* η_F *such that* $\|F(v)\| \le \eta_F$ *for all* $v \in \mathbb{R}^n$.

(H4) $G : K \to \mathbb{R}^m$ *is a continuous function such that*

(1) G *is monotone on* K, *i.e.*

$$\langle u - v, G(u) - G(v)\rangle \ge 0, \qquad \forall u, v \in K;$$

(2) *there exists* $v_0 \in K$ *such that*

$$\lim_{v\in K,\, \|v\|\to\infty} \frac{\langle v - v_0, G(v)\rangle}{\|v\|^2} > 0.$$

(H5) $h : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and there are positive constants* η_h, ζ_h *such that*

$$\|h(u)\| \le \eta_h\|u\| + \zeta_h, \qquad \forall u \in \mathbb{R}^n.$$

We now give the definition of a solution for the DVI (1.1)–(1.3).

DEFINITION 3.1. A continuous function $x : [-\tau, T] \to \mathbb{R}^n$ is called a *solution* of (1.1)–(1.3) if there exists an integrable function $u : J \to K$ such that

$$x(t) = e^{tA}\varphi(0) + \int_0^t e^{(t-s)A}B(x(s), x_s)u(s)\,ds + \int_0^t e^{(t-s)A}h(x(s))\,ds, \qquad t \in J,$$

$$\langle v - u(t), F(x(t)) + G(u(t))\rangle \ge 0 \quad \text{for a.e. } t \in J \text{ and all } v \in K,$$

$$x(s) = \varphi(s), \qquad s \in [-\tau, 0].$$

We denote

$$\text{(3.1)} \qquad \mathrm{SOL}(K, Q) = \{v \in K : \langle w - v, Q(v)\rangle \ge 0,\ \forall w \in K\},$$

where $Q : \mathbb{R}^m \to \mathbb{R}^m$ is a given mapping.

Due to [15, Proposition 6.2], we get the following result.

LEMMA 3.2. *Let* (H4) *hold. Then for each* $z \in \mathbb{R}^m$, *the solution set* $\mathrm{SOL}(K, z + G(\cdot))$ *is nonempty, convex and closed. Moreover, there exists* $\eta_G > 0$ *such that*

$$\text{(3.2)} \qquad \|v\| \le \eta_G(1 + \|z\|), \qquad \forall v \in \mathrm{SOL}(K, z + G(\cdot)).$$

In order to solve (1.1)–(1.3), we convert it into a differential inclusion. Let

$$U(z) = \mathrm{SOL}(K, z + G(\cdot)), \qquad z \in \mathbb{R}^m.$$

Then $U : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^m)$ has compact convex values, thanks to Lemma 3.2. Moreover, it is easy to verify that U is a closed map. By (3.2), we see that U is locally bounded, so it is u.s.c.

Now we define $\Phi : \mathbb{R}^n \times C_\tau \to \mathcal{P}(\mathbb{R}^n)$ as follows:

$$\Phi(v, w) = \{B(v, w)y + h(v) : y \in U(F(v))\}. \tag{3.3}$$

Since $B(v, w)$ is a linear operator for all $v \in \mathbb{R}^n$ and $w \in C_\tau$, and U has compact convex values, Φ also has compact convex values. Furthermore, thanks to the continuity of B, F, h and the fact that U is u.s.c., the composition multimap Φ is u.s.c. as well.

Due to the above setting, DVI (1.1)–(1.3) is converted into the following differential inclusion:

$$x'(t) \in Ax(t) + \Phi(x(t), x_t), \quad t \in J, \tag{3.4}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0]. \tag{3.5}$$

Denote

$$\mathcal{P}_\Phi(x) = \{f \in L^1(J; \mathbb{R}^n) : f(t) \in \Phi(x(t), x_t)\} \quad \text{for } x \in \mathcal{C}. \tag{3.6}$$

Then we deduce that a solution $x \in \mathcal{C}$ of the DVI (1.1)–(1.3) is given by

$$x(t) = e^{tA}\varphi(0) + \int_0^t e^{(t-s)A} f(s)\, ds, \quad f \in \mathcal{P}_\Phi(x),\ t \in J, \tag{3.7}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0]. \tag{3.8}$$

For $y \in C_T$ and $\varphi \in C_\tau$, we define $y[\varphi] \in \mathcal{C}$ as follows:

$$y[\varphi](t) = \begin{cases} y(t) & \text{if } t \in [0, T], \\ \varphi(t) & \text{if } t \in [-\tau, 0]. \end{cases}$$

By defining

$$\mathcal{W} : L^1(J; \mathbb{R}^n) \to C_T, \quad \mathcal{W}(f)(t) = \int_0^t e^{(t-s)A} f(s)\, ds, \tag{3.9}$$

we construct a solution operator $\mathcal{F} : C_T \to \mathcal{P}(C_T)$ as follows:

$$\mathcal{F}(y)(t) = \{e^{tA}\varphi(0) + \mathcal{W}(f)(t) : f \in \mathcal{P}_\Phi(y[\varphi])\}, \quad t \in J.$$

It is obvious that $y \in C_T$ is a fixed point of $\mathcal{F}$ iff $y[\varphi]$ is a solution of (1.1)–(1.3).

LEMMA 3.3. *Under the assumptions* (H2)–(H5), $\mathcal{P}_\Phi$ *is well-defined and weakly u.s.c.*

Proof. Using the assumptions and the result of Lemma 3.2, we get

$$\begin{aligned}
\|\Phi(v,w)\| &:= \sup\{\|z\| : z \in \Phi(v,w)\} \\
&\le \|B(v,w)\|\eta_G(1+\|F(v)\|) + \|h(v)\| \\
&\le \eta_G(1+\eta_F)[\eta_B(\|v\|+\|w\|_{C_\tau})+\zeta_B] + \eta_h\|v\| + \zeta_h.
\end{aligned} \tag{3.10}$$

Since Φ is u.s.c. with compact convex values, the multimap $\Lambda(t) = \Phi(x(t), x_t)$ is strongly measurable due to [12, Proposition 1.3.1]. Therefore it has a Castaing representation (see [12, Definition 1.3.3]), and hence $\mathcal{P}_\Phi(x) \neq \emptyset$ for $x \in \mathcal{C}$.

We prove the second assertion by using Lemma 2.5. Let $\{x_k\} \subset \mathcal{C}$ be such that $x_k \to x^*$ and $f_k \in \mathcal{P}_\Phi(x_k)$. Then $\{f_k(t)\} \subset C(t) := \overline{\Phi(\{x_k(t), (x_k)_t\})}$, and $C(t)$ is a compact set for each $t \in J$. Furthermore, by (3.10), $\{f_k\}$ is integrably bounded (bounded by an integrable function). Thus $\{f_k\}$ is weakly relatively compact in $L^1(J;\mathbb{R}^n)$ (see [7, Corollary 2.6]). Let $f_k \rightharpoonup f^*$ in $L^1(J;\mathbb{R}^n)$. Then by Mazur's lemma (see e.g. [8]) there are $\bar{f}_k \in \mathrm{co}\{f_i : i \ge k\}$ such that $\bar{f}_k \to f^*$ in $L^1(J;\mathbb{R}^n)$, and so $\bar{f}_k(t) \to f^*(t)$ for a.e. $t \in J$, up to a subsequence. Observe that in our case, the upper semicontinuity of Φ implies that for a given $\epsilon > 0$,

$$\Phi(x_k(t), (x_k)_t) \subset \Phi(x^*(t), x_t^*) + B_\epsilon \quad \text{for all large } k,$$

where B_ϵ is the ball in $\mathbb{R}^n$ centered at origin with radius ϵ. So

$$f_k(t) \in \Phi(x^*(t), x_t^*) + B_\epsilon \quad \text{for a.e. } t \in J$$

and

$$\bar{f}_k(t) \in \Phi(x^*(t), x_t^*) + B_\epsilon \quad \text{for a.e. } t \in J,$$

thanks to the convexity of $\Phi(x^*(t), x_t^*) + B_\epsilon$. The last inclusion implies that $f^*(t) \in \Phi(x^*(t), x_t^*) + B_\epsilon$ for a.e. $t \in J$. Since ϵ is arbitrary, one obtains $f^* \in \mathcal{P}_\Phi(x^*)$. ∎

LEMMA 3.4. *The operator $\mathcal{W}$ defined by* (3.9) *is compact.*

Proof. We have to show that $\mathcal{W}(\Omega)$ is relatively compact in C_T for any bounded set $\Omega \subset L^1(J;\mathbb{R}^n)$. Obviously, $\mathcal{W}(\Omega)(t)$ is bounded in $\mathbb{R}^n$. In addition, $\mathcal{W}(\Omega)$ is equicontinuous since $S(t) = e^{tA}$ is a norm-continuous semigroup. So we get the conclusion by using the Arzelà–Ascoli theorem. ∎

LEMMA 3.5. *Let* (H1)–(H5) *hold. Then the solution operator $\mathcal{F}$ is compact and has a closed graph.*

Proof. Since $\mathcal{W}$ is compact, it is easy to check that $\mathcal{F}(B)$ is relatively compact for any bounded set $B \subset C_T$. So $\mathcal{F}$ is a compact multimap.

Now let $\{x_k\} \subset C_T$, $x_k \to x^*$, $y_k \in \mathcal{F}(x_k[\varphi])$ and $y_k \to y^*$. We will verify that $y^* \in \mathcal{F}(x^*)$. By the formulation of $\mathcal{F}$, one can take $f_k \in \mathcal{P}_\Phi(x_k[\varphi])$ such that

$$y_k(t) = e^{tA}\varphi(0) + \mathcal{W}(f_k)(t), \quad t \in J. \tag{3.11}$$

Since $\mathcal{P}_\Phi$ is weakly u.s.c. and $\{x_k\}$ is relatively compact, it follows that $\{f_k\}$ is weakly relatively compact, so we can assume that $f_k \rightharpoonup f^*$ in $L^1(J;\mathbb{R}^n)$, up to a subsequence. Moreover, $f^* \in \mathcal{P}_\Phi(x^*[\varphi])$. By the compactness of $\mathcal{W}$, we obtain $\mathcal{W}(f_k) \to \mathcal{W}(f^*)$ in C_T. Taking the limit of (3.11) as $k \to \infty$, we get

$$y^*(t) = e^{tA}\varphi(0) + \mathcal{W}(f^*)(t), \quad t \in J.$$

Thus $y^* \in \mathcal{F}(x^*)$. ■

THEOREM 3.6. *Assume* (H1)–(H5). *Then problem* (3.4)–(3.5) *has at least one solution on* $[-\tau, T]$. *Moreover, the solution set is compact.*

Proof. Using Theorem 2.6 we will prove that $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$. According to Lemma 3.5, it suffices to show that there exists a bounded closed convex set $\mathcal{M}_0 \subset C_T$ such that $\mathcal{F}(\mathcal{M}_0) \subset \mathcal{M}_0$. Let $y \in \mathcal{F}(x)$. Then it follows from the definition of the solution operator and estimate (3.10) that there exists $f \in \mathcal{P}_\Phi(x[\varphi])$ such that

$$\begin{aligned}\|y(t)\| &= \Big\|e^{tA}\varphi(0) + \int_0^t e^{(t-s)A} f(s)\,ds\Big\| \le M\|\varphi(0)\| + \int_0^t \|e^{(t-s)A}\|\,\|f(s)\|\,ds \\ &\le M\|\varphi(0)\| + M\int_0^t \big[(\eta+\eta_h)\|x(s)\| + \eta\|x[\varphi]_s\|_{C_\tau} + \zeta\big]\,ds, \quad \forall t \in J,\end{aligned}$$

where $M = \sup_{t\in J}\|e^{tA}\|$, $\eta = \eta_G(1+\eta_F)\eta_B$ and $\zeta = \eta_G(1+\eta_F)\zeta_B + \zeta_h$.

On the other hand, due to the estimate

$$\|x[\varphi]_s\|_{C_\tau} = \sup_{\theta\in[-\tau,0]} \|x[\varphi](s+\theta)\| \le \|\varphi\|_{C_\tau} + \sup_{\rho\in[0,s]} \|x(\rho)\|,$$

one has

$$\begin{aligned}\|y(t)\| &\le M_1 + M\int_0^t \Big((\eta+\eta_h)\|x(s)\| + \eta \sup_{\rho\in[0,s]}\|x(\rho)\|\Big)\,ds \\ &\le M_1 + M(2\eta+\eta_h)\int_0^t \sup_{\rho\in[0,s]}\|x(\rho)\|\,ds,\end{aligned} \tag{3.12}$$

where $M_1 = M\|\varphi(0)\| + MT\,[\eta\|\varphi\|_{C_\tau} + \zeta]$. Since the last term of (3.12) is nondecreasing in t, we have

$$\sup_{\rho\in[0,t]}\|y(\rho)\| \le M_1 + M(2\eta+\eta_h)\int_0^t \sup_{\rho\in[0,s]}\|x(\rho)\|\,ds. \tag{3.13}$$

Denote

$$\mathcal{M}_0 = \Big\{x \in C_T : \sup_{s\in[0,t]} \|x(s)\| \le \psi(t),\ t \in [0,T]\Big\},$$

where ψ is the unique solution of the integral equation

$$\psi(t) = M_1 + M(2\eta + \eta_h)\int_0^t \psi(s)\,ds, \qquad t \in J.$$

It is clear that $\mathcal{M}_0$ is a bounded closed convex subset of C_T, and estimate (3.13) ensures that $\mathcal{F}(\mathcal{M}_0) \subset \mathcal{M}_0$. ∎

4. Decaying solutions. In this section, we consider the solution operator $\mathcal{F}$ on $BC(0,\infty;\mathbb{R}^n)$. For a positive number γ and $\varphi \in C_\tau$, denote

$$B^\gamma_\varphi(R) = \{x \in C([0,\infty);\mathbb{R}^n) : x(0) = \varphi(0),\ e^{\gamma t}\|x(t)\| \le R \text{ for all } t \ge 0\}.$$

Then $B^\gamma_\varphi(R)$ is a bounded closed convex subset of $BC(0,\infty;\mathbb{R}^n)$. We need to replace the assumptions (H1), (H2) and (H5) by stronger ones:

(H1*) *A is a linear operator on $\mathbb{R}^n$ such that there exists $a > 0$ satisfying $\langle -Az, z\rangle \ge a\|z\|^2$ for all $z \in \mathbb{R}^n$.*
(H2*) *B satisfies* (H2) *with $\zeta_B = 0$.*
(H5*) *h obeys* (H5) *with $\zeta_h = 0$.*

LEMMA 4.1. *Under the assumptions* (H1*), (H2*), (H3)–(H4) *and* (H5*), *we have $\mathcal{F}(B^\gamma_\varphi(R)) \subset B^\gamma_\varphi(R)$ for some $R > 0$, provided that*

$$\eta_G(1+\eta_F)\eta_B(1+e^{\gamma\tau}) + \eta_h + \gamma < a. \tag{4.1}$$

Proof. By (H1*) we have

$$\|e^{tA}\| \le e^{-at}, \qquad t \ge 0. \tag{4.2}$$

Assume the opposite: for each $n \in \mathbb{N}$ there exist $x_n \in B^\gamma_\varphi(n)$ and $y_n \in \mathcal{F}(x_n)$ with $y_n \notin B^\gamma_\varphi(n)$. Then one can find $f_n \in \mathcal{P}_\Phi(x_n[\varphi])$ such that

$$y_n(t) = e^{tA}\varphi(0) + \int_0^t e^{(t-s)A} f_n(s)\,ds, \qquad \forall t \ge 0.$$

Using (4.2) and estimate (3.10), we get

$$\begin{aligned} \|y_n(t)\| \le{}& e^{-at}\|\varphi\|_{C_\tau} \\ &+ \eta_G(1+\eta_F)\eta_B\int_0^t e^{-a(t-s)}(\|x_n(s)\| + \|x_n[\varphi]_s\|_{C_\tau})\,ds \\ &+ \eta_h\int_0^t e^{-a(t-s)}\|x_n(s)\|\,ds. \end{aligned} \tag{4.3}$$

Now one observes that $e^{\gamma t}\|x_n(t)\| \le n$ for all $t \ge 0$. Then for all $t \ge \tau$,

$$\begin{aligned} e^{\gamma t}\|x_n[\varphi]_t\|_{C_\tau} &= e^{\gamma t} \sup_{\rho\in[-\tau,0]} \|x_n(t+\rho)\| \\ &= e^{\gamma t} \sup_{\rho\in[-\tau,0]} e^{-\gamma(t+\rho)} e^{\gamma(t+\rho)} \|x_n(t+\rho)\| \\ &\le e^{\gamma t} e^{-\gamma(t-\tau)} \sup_{\rho\in[-\tau,0]} e^{\gamma(t+\rho)}\|x_n(t+\rho)\| \le n e^{\gamma\tau}. \end{aligned}$$

On the other hand, for $t \in [0,\tau]$ one has $e^{\gamma t}\|x_n[\varphi]_t\|_{C_\tau} \le e^{\gamma\tau}\|\varphi\|_{C_\tau}$. Hence

$$e^{\gamma t}\|x_n[\varphi]_t\|_{C_\tau} \le e^{\gamma\tau}(n + \|\varphi\|_{C_\tau}) \quad \text{for all } t \ge 0.$$

So it can be deduced from (4.3) that

$$\begin{aligned} e^{\gamma t}\|y_n(t)\| &\le e^{-(a-\gamma)t}\|\varphi\|_{C_\tau} + [\eta_G(1+\eta_F)\eta_B + \eta_h]\int_0^t e^{-(a-\gamma)(t-s)} e^{\gamma s}\|x_n(s)\|\,ds \\ &\quad + \eta_G(1+\eta_F)\eta_B \int_0^t e^{-(a-\gamma)(t-s)} e^{\gamma s}\|x_n[\varphi]_s\|_{C_\tau}\,ds \\ &\le \|\varphi\|_{C_\tau} + \{n[\eta_G(1+\eta_F)\eta_B + \eta_h] + (n + \|\varphi\|_{C_\tau})e^{\gamma\tau}\eta_G(1+\eta_F)\eta_B\} I, \end{aligned}$$

where

$$I = \int_0^t e^{-(a-\gamma)(t-s)}\,ds = \frac{1}{a-\gamma}(1 - e^{-(a-\gamma)t}).$$

Therefore

$$\frac{1}{n}\sup_{t\ge 0} e^{\gamma t}\|y_n(t)\| \le \frac{1}{a-\gamma}\big[\eta_G(1+\eta_F)\eta_B(1+e^{\gamma\tau}) + \eta_h\big] + \frac{C}{n}, \tag{4.4}$$

where

$$C = \|\varphi\|_{C_\tau} + \frac{1}{a-\gamma}\big[\|\varphi\|_{C_\tau}\eta_G(1+\eta_F)\eta_B e^{\gamma\tau}\big].$$

Taking the limit of (4.4) as $n \to \infty$, we get a contradiction of (4.1). ∎

We now prove the main result of this section.

THEOREM 4.2. *Assume that* (H1*)–(H2*), (H3)–(H4) *and* (H5*) *hold, and there exists* $\gamma > 0$ *such that*

$$\eta_G(1+\eta_F)\eta_B(1+e^{\gamma\tau}) + \eta_h + \gamma < a.$$

Then the DVI (1.1)–(1.3) *has a nonempty compact set of solutions on* $[-\tau,\infty)$ *satisfying*

$$e^{\gamma t}\|x(t)\| = O(1) \quad \textit{as } t \to \infty.$$

Proof. By Lemma 4.1, one can consider $\mathcal{F} : B^\gamma_\varphi(R) \to \mathcal{P}(B^\gamma_\varphi(R))$ for a number $R > 0$. Due to Theorem 2.6 it remains to show that $\mathcal{F}$ is compact and u.s.c. We first prove that $\mathcal{F}$ is a compact multimap. Let $D \subset B^\gamma_\varphi(R)$.

Recall that $\chi^*(D) = \chi_\infty(D) + d_\infty(D)$, where χ_∞ and d_∞ are defined in Section 2.

By the same arguments as in the proof of Lemma 3.5, we have

$$\chi_T(\pi_T(\mathcal{F}(D))) = 0.$$

Therefore

$$\chi_\infty(\mathcal{F}(D)) = 0. \tag{4.5}$$

To show that $d_\infty(\mathcal{F}(D)) = 0$, let $x \in D$ and $y \in \mathcal{F}(x)$. Then in view of Lemma 4.1 we have

$$\|y(t)\| \le Ce^{-\gamma t}, \quad \forall t \ge 0,$$

where $C = C(R, a, \gamma, \eta_B, \eta_F, \eta_G, \eta_h)$. Thus for $T > 0$,

$$\sup_{t \ge T} \|y(t)\| \le Ce^{-\gamma T}, \quad \forall y \in D.$$

This implies $d_T(D) \le Ce^{-\gamma T}$, so $d_\infty(D) = \lim_{T\to\infty} d_T(D) = 0$. Combining this with (4.5) yields

$$\chi^*(\mathcal{F}(D)) = 0.$$

Since the MNC χ^* is regular, we conclude that $\mathcal{F}(D)$ is relatively compact.

To prove that $\mathcal{F}$ is u.s.c., it suffices to show that $\mathcal{F}$ has closed graph. This is done as in the proof of Lemma 3.5. ■

5. Existence of a global attractor. The m-semiflow governed by the DVI (1.1)–(1.3) is defined as follows:

$$G : \mathbb{R}^+ \times C_\tau \to \mathcal{P}(C_\tau),$$
$$G(t, \varphi) = \{x_t : x[\varphi] \text{ is a solution of (1.1)–(1.3) on } [-\tau, T] \text{ for any } T > 0\}.$$

By the same argument as in [6], we see that

$$G(t_1 + t_2, \varphi) = G(t_1, G(t_2, \varphi)) \quad \text{for all } t_1, t_2 \in \mathbb{R}^+,\ \varphi \in C_\tau.$$

For each $\varphi \in C_\tau$ we denote

$$\Sigma(\varphi) = \big\{x \in C([0,\infty); \mathbb{R}^n) : x[\varphi] \text{ is a solution of (1.1)–(1.3)} \\ \text{on } [-\tau, T] \text{ for any } T > 0\big\}.$$

It is clear that

$$\pi_t \circ \Sigma(\varphi) \subset S(\cdot)\varphi(0) + \mathcal{W} \circ \mathcal{P}_\Phi(\pi_t \circ \Sigma(\varphi)[\varphi]). \tag{5.1}$$

In addition, $G(t, \varphi) = \{x[\varphi]_t : x \in \Sigma(\varphi)\}$. On the other hand, by Theorem 3.6, $\pi_t \circ \Sigma(\varphi)$ is a compact set in $C([0, t]; \mathbb{R}^n)$ for any $t > 0$. It follows that $G(t, \varphi)$ is compact in C_τ, and so $G(t, \cdot)$ has compact values. In fact, we have the following result.

LEMMA 5.1. *Let the hypotheses* (H1)–(H5) *hold. Then $G(t, \cdot)$ is a compact multimap for each $t > \tau$.*

Proof. Let $\Omega \subset C_\tau$ be a bounded set and $\{z_n\} \subset G(t,\Omega)$ be a sequence. Then for each n one can find a function $\varphi_n \in \Omega$ and $x_n \in \Sigma(\varphi_n)$ such that $z_n = x_n[\varphi_n]_t$.

Since $t > \tau$, we have

$$z_n = x_n(t+\cdot) = e^{(t+\cdot)A}\varphi_n(0) + \mathcal{W}(f_n)(t+\cdot),$$

where $f_n \in \mathcal{P}_\Phi(x_n[\varphi_n])$. Since $\{\varphi_n(0)\} \subset \mathbb{R}^n$ is a bounded set, the set $\{e^{(t+\cdot)A}\varphi_n(0)\}$ is relatively compact in C_τ. On the other hand, it is easily seen that $\{x_n\}$ is a bounded sequence, and $\{f_n\}$ is integrably bounded due to estimate (3.10). Since $\mathcal{W}$ is a compact operator, we see that $\{\mathcal{W}(f_n)(t+\cdot)\}$ is relatively compact in C_τ as well. Thus $\{z_n\}$ is relatively compact, as desired. ∎

COROLLARY 5.2. *Let the hypotheses* (H1)–(H5) *hold. Then the m-semiflow G is asymptotically upper semicompact.*

Proof. Taking $t_1 > \tau$, we find that $G(t_1,\cdot)$ is compact, due to Lemma 5.1. Then the conclusion follows from [14, Proposition 1]. ∎

LEMMA 5.3. *Let the hypotheses* (H1)–(H5) *hold. Then $G(t,\cdot)$ is u.s.c. for each $t \geq 0$.*

Proof. Since $G(t,\cdot)$ is a compact multimap with compact values, it suffices to prove that $G(t,\cdot)$ is closed for each $t \geq 0$, thanks to Lemma 2.4. Let $\varphi_n \to \varphi^*$ in C_τ and $z_n \in G(t,\varphi_n)$ be such that $z_n \to z^*$. We show that $z^* \in G(t,\varphi^*)$, i.e. $z^* = x^*[\varphi^*]_t$ for an $x^* \in \Sigma(\varphi^*)$. Taking $x_n \in \Sigma(\varphi_n)$ such that $z_n = x_n[\varphi_n]_t$, one can find $f_n \in \mathcal{P}_\Phi(x_n[\varphi_n])$ satisfying

$$x_n = e^{(\cdot)A}\varphi_n(0) + \mathcal{W}(f_n). \tag{5.2}$$

Since $\{\varphi_n\}$ is bounded in C_τ, $\{x_n\}$ is a bounded sequence in $C([0,T];\mathbb{R}^n)$ for any $T > 0$. Thus $\{f_n\}$ is integrably bounded in $L^1(0,T;\mathbb{R}^n)$. The compactness of $\mathcal{W}$ implies that $\{\mathcal{W}(f_n)\}$ is relatively compact in $C([0,T];\mathbb{R}^n)$. In addition, $\{e^{(\cdot)A}\varphi_n(0)\}$ is a convergent sequence in $C([0,T];\mathbb{R}^n)$, so taking into account (5.2), we see that $\{x_n\}$ has a convergent subsequence (still denoted by $\{x_n\}$). Let $x^* = \lim_{n\to\infty} x_n$ in $C([0,T];\mathbb{R}^n)$. Then $x_n[\varphi_n] \to x^*[\varphi^*]$ in $C([-\tau,T];\mathbb{R}^n)$. Since $\mathcal{P}_\Phi$ is weakly u.s.c., we have $f_n \rightharpoonup f^* \in \mathcal{P}_\Phi(x^*[\varphi^*])$ up to a subsequence, thanks to Lemma 2.5. Therefore one can take the limit of (5.2) to get

$$x^* = e^{(\cdot)A}\varphi^*(0) + \mathcal{W}(f^*)$$

for a selection $f^* \in \mathcal{P}_\Phi(x^*[\varphi^*])$. That is, $x^*[\varphi^*]$ is a solution of (1.1)–(1.3) and then $x^*[\varphi^*]_t \in G(t,\varphi^*)$. Obviously, $z_n = x_n[\varphi_n]_t \to z^* = x^*[\varphi^*]_t$ and $z^* \in G(t,\varphi^*)$. ∎

In order to apply Theorem 2.12, it remains to show that G has an absorbing set in C_τ. To this end, we make use of the following result (see [11]).

PROPOSITION 5.4 (Halanay's inequality). *Let $f : [t_0 - \tau, T) \to \mathbb{R}^+$, $0 \le t_0 < T < \infty$, satisfy the functional differential inequality*

$$f'(t) \le -\gamma f(t) + \nu \sup_{s\in[t-\tau,t]} f(s)$$

for $t \ge t_0$, where $\gamma > \nu > 0$. Then

$$f(t) \le \kappa e^{-\ell(t-t_0)}, \qquad t \ge t_0,$$

where $\kappa = \sup_{s\in[t_0-\tau,t_0]} f(s)$ and ℓ is the solution of $\gamma = \ell + \nu e^{\ell\tau}$.

Using Halanay's inequality, we prove the following result.

LEMMA 5.5. *Let the assumptions* (H1*) *and* (H2)–(H5) *hold. Then the m-semiflow G admits an absorbing set provided that*

$$2\eta_B\eta_G(1+\eta_F) + \eta_h < a.$$

Proof. For $t > 0$ and $\varphi \in C_\tau$, we consider the solution $x[\varphi]$ given by

$$x(t) = e^{tA}\varphi(0) + \int_0^t e^{(t-s)A} f(s)\, ds$$

for $f \in \mathcal{P}_\Phi(x[\varphi])$. Using (H1*) and estimate (3.10), we obtain

$$\|x(t)\| \le e^{-at}\|\varphi(0)\| + \int_0^t e^{-a(t-s)}\big[(\eta+\eta_h)\|x(s)\| + \eta\|x[\varphi]_s\|_{C_\tau} + \eta_G(1+\eta_F)\zeta_B + \zeta_h\big]\, ds,$$

where $\eta = \eta_B\eta_G(1+\eta_F)$. Since $a-(2\eta+\eta_h) > 0$, one can choose $R > 0$ such that $\eta+(\eta_G(1+\eta_F)\zeta_B+\zeta_h)/R = d < a-(\eta+\eta_h)$. Firstly, we prove that for $\varphi \in C_\tau$ satisfying $\|\varphi\|_{C_\tau} \le C$, there exists $t_0 > 0$ such that $\|x[\varphi]_{t_0}\|_{C_\tau} \le R$. Assume to the contrary that $\|x[\varphi]_t\|_{C_\tau} > R$ for all $t > 0$. Then

$$\eta\|x[\varphi]_s\|_{C_\tau} + \eta_G(1+\eta_F)\zeta_B + \zeta_h \le d\|x[\varphi]_s\|_{C_\tau}, \qquad \forall s \ge 0.$$

Therefore

$$\|x(t)\| \le e^{-at}\|\varphi(0)\| + \int_0^t e^{-a(t-s)}\big[(\eta+\eta_h)\|x(s)\| + d\|x[\varphi]_s\|_{C_\tau}\big]\, ds, \qquad t \ge 0.$$

Let

$$y(t) = \begin{cases} e^{-at}\|\varphi(0)\| + \int_0^t e^{-a(t-s)}[(\eta+\eta_h)\|x(s)\| + d\|x_s\|_{C_\tau}]\, ds, & t \ge 0, \\ \|x(t)\|, & t \in [-\tau, 0]. \end{cases}$$

Then $\|x(t)\| \le y(t)$ for all $t \ge -\tau$, and

$$y'(t) \le -[a-(\eta+\eta_h)]y(t) + d \sup_{s\in[t-\tau,t]} y(s).$$

Applying Halanay's inequality yields

$$\|x(t)\| \le \|\varphi\|_{C_\tau} e^{-\ell t} \le Ce^{-\ell t}, \quad \forall t \ge 0,$$

where ℓ is a positive number. Then

$$R < \|x_t\|_{C_\tau} = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\| \le Ce^{\ell\tau}e^{-\ell t}, \quad \forall t \ge 0.$$

This indicates that $\|x_t\|_{C_\tau}$ tends to zero as $t \to \infty$, so one can find $t_1 > 0$ such that $\|x_{t_1}\|_{C_\tau} < R$, a contradiction.

We have just proved that if $\|\varphi\|_{C_\tau} \le C$, then there exists $t_0 > 0$ such that $\|x_{t_0}\|_{C_\tau} \le R$. We claim that $\|u_t\|_{C_\tau} \le R$ for all $t \ge t_0$. Assume the opposite: there exists $t_1 \ge t_0$ satisfying

$$\|x_{t_1}\|_{C_\tau} \le R \quad \text{but} \quad \|x_t\|_{C_\tau} > R \text{ for all } t \in (t_1, t_1+\theta),$$

where $\theta > 0$. Regarding the solution $x[\varphi]$ on $[t_1, t_1+\theta)$, we have

$$x(t) = e^{(t-t_1)A}x(t_1) + \int_{t_1}^{t} e^{(t-s)A} f(s)\,ds, \quad f \in \mathcal{P}_\Phi(x[\varphi]).$$

Then

$$\|x(t)\| \le e^{-a(t-t_1)}\|\varphi(0)\| + \int_{t_1}^{t} e^{-a(t-s)}[(\eta+\eta_h)\|x(s)\| + d\|x_s\|_{C_\tau}]\,ds$$

for $t \in [t_1, t_1+\theta)$. Using the same arguments as above, we see that

$$\|x(t)\| \le \|x_{t_1}\|_{C_\tau} e^{-\ell(t-t_1)} \le \|x_{t_1}\|_{C_\tau} \le R, \quad \forall t \in [t_1, t_1+\theta).$$

Hence for $t \in [t_1, t_1+\theta)$ we have

$$\begin{aligned}
\|x_t\|_{\mathcal{C}_\tau} &= \sup_{s\in[-\tau,0]} \|x(t+s)\| = \sup_{r\in[t-\tau,t]} \|x(r)\| \\
&\le \sup_{r\in[t_1-\tau,t]} \|x(r)\| = \max\Big\{ \sup_{r\in[t_1-\tau,t_1]} \|x(r)\|, \sup_{r\in[t_1,t]} \|x(r)\| \Big\} \\
&= \max\Big\{ \|x_{t_1}\|_{C_\tau}, \sup_{r\in[t_1,t]} \|x(r)\| \Big\} \le R,
\end{aligned}$$

a contradiction. In summary, we can take a ball centered at origin with radius R as an absorbing set for the m-semiflow G, where R is chosen such that

$$R > \frac{\eta_G(1+\eta_F)\zeta_B + \zeta_h}{a-(2\eta+\eta_h)}. \ \blacksquare$$

THEOREM 5.6. *Let the assumptions* (H1*) *and* (H2)–(H5) *hold. Then the m-semiflow* G *generated by* (1.1)–(1.3) *admits a compact global attractor provided that*

$$2\eta_B\eta_G(1+\eta_F)+\eta_h < a.$$

Proof. The conclusion follows from Corollary 5.2 and Lemmas 5.3, 5.5. ∎

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