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Strict plurisubharmonicity of Bergman kernels on generalized annuli

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Abstract. Let $A_\zeta = \Omega - \overline{\rho(\zeta)\cdot\Omega}$ be a family of generalized annuli over a domain U. We show that the logarithm of the Bergman kernel $K_\zeta(z)$ of A_ζ is plurisubharmonic provided $\rho \in \mathrm{PSH}(U)$. It is remarkable that A_ζ is non-pseudoconvex when the dimension of A_ζ is larger than one. For standard annuli in $\mathbb{C}$, we obtain an interesting formula for $\partial^2 \log K_\zeta/\partial\zeta\partial\bar{\zeta}$, as well as its boundary behavior.

1. Introduction and results. In 2004, F. Maitani and H. Yamaguchi [MY] brought a new viewpoint by studying the variation of the Bergman metrics on the Riemann surfaces. Let us briefly recall their results.

Let B be a disk in the complex ζ-plane, D be a domain in the product space $B \times \mathbb{C}_z$, and π be the first projection from $B \times \mathbb{C}_z$ to B, which is proper and smooth. Let $D_\zeta = \pi^{-1}(\zeta)$ be a domain in $\mathbb{C}_z$. Let K_ζ denote the Bergman kernel of D_ζ. Set $\partial D = \bigcup_{\zeta\in B}(\zeta, \partial D_\zeta)$.

Theorem 1.1 ([MY]). *If D is a pseudoconvex domain over $B\times\mathbb{C}_z$ with smooth boundary, then $\log K_\zeta(z)$ is plurisubharmonic on D.*

Theorem 1.2 ([MY]). *If D is a pseudoconvex domain over $B \times \mathbb{C}_z$ with smooth boundary, and for each $\zeta \in B$, ∂D has at least one strictly pseudoconvex point, then $\log K_\zeta(z)$ is strictly plurisubharmonic on D.*

In 2006, B. Berndtsson [B] made a striking generalization of Theorem 1.1 to higher-dimensional case, by using Hörmander's L^2-estimates for $\bar{\partial}$:

Theorem 1.3 ([B]). *Let D be a pseudoconvex domain in $\mathbb{C}^k_\zeta \times \mathbb{C}^n_z$ and ϕ a plurisubharmonic function on D. For each ζ, let D_ζ denote the n-dimensional slice $D_\zeta := \{z \in \mathbb{C}^n : (\zeta, z) \in D\}$ and ϕ^ζ the restriction of ϕ to D_ζ. Let $K_\zeta(z)$ be the Bergman kernel of the Bergman space $H^2(D_\zeta, e^{-\phi^\zeta})$. Then $\log K_\zeta(z)$ is plurisubharmonic or identically equal to $-\infty$ on D.*

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The above mentioned works rely heavily upon the pseudoconvexity of the total space D. In this paper, we obtain the plurisubharmonicity of $\log K_\zeta(z)$ for certain *non-pseudoconvex* domains.

We consider the following family of generalized annuli:

$$A_\zeta = \Omega - \overline{\Omega_\zeta},$$

where $\Omega \subset \mathbb{C}^n$ is a bounded complete circular domain and

$$\Omega_\zeta = \rho(\zeta) \cdot \Omega := \{\rho(\zeta) z : z \in \Omega\}$$

with $0 < \rho < 1$ being an upper semicontinuous function on a domain U in $\mathbb{C}^m$. Let $K_\zeta(z)$ denote the Bergman kernel of A_ζ.

MAIN THEOREM 1.4. *If $n \geq 2$ and $\rho \in \mathrm{PSH}(U)$, then $\log K_\zeta(z)$ is a plurisubharmonic function on $U \times \Omega$. Furthermore, if ρ is strictly plurisubharmonic on U, then $\log K_\zeta(z)$ is strictly plurisubharmonic on $U \times \Omega$.*

The plurisubharmonicity of $\log K_\zeta(z)$ does not imply the pseudoconvexity of the total space even when the slices are planar domains. A simple example may be constructed as follows: let $D = \mathbb{D}^2 - \Gamma_f$, where f is a *non-holomorphic* continuous self-map of the unit disc $\mathbb{D}$ and Γ_f is the graph of f. Since $\log K_\zeta(z) = \log K_{\mathbb{D}}(z)$, it is naturally plurisubharmonic, yet D is not pseudoconvex, in view of the theorem of Hartogs on holomorphicity of pseudoconcave continuous graphs. Nevertheless, it is still worthwhile to ask the following question:

QUESTION 1.5. *Suppose D is a bounded domain over $U \times \mathbb{C}$ where U is a domain in $\mathbb{C}$. Let K_ζ denote the Bergman kernel of the slice D_ζ, and suppose $\log K_\zeta(z)$ is a plurisubharmonic function on D. Under which conditions is D pseudoconvex?*

It is the case when $K_\zeta(z) \to \infty$ as $z \to \partial D_\zeta$ (note that $\log K_\zeta(z)$ is plurisubharmonic, in particular, upper semicontinuous on D). We remind the readers that Zwonek [Z] gave a complete characterization of Bergman exhaustiveness of bounded planar domains in terms of log capacities.

For standard annuli, i.e., Ω is the unit disc $\mathbb{D}$, U is the punctured disc $\mathbb{D}^*$, and $\rho(\zeta) = |\zeta|$, we have an interesting formula for $\partial^2 \log K_\zeta / \partial\zeta\partial\bar{\zeta}$:

MAIN THEOREM 1.6.

$$\frac{\partial^2 \log K_\zeta(z)}{\partial\zeta\partial\overline{\zeta}} = e^{2\omega_1} \frac{(2\mathcal{P}(u) - \mathcal{P}(\omega_1) + c)(\mathcal{P}(\omega_1) + c)}{4\omega_1^2(\mathcal{P}(u) + c)^2},$$

where $u = -2\log|z|$, $\omega_1 = -\log|\zeta|$, $c(\omega_1) = \zeta(\omega_1)/\omega_1$, $\mathcal{P}(\cdot)$ is the Weierstrass elliptic function with periods $2\omega_1$ and $2\pi i$, and $\zeta(\cdot)$ is the Weierstrass zeta function.

As a consequence, we obtain

COROLLARY 1.7. *$\partial^2 \log K_\zeta(z)/\partial\zeta\partial\overline{\zeta} \to 0$ as $D \ni (\zeta, z) \to \partial D$ in a non-trivial way, that is, at first $\zeta \to \zeta_0$, then $z \to \partial A_{\zeta_0}$.*

2. Proof of Main Theorem 1.4. It is well-known that every holomorphic function f on a bounded complete circular domain Ω admits a power series expansion as follows:

$$f(z) = \sum_{j\ge 0} p_j(z),$$

where $p_j(z)$ is a holomorphic polynomial of degree j, in the sense of locally uniform convergence. Thus the Bergman space $H^2(\Omega)$ of Ω admits a complete orthogonal basis

$$p_{j_1}, \ldots, p_{j_{m_j}} \in L_j, \quad j = 0, 1, \ldots,$$

where L_j is the linear space spanned by homogeneous polynomials of degree j, and $m_j = \dim_{\mathbb{C}} L_j$. Since

$$\int_{\Omega_\zeta} p_{j,r}\overline{p_{k,s}} = \rho(\zeta)^{2j+2k+2n} \int_{\Omega} p_{j,r}\overline{p_{k,s}} = 0$$

for any pair $(j, r) \neq (k, s)$, it follows that

$$\int_{A_\zeta} p_{j,r}\overline{p_{k,s}} = \int_{\Omega} p_{j,r}\overline{p_{k,s}} - \int_{\Omega_\zeta} p_{j,r}\overline{p_{k,s}} = 0.$$

By Hartogs' extension theorem, every holomorphic function on A_ζ can be extended to a holomorphic function on Ω. Thus

$$K_\zeta(z) = \sum_{j=0}^{\infty}\sum_{r=1}^{m_j} c_{j,r}|p_{j,r}(z)|^2, \tag{2.1}$$

where

$$c_{j,r}^{-1} = \int_{A_\zeta} |p_{j,r}(z)|^2 = \int_{\Omega} |p_{j,r}(z)|^2 - \int_{\Omega_\zeta} |p_{j,r}(z)|^2 = 1 - \rho(\zeta)^{2j+2n}.$$

That is,

$$K_\zeta(z) = \sum_{j=0}^{\infty}\sum_{r=1}^{m_j} \frac{|p_{j,r}(z)|^2}{1 - \rho(\zeta)^{2j+2n}} \tag{2.2}$$

for any $z \in A_\zeta$. Set

$$K_\zeta^k(z) = \sum_{j=0}^{k}\sum_{r=1}^{m_j} \frac{|p_{j,r}(z)|^2}{1 - \rho(\zeta)^{2j+2n}}.$$

Since $K_\zeta^k \in \mathrm{PSH}(\Omega)$, we infer from the maximum principle that

$$\max_{z\in M} K_\zeta^k(z) \le \max_{z\in\partial G} K_\zeta^k(z) \le \max_{z\in\partial G} K_\zeta(z),$$

where M is a compact set whose interior contains $\overline{\Omega_\zeta}$ and G is a domain such that $M \subset G \subset\subset \Omega$. It follows immediately that the power series (2.2) converges uniformly on compact subsets of Ω, so that K_ζ can be extended to a smooth real function on $U \times \Omega$. It is easy to verify that

$$u_j(\zeta, z) = \log \sum_{r=1}^{m_j} |p_{j,r}(z)|^2 - \log(1 - \rho(\zeta)^{2j+2n})$$

is a plurisubharmonic function on Ω. Since

$$K_\zeta^k(z) = \sum_{j=0}^{k} e^{u_j(\zeta,z)} \tag{2.3}$$

and

$$\chi(t_0, \dots, t_k) := \log(e^{t_0} + \dots + e^{t_k})$$

is a convex function which is non-decreasing in each t_j, we conclude that $\log K_\zeta^k(z)$ is plurisubharmonic on $U \times \Omega$ (see [D, Theorem 4.16]). Since $\{\log K_\zeta^k(z)\}_{k=0}^\infty$ is an increasing sequence of plurisubharmonic functions on $U \times \Omega$ whose limit is the *continuous* function $\log K_\zeta(z)$, it follows that $\log K_\zeta(z)$ has to be plurisubharmonic on $U \times \Omega$.

Now suppose ρ is strictly plurisubharmonic on U. Without loss of generality, we may assume that the volume of Ω equals 1. Then

$$u_0(\zeta, z) = u_0(\zeta) = -\log(1 - \rho(\zeta)^{2n})$$

is also strictly plurisubharmonic on U. Since χ is convex and non-decreasing in each t_j,

$$\partial\bar\partial \log K_\zeta^k(z) \ge \frac{e^{u_0}}{K_\zeta^k(z)} \partial\bar\partial u_0(\zeta).$$

Letting $k \to \infty$ we get

$$\partial\bar\partial \log K_\zeta(z) \ge \frac{e^{u_0}}{K_\zeta(z)} \partial\bar\partial u_0(\zeta),$$

so that for every $\xi = (\xi_1, \dots, \xi_m, \xi_{m+1}, \dots, \xi_{m+n})$ with $(\xi_1, \dots, \xi_m) \ne 0$, the Levi form satisfies $L(\log K_\zeta(z); \xi) > 0$. Moreover, for every non-zero vector $\xi = (0, \dots, 0, \xi_{m+1}, \dots, \xi_{m+n})$, we have

$$L(\log K_\zeta(z); \xi) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log K_\zeta(z)}{\partial z_j \partial \bar z_k} \xi_{m+\alpha} \overline{\xi_{m+\beta}} > 0.$$

Thus $\log K_\zeta(z)$ is strictly plurisubharmonic on $U \times \Omega$.

REMARK. Since $\log K_\zeta(0) = u_0(\zeta)$, we conclude that $\log K_\zeta(z)$ will not be plurisubharmonic on $U \times \Omega$ if $u_0(\zeta)$ is not plurisubharmonic.

3. Proof of Main Theorem 1.6 and Corollary 1.7

Proof of Theorem 1.2. It is known from [S] that

$$\pi K_\zeta(z) = \frac{\mathcal{P}(-2\log|z|) + \eta/(-\log|\zeta|)}{|z|^2}, \tag{3.1}$$

where

$$2\eta = \zeta(u - 2\log|\zeta|) - \zeta(u), \tag{3.2}$$

$u = -2\log|z|$, $\mathcal{P}(\cdot)$ is the Weierstrass elliptic function with periods $-2\log|\zeta|$ and $2\pi i$, and $\zeta(\cdot)$ is the Weierstrass zeta function. If we let $\omega_1 = -\log|\zeta|$, then (3.1) changes to

$$\pi K_\zeta(z) = \frac{\mathcal{P}(u) + \eta/\omega_1}{|z|^2}. \tag{3.3}$$

Since $\zeta'(\cdot) = -\mathcal{P}(\cdot)$, we have

$$\zeta'(\cdot + 2\omega_1) = \zeta'(\cdot),$$

so that

$$\zeta(\cdot + 2\omega_1) = \zeta(\cdot) + C.$$

Take $u = -\omega_1$. Then we get $C = 2\zeta(\omega_1)$ and

$$\zeta(\cdot + 2\omega_1) = \zeta(\cdot) + 2\zeta(\omega_1). \tag{3.4}$$

By (3.2) and (3.4), we obtain $\eta = \zeta(\omega_1)$. Hence, (3.3) changes to

$$K_\zeta(z) = \frac{\mathcal{P}(u) + c(\omega_1)}{\pi|z|^2}, \tag{3.5}$$

where $u = (0, 2\omega_1)$ and $c(\omega_1) = \zeta(\omega_1)/\omega_1$.

Now we turn to calculating $\partial^2 \log K_\zeta(z)/\partial\zeta\partial\overline{\zeta}$. A straightforward calculation yields

$$\begin{aligned}
\frac{\partial c(\omega_1)}{\partial \zeta} &= \frac{\partial c(\omega_1)}{\partial \omega_1}\frac{\partial \omega_1}{\partial \zeta} = \frac{1}{2\zeta}\frac{\mathcal{P}(\omega_1) + c(\omega_1)}{\omega_1},\\
\frac{\partial c(\omega_1)}{\partial \overline{\zeta}} &= \frac{\partial c(\omega_1)}{\partial \omega_1}\frac{\partial \omega_1}{\partial \overline{\zeta}} = \frac{1}{2\overline{\zeta}}\frac{\mathcal{P}(\omega_1) + c(\omega_1)}{\omega_1},\\
\frac{\partial^2 c(\omega_1)}{\partial\zeta\partial\overline{\zeta}} &= \frac{\partial^2 c(\omega_1)}{\partial \omega_1^2}\frac{\partial \omega_1}{\partial \zeta}\frac{\partial \omega_1}{\partial \overline{\zeta}} + \frac{\partial c(\omega_1)}{\partial \omega_1}\frac{\partial^2 \omega_1}{\partial\zeta\partial\overline{\zeta}}\\
&= \frac{1}{4|\zeta|^2}\frac{\mathcal{P}(\omega_1) + c(\omega_1) - \omega_1(\mathcal{P}'(\omega_1) + c'(\omega_1))}{\omega_1^2}.
\end{aligned}$$

We claim that $\mathcal{P}'(\omega_1) = 0$. To see this, simply note that $\mathcal{P}$ is an even function, hence $\mathcal{P}'(-\omega_1) = -\mathcal{P}'(\omega_1)$. Since $\mathcal{P}'(\omega_1) = \mathcal{P}'(-\omega_1)$ by periodicity, we have $\mathcal{P}(\omega_1) = 0$. It follows that

$$\frac{\partial^2 c(\omega_1)}{\partial\zeta\partial\overline{\zeta}} = \frac{1}{4|\zeta|^2}\frac{2(\mathcal{P}(\omega_1) + c(\omega_1))}{\omega_1^2}.$$

So

$$\frac{\partial^2 \log K_\zeta(z)}{\partial\zeta\partial\overline{\zeta}} = e^{2\omega_1}\frac{(2\mathcal{P}(u) - \mathcal{P}(\omega_1) + c)(\mathcal{P}(\omega_1) + c)}{4\omega_1^2(\mathcal{P}(u) + c)^2}. \ \blacksquare$$

Proof of Corollary 1.7. It is easy to see that $\mathcal{P}(0) = \infty$ and $\mathcal{P}(u)$ decreases in $(0, \omega_1)$. We also know that $\mathcal{P}(2\omega_1 - u) = \mathcal{P}(u)$ and $\omega_1^2\mathcal{P}(\omega_1) = \pi^2/6$. So $\mathcal{P}(u) > 0$ in $(0, 2\omega_1)$. Note that

$$\mathcal{P}(u) = u^{-2}(1 + O(u^2))$$

as $u \to 0$. Thus,

$$2\mathcal{P}(u) - \mathcal{P}(\omega_1) + c = 2u^{-2}(1 + O(u^2)), \qquad (\mathcal{P}(u) + c)^2 = u^{-4}(1 + O(u^2)).$$

If $|z| \to 1$, then $u \to 0$. Hence,

$$\lim_{|z|\to 1} \frac{\partial^2 \log K_\zeta(z)}{\partial\zeta\partial\overline{\zeta}} = 0.$$

Using the periodicity of $\mathcal{P}(u)$, we conclude that

$$\lim_{|z|\to|\zeta|} \frac{\partial^2 \log K_\zeta(z)}{\partial\zeta\partial\overline{\zeta}} = 0. \ \blacksquare$$

REMARK. The proof of Main Theorem 1.6 implies that although the Levi form of $\log K_\zeta(z)$ with respect to ζ approaches 0 when (ζ, z) tends to the boundary of the domain, $\log K_\zeta(z)$ is a strictly plurisubharmonic function on D. So, in Theorem 1.2, the condition that for each $\zeta \in B$, ∂D has at least one strictly pseudoconvex point is only a sufficient condition for $\log K_\zeta(z, z)$ to be strictly plurisubharmonic on D.

REMARK. The proof of Main Theorem 1.6 also yields the equation

$$\frac{\partial^2 K_\zeta(z)}{\partial\zeta\partial\overline{\zeta}} = \frac{\partial K_\zeta(z)}{\partial\zeta}\,\frac{\partial K_\zeta(z)}{\partial\overline{\zeta}}.$$

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