ANNALES
POLONICI MATHEMATICI
111.3 (2014)

On the principle of real moduli flexibility: perfect parametrizations

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Abstract. Let V be a real algebraic manifold of positive dimension. The aim of this paper is to show that, for every integer b (arbitrarily large), there exists a trivial Nash family $\mathscr{V} = \{V_y\}_{y \in R^b}$ of real algebraic manifolds such that $V_0 = V$, $\mathscr{V}$ is an algebraic family of real algebraic manifolds over $y \in R^b \setminus \{0\}$ (possibly singular over $y = 0$) and $\mathscr{V}$ is perfectly parametrized by R^b in the sense that V_y is birationally nonisomorphic to V_z for every $y, z \in R^b$ with $y \neq z$. A similar result continues to hold if V is a singular real algebraic set.

1. Introduction and main results. This paper deals with the following principle.

Principle of real moduli flexibility. *The algebraic structure of every real algebraic manifold of positive dimension can be deformed by an arbitrarily large number of effective parameters.*

In [4], we proved the validity of this principle via the notion of almost perfectly parametrized algebraic real-deformation. Let us recall our result.

Let R be a real closed field. By a *real algebraic set*, we mean an algebraic subset of some R^n. A *real algebraic manifold* is an irreducible nonsingular real algebraic set. Let X and Y be real algebraic sets. If there exists a biregular isomorphism from a Zariski dense Zariski open subset of X to a Zariski dense Zariski open subset of Y, then X and Y are said to be *birationally isomorphic*, and we write $X \sim Y$. If X and Y are not birationally isomorphic, then we say that X and Y are birationally nonisomorphic and we write $X \nsim Y$. We will use standard notions from real semialgebraic and Nash geometry (see [5]).

Let $f : X \to R^b$ be a surjective regular map from the real algebraic set X to some R^b. By identifying f with the family $\{f^{-1}(y)\}_{y \in R^b}$ of its fibers, we can assert that f is parametrized by R^b. Define the subset $\mathcal{S}_f$ of $R^b \times R^b$

2010 *Mathematics Subject Classification*: Primary 14P20; Secondary 14P05, 14P25.
Key words and phrases: Nash manifolds, real algebraic manifolds, purely real deformations.

DOI: 10.4064/ap111-3-3

and the map $\rho_b : R^b \times R^b \to R^b$ by setting

$$\mathcal{S}_f := \{(y,z) \in R^b \times R^b \mid f^{-1}(y) \sim f^{-1}(z)\} \quad \text{and} \quad \rho_b(y,z) := y.$$

We say that f is *perfectly parametrized by* R^b if $f^{-1}(y) \nsim f^{-1}(z)$ for each $y, z \in R^b$ with $y \neq z$. We say that f is *almost perfectly parametrized by* R^b if there exists a semialgebraic subset $\mathcal{T}$ of $R^b \times R^b$ such that $\mathcal{T}$ contains $\mathcal{S}_f$ and each fiber of the restriction of ρ_b to $\mathcal{T}$ is finite.

Consider now an algebraic family of real algebraic manifolds parametrized by R^b; that is, a surjective submersive regular map $\pi : \mathcal{V} \to R^b$ from a real algebraic manifold $\mathcal{V}$ to R^b with irreducible fibers. Given a real algebraic manifold V, we say that π is an *algebraic real-deformation of* V if there exists a regular map $\pi' : \mathcal{V} \to V$ such that the map $\pi \times \pi' : \mathcal{V} \to R^b \times V$ is a Nash isomorphism and the restriction of π' to $\pi^{-1}(0)$ is a biregular isomorphism.

In [4], we proved the following result, which is a manifestation of the principle stated above.

Theorem $\mathcal{F}$. *Every real algebraic manifold V of positive dimension has the following property: for each nonnegative integer b, there exists an algebraic real-deformation of V almost perfectly parametrized by R^b.*

In the same paper, we conjectured that, in the preceding statement, one can replace "almost perfectly parametrized" with "perfectly parametrized" (see also [9, Remark 1.10(i)]).

The aim of this paper is to show that, on relaxing suitably the notion of algebraic real-deformation, this conjecture is true. Furthermore, we will show that, in the singular case, the conjecture holds with a quite natural notion of "topological" real-deformation.

Let us present our results. We begin by introducing the concept of real algebro-Nash manifold. By a *real algebro-Nash manifold*, we mean an irreducible algebraic subset of some R^n, which is also a Nash submanifold of R^n. This hybrid notion makes sense. In fact, it determines an intermediate class between the class of real algebraic manifolds and that of Nash manifolds. The algebraic curve of R^2 given by the equation $y^3 - x^3(x^2+1) = 0$ is a simple example of a real algebro-Nash manifold, which is not a real algebraic manifold, having a singularity at $(0,0)$.

Let $\mathcal{N}$ be a real algebro-Nash manifold and let $\varpi : \mathcal{N} \to R^b$ be a surjective submersive regular map with irreducible fibers. Denote by $\mathrm{Reg}(\mathcal{N})$ the set of nonsingular points of $\mathcal{N}$ of maximum dimension, that is, of dimension $\dim \mathcal{N}$. We say that ϖ is an *algebro-Nash real-deformation of* the real algebraic manifold V if $\varpi^{-1}(R^b \setminus \{0\}) \subset \mathrm{Reg}(\mathcal{N})$ and there exists a regular map $\varpi' : \mathcal{N} \to V$ such that the map $\varpi \times \varpi' : \mathcal{N} \to R^b \times V$ is a Nash isomorphism and the restriction of ϖ' to $\varpi^{-1}(0)$ is a biregular isomorphism.

The reader will observe that, if ϖ has these properties, then its fibers are real algebraic manifolds, Nash isomorphic to V. In this way, an algebro-Nash real-deformation of V, parametrized by R^b, can be interpreted as a family of real algebraic structures on the Nash manifold underlying V, which coincides with the real algebraic structure of V itself on $y = 0$, and which depends on $y \in R^b$ with Nash regularity and on $y \in R^b \setminus \{0\}$ with algebraic regularity. Such a family may be singular over $y = 0$ in the algebraic sense, but not in the Nash sense.

The first main result of this paper is as follows.

THEOREM 1.1. *Every real algebraic manifold V of positive dimension has the following property: for each nonnegative integer b, there exists an algebro-Nash real-deformation of V perfectly parametrized by R^b.*

Let W and $\mathcal{W}$ be real algebraic sets, and let $\Pi : \mathcal{W} \to R^b$ be a surjective regular map. We call Π a *semialgebraic real-deformation* of W if there exists a regular map $\Pi' : \mathcal{W} \to W$ such that $\Pi \times \Pi' : \mathcal{W} \to R^b \times W$ is a semialgebraic homeomorphism and the restriction of Π' to $\Pi^{-1}(0)$ is a biregular isomorphism. Moreover, we say that Π is a *good semialgebraic real-deformation* of W if, for each $y \in R^b \setminus \{0\}$, the restriction Π'_y of Π' to $W_y := \Pi^{-1}(y)$ has the following additional property: $(\Pi'_y)^{-1}(\mathrm{Reg}(W)) \subset \mathrm{Reg}(W_y)$ and the restriction of Π'_y from $(\Pi'_y)^{-1}(\mathrm{Reg}(W))$ to $\mathrm{Reg}(W)$ is a Nash isomorphism.

The next is our second main result (see also Remark 3.1).

THEOREM 1.2. *Every real algebraic set W of positive dimension has the following property: for each nonnegative integer b, there exists a good semialgebraic real-deformation of W perfectly parametrized by R^b.*

Let N be an arbitrary connected (affine) Nash manifold over R. It is well-known that N is Nash isomorphic to a real algebraic manifold (see [7]). Thanks to Theorem 1.1, it follows at once that, if $\dim N \geq 1$, then the set of birationally nonisomorphic real algebraic manifolds which are Nash isomorphic to N has the cardinality of R. This was proved in [10, Corollary 2] by a different argument (see also [3, 6, 11, 12]).

We will give the proof of Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. We conclude this introductory section by presenting the idea of these proofs.

Sketch of proof of Theorem 1.1. Let V be a real algebraic manifold of positive dimension and let b be a nonnegative integer. By Theorem $\mathcal{F}$, there is an algebraic real-deformation $\pi : \mathcal{V} \to R^{b+1}$ of V almost perfectly parametrized by R^{b+1}. For each $u \in R^{b+1}$, define $V_u := \pi^{-1}(u)$. Let $\mathcal{T}$ be a semialgebraic subset of $R^{b+1} \times R^{b+1}$ containing $\mathcal{S}_\pi = \{(u, v) \in R^{b+1} \times R^{b+1} \mid V_u \sim V_v\}$ such that the map $\rho_{\mathcal{T}} : \mathcal{T} \to R^{b+1}$ sending (u, v) to u has finite fibers. By combining this property of $\rho_{\mathcal{T}}$ with the fact that $\mathcal{T}$ contains the diagonal Δ

of $R^{b+1}\times R^{b+1}$, we infer that $\dim\mathcal{T}=b+1$. It follows that Δ is an irreducible component of the Zariski closure $\overline{\mathcal{T}}$ of $\mathcal{T}$ in $R^{b+1}\times R^{b+1}$. Since the intersection of Δ with the other irreducible components of $\overline{\mathcal{T}}$ is a proper closed subset of Δ, there exist $u_0\in R^{b+1}$ and a sufficiently small neighborhood B of u_0 in R^{b+1} such that $(\mathrm{B}\times\mathrm{B})\cap\mathcal{S}_\pi\subset\Delta$. Thanks to this inclusion, it is easy to find a sphere S of R^{b+1} contained in B such that $u_0\in\mathrm{S}$, $V_u\not\sim V$ for each $u\in\mathrm{S}\setminus\{u_0\}$ and $V_u\not\sim V_v$ for each $u,v\in\mathrm{S}\setminus\{u_0\}$ with $u\neq v$. Choose $u_1\in\mathrm{S}\setminus\{u_0\}$ and a biregular isomorphism $\lambda:R^b\to\mathrm{S}\setminus\{u_1\}$ such that $\lambda(0)=u_0$. It is evident that the family $\mathscr{V}':=\{V_{\lambda(t)}\}_{t\in R^b}$ is an algebraic real-deformation of V_{u_0} perfectly parametrized by R^b. Now, the problem is that V_{u_0} could not be biregularly isomorphic to V. To overcome this problem, we use a suitable version of a blowing down theorem due to Akbulut and King, which allows one to make algebraic the operation of topological adjunction. In order to be able to apply the result of Akbulut and King, we must work with real algebraic sets satisfying certain "projective closedness" conditions. This leads to some technical difficulties in the proof.

Sketch of proof of Theorem 1.2. Let W be a real algebraic set of positive dimension and let b be as above. First, suppose W irreducible. We apply Hironaka's desingularization theorem to W, obtaining a real algebraic manifold W^*. By Theorem 1.1, there exists an algebro-Nash real-deformation $\{W^*(u)\}_{u\in R^{b+1}}$ of W^* perfectly parametrized by R^{b+1}. By suitably using the aforementioned blowing down theorem of Akbulut and King, we obtain a good semialgebraic real-deformation $\{\mathcal{P}_u:\mathcal{W}(u)\to W\}_{u\in R^{b+1}}$ of W such that $\mathcal{W}(u)\sim W^*(u)$ for each $u\in R^{b+1}\setminus\{0\}$. Since $W^*(u)\not\sim W^*(v)$ for each $u,v\in R^{b+1}\setminus\{0\}$ with $u\neq v$, there exists a sphere S of R^{b+1} containing 0 such that $\mathcal{W}(u)\not\sim\mathcal{W}(v)$ for each $u,v\in\mathrm{S}$ with $u\neq v$. Let u_0 be a point of $\mathrm{S}\setminus\{0\}$ and let $\lambda:R^b\to\mathrm{S}\setminus\{u_0\}$ be a biregular isomorphism such that $\lambda(0)=0$. The family $\{\Pi'_y:=\mathcal{P}_{\lambda(y)}\}_{y\in R^b}$ is the desired good semialgebraic real-deformation of W perfectly parametrized by R^b. In case W is reducible, the proof is more complicated. In fact, it requires an improved version of Theorem 1.1 (see Theorem 2.2 below).

2. Proof of Theorem 1.1. This section is subdivided into two subsections. In the first, we present a version of the real algebraic blowing down theorem of Akbulut and King (see [1, Proposition 3.1] and [2, Section 6 of Chapter II]). In the second, we use this result and a strong version of Theorem $\mathcal{F}$ to prove Theorem 1.1, first in the bounded case and then in the unbounded one.

2.1. Real algebraic blowing down. Let $n\in\mathbb{N}^*:=\mathbb{N}\setminus\{0\}$, let $\ell\in\{0,1,\dots,n-1\}$ and let $\chi_{n-\ell}:R^{n-\ell}\to\mathbb{P}^{n-\ell}(R)$ be the coordinate chart sending x'' into $[1,x'']$. We say that a real algebraic subset W of $R^n=$

$R^\ell \times R^{n-\ell}$ is *projectively closed with respect to* $\{0\} \times R^{n-\ell}$ if $(\mathrm{id}_{R^\ell} \times \chi_{n-\ell})(W)$ is Zariski closed in $R^\ell \times \mathbb{P}^{n-\ell}(R)$. Denote by X' and X'' the indeterminates $(X_1, \ldots, X_\ell)$ and $(X_{\ell+1}, \ldots, X_n)$, respectively. Let P be a polynomial in $R[X] = R[X', X'']$. Write P as follows: $P(X', X'') = P_0(X', X'') + \cdots + P_e(X', X'')$, where e is the degree of P with respect to X'' and each polynomial P_j is homogeneous of degree j with respect to X''. We call P_e the *principal part of* P *with respect to* X'' and we say that P is *overt with respect to* X'' if P_e is nowhere null (in the case $e = 0$) or it vanishes only on $R^\ell \times \{0\}$ (in the case $e > 0$). One can easily verify that W is projectively closed with respect to $\{0\} \times R^{n-\ell}$ if and only if there exists a polynomial in $R[X] = R[X', X'']$ overt with respect to X'', whose zero set is W. If $\ell = 0$, then the preceding notions reduce to the standard ones of projectively closed real algebraic set and of overt polynomial (see [2, p. 34]).

The next result is the version of the Akbulut–King real algebraic blowing down theorem we need below. For the reader's convenience, we include the simple and short proof.

LEMMA 2.1. *Let $n, k \in \mathbb{N}^*$, let $\mathcal{W}$ be a nonempty real algebraic subset of $R^n \times R^k$, let $\pi : R^n \times R^k \to R^n$ be the projection sending (x, z) into x, let W' be a real algebraic subset of R^n and let $K \in R[X] = R[X_1, \ldots, X_n]$ be a polynomial having W' as zero set. Suppose that $\mathcal{W}$ is projectively closed with respect to $\{0\} \times R^k$. Let L be a polynomial in $R[X, Z] = R[X, Z_1, \ldots, Z_k]$ overt with respect to Z whose zero set is $\mathcal{W}$, let $e > 0$ be the degree of L with respect to Z and let L_Z be the principal part of L with respect to Z. Define the regular map $\Sigma : R^n \times R^k \to R^n \times R^k$ and the subset $\mathbf{W}$ of $R^n \times R^k$ by setting $\Sigma(x, z) := (x, z \cdot K(x))$ and $\mathbf{W} := (W' \times \{0\}) \cup \Sigma(\mathcal{W})$, respectively.*

Then $\mathbf{W}$ is Zariski closed in $R^n \times R^k$. More precisely, there exists a polynomial $L' \in R[X, Z]$ such that the degree of L' with respect to Z is $< e$ and

$$\mathbf{W} = \{(x, z) \in R^n \times R^k \mid L_Z(x, z) + K(x) L'(x, z) = 0\}. \tag{2.1}$$

In this way, $\mathbf{W}$ is a real algebraic subset of $R^n \times R^k$, projectively closed with respect to $\{0\} \times R^k$.

Proof. The restriction of Σ from $(R^n \setminus W') \times R^k$ into itself is a biregular isomorphism whose inverse sends (x, z) to $(x, z/K(x))$. It follows that

$$\mathbf{W} \setminus (W' \times R^k) = \{(x, z) \in (R^n \setminus W') \times R^k \mid L(x, z/K(x)) = 0\}. \tag{2.2}$$

Furthermore,

$$\mathbf{W} \cap (W' \times R^k) = W' \times \{0\}. \tag{2.3}$$

By clearing denominators in the expression $L(X, Z/K(X))$, one obtains a polynomial $L' \in R[X, Z]$ whose degree with respect to Z is $< e$ and

$$(K(x))^e L(x, z/K(x)) = L_Z(x, z) + K(x) L'(x, z) \tag{2.4}$$

for each $(x,z) \in (R^n \setminus W') \times R^k$. By combining (2.2), (2.3), (2.4) and the fact that L_Z vanishes only on $R^n \times \{0\}$, we immediately obtain (2.1). In particular, $\mathbf{W}$ is a real algebraic subset of $R^n \times R^k$, projectively closed with respect to $\{0\} \times R^k$. ∎

Let V be a real algebraic submanifold of R^n and let $C = R[s]/(s^2+1)$ be the algebraic closure of R. An irreducible projective complex algebraic manifold $Z \subset \mathbb{P}^m(C)$ is called a *nonsingular complexification* of V if it is defined over R (that is, Z is invariant under the complex conjugation of $\mathbb{P}^m(C)$) and the real part $Z \cap \mathbb{P}^m(R)$ of Z is biregularly isomorphic to V. If V is bounded, that is, contained in some open ball of R^n, then the existence of nonsingular complexifications of V is ensured by Hironaka's desingularization theorem [13] (see also [4, Lemma 2.1]).

Let us recall the statement of Theorem 3.1 of [4] (see also [4, Remark 3.2]).

Theorem $\mathcal{F}^+$. *Let V be a bounded real algebraic manifold of positive dimension r, let $b,\ell \in \mathbb{N}^*$ and let d be an odd integer ≥ 3. Then there exist $M,c \in \mathbb{N}^*$, a real algebraic submanifold $\mathcal{V}$ of $R^b \times V \times R^M \times R \times R \times R$ and regular maps $\phi_1 : R^b \to R^M$, $\phi_2 : R^b \to R$, $G_1 : R^b \times V \to R$ and $G_2 : R^b \times V \times R \to R$ with the following properties:*

(i) *$G_1(R^b \times V) \subset (0,2)$ and $G_2(R^b \times V \times (0,2)) \subset \{r \in R \mid r > 0\}$.*
(ii) *$G_1(0,x) = 1$ and $G_2(0,x,1) = 1$ for each $x \in V$.*
(iii) *$\mathcal{V}$ is equal to the subset of $R^b \times V \times R^M \times R \times R \times R$ consisting of points $(y,x,\mathfrak{a},s,t,v)$ such that $\mathfrak{a} = \phi_1(y)$, $s = \phi_2(y)$, $t^d = G_1(y,x)$ and $v^d = G_2(y,x,t)$. In particular, $\mathcal{V}$ is the graph of a Nash map from $R^b \times V$ to $R^M \times R \times R \times R$.*
(iv) *The projection $\pi : \mathcal{V} \to R^b$, sending $(y,x,\mathfrak{a},s,t,v)$ into y, is an algebraic real-deformation of V almost perfectly parametrized by R^b.*
(v) *$c \geq \ell$ and, for each $y \in R^b \setminus \{0\}$, $\pi^{-1}(y)$ admits a nonsingular complexification Z_y with ample canonical complex line bundle ω_{Z_y} such that $\omega^r_{Z_y} = c$.*

We denote by $\mathrm{id}_T : T \to T$ the identity map on the set T and by S^{m-1} the standard unit sphere of R^m.

We are now in a position to prove the next result, which is a strong form of Theorem 1.1 for bounded real algebraic manifolds.

Theorem 2.2. *Let V be a bounded real algebraic submanifold of R^n of positive dimension r and let $b,\ell \in \mathbb{N}^*$. Then there exist $m,c \in \mathbb{N}^*$ and a real algebro-Nash submanifold $\mathcal{N}$ of $R^b \times V \times R^m$ with the following properties:*

(i) *$\mathcal{N}$ is the graph of a Nash map from $R^b \times V$ to R^m.*
(ii) *$\mathcal{N} \subset R^b \times V \times S^{m-1}$.*

(iii) *If $\varpi : \mathcal{N} \to R^b$ denotes the projection sending (y, x, u) to y, then ϖ is an algebro-Nash real-deformation of V perfectly parametrized by R^b.*

(iv) *$c \geq \ell$ and, for each $y \in R^b \setminus \{0\}$, $\varpi^{-1}(y)$ admits a nonsingular complexification Z_y with ample canonical complex line bundle ω_{Z_y} such that $\omega^r_{Z_y} = c$.*

Proof. We divide the proof into five steps.

Step I. Fix an odd integer $d \geq 3$. We apply Theorem $\mathcal{F}^+$ with $b+1$ in place of b, obtaining M, $\mathcal{V}$, ϕ_1, ϕ_2, G_1, G_2 and $\pi : \mathcal{V} \to R^{b+1}$ with the properties described in its statement. In particular, $\mathcal{V}$ is equal to the subset of $\mathscr{R} := R^{b+1} \times V \times R^M \times R \times R \times R \subset R^{b+1} \times R^n \times R^{M+3}$ consisting of all points $(u, x, \mathfrak{a}, s, t, v)$ such that

$$\mathfrak{a} = \phi_1(u), \qquad s = \phi_2(u), \quad t^d = G_1(u, x), \qquad v^d = G_2(u, x, t). \tag{2.5}$$

Moreover, by Theorem $\mathcal{F}^+$, (v), there exists an integer $c \geq \ell$ such that, for each $u \in R^{b+1} \setminus \{0\}$, $\pi^{-1}(u)$ admits a nonsingular complexification Z_u with ample canonical complex line bundle ω_{Z_u} such that $\omega^r_{Z_u} = c$.

Let us prove that, by replacing $M+3$ with a larger integer k, we may suppose that $\mathcal{V}$ is projectively closed with respect to $\{0\} \times \{0\} \times R^k$.

Let $\chi : R \to \mathbb{P}^1(R)$ be the coordinate chart sending t into $[1, t]$, let $\chi_* : \mathscr{R} \to \mathscr{R}_1 := R^{b+1} \times V \times R^M \times R \times \mathbb{P}^1(R) \times \mathbb{P}^1(R)$ be the map $\mathrm{id}_{R^{b+1}} \times \mathrm{id}_V \times \mathrm{id}_{R^M} \times \mathrm{id}_R \times \chi \times \chi$, let $\varphi : \mathbb{P}^1(R) \to R^2$ be the biregular embedding sending $[t_0, t]$ to $(t_0^2 - t^2, 2t_0 t)/(t_0^2 + t^2)$ and let $\varphi_* : \mathscr{R}_1 \to \mathscr{R}_2 := R^{b+1} \times V \times R^M \times R \times R^2 \times R^2$ be the biregular embedding $\mathrm{id}_{R^{b+1}} \times \mathrm{id}_V \times \mathrm{id}_{R^M} \times \mathrm{id}_R \times \varphi \times \varphi$. Thanks to the last two equations of (2.5), it is immediate to verify that $\chi_*(\mathcal{V})$ is Zariski closed in $\mathscr{R}_1$ and hence $\mathcal{V}_* := \varphi_*(\chi_*(\mathcal{V}))$ is Zariski closed in $\mathscr{R}_2$. It follows that $\mathcal{V}_*$ is a real algebraic submanifold of $\mathscr{R}_2$ biregularly isomorphic to $\mathcal{V}$ (via $\varphi_* \circ \chi_*$). Moreover, since $\varphi(\mathbb{P}^1(R)) = S^1$, we have

$$\mathcal{V}_* \subset R^{b+1} \times V \times R^M \times R \times S^1 \times S^1. \tag{2.6}$$

Denote by $(u, x, \mathfrak{a}, s, p_1, q_1, p_2, q_2) = (u, x, \mathfrak{a}, s, (p_1, q_1), (p_2, q_2))$ the coordinates of $\mathscr{R}_2$, where $u = (u_1, \ldots, u_{b+1})$, $x = (x_1, \ldots, x_n)$ and $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_M)$. Choose a polynomial L_* in $R[U, X, \mathfrak{A}, S, P_1, Q_1, P_2, Q_2]$ having $\mathcal{V}_*$ as zero set. Here U, X and $\mathfrak{A}$ denote the indeterminates $(U_1, \ldots, U_{b+1})$, $(X_1, \ldots, X_n)$ and $(\mathfrak{A}_1, \ldots, \mathfrak{A}_M)$, respectively. Let e_* be the degree of L_* with respect to the indeterminates $Z := (\mathfrak{A}, S, P_1, Q_1, P_2, Q_2)$ and let e be the smallest even integer $> e_*$.

By Proposition 2.1.1 of [2], there exist polynomials $\phi'_{1,1}, \ldots, \phi'_{1,M}$, ϕ''_1, ϕ'_2, ϕ''_2 in $R[U]$ such that ϕ''_1 and ϕ''_2 are nowhere null on R^{b+1} and

$$\phi_1(u) = (\phi'_{1,1}(u), \ldots, \phi'_{1,M}(u))/\phi''_1(u) \quad \text{and} \quad \phi_2(u) = \phi'_2(u)/\phi''_2(u)$$

for each $u \in R^{b+1}$. Define the polynomial $L \in R[U,X,Z]$ by setting

$$L(U,X,Z) := (L_*(U,X,Z))^2 + \sum_{j=1}^{M}(\phi_1''(U)\mathfrak{A}_j - \phi_{1,j}'(U))^{2e} + (\phi_2''(U)S - \phi_2'(U))^{2e} + (P_1^2+Q_1^2-1)^e + (P_2^2+Q_2^2-1)^e.$$

By combining (2.6) and the first two equations of (2.5), we infer that the zero set of L in $R^{b+1}\times R^n \times R^M \times R \times R^2 \times R^2$ coincides with $\mathcal{V}_*$.

For simplicity, we rename $\mathcal{V}^*$ as $\mathcal{V}$ and Z as $(Z_1,\dots,Z_k)$, where $k := M+5$. Denote by L_{2e} the principal part of L with respect to Z. The definition of L implies that

$$L_{2e}(U,X,Z) = (\phi_1''(U))^{2e}\Big(\sum_{j=1}^{k-5} Z_j^{2e}\Big) + (\phi_2''(U))^{2e}Z_{k-4}^{2e} + (Z_{k-3}^2+Z_{k-2}^2)^e + (Z_{k-1}^2+Z_k^2)^e. \tag{2.7}$$

Since L_{2e} vanishes only on $R^{b+1}\times R^n \times \{0\}$, we infer at once that

(2.8) $\mathcal{V}$ is projectively closed with respect to $\{0\}\times\{0\}\times R^k$.

Step II. Let $\rho : R^{b+1}\times R^{b+1} \to R^{b+1}$ be the projection onto the first factor and let $\mathcal{T}$ be a semialgebraic subset of $R^{b+1}\times R^{b+1}$ containing $\mathcal{S}_\pi$ such that the restriction $\rho_{\mathcal{T}}$ of ρ to $\mathcal{T}$ has finite fibers. Such a $\mathcal{T}$ exists because π is an algebraic real-deformation of V almost perfectly parametrized by R^{b+1}. The finiteness of the fibers of $\rho_{\mathcal{T}}$ implies that $\dim \mathcal{T} = b+1$ (recall that $\mathcal{T}$ contains the diagonal Δ of $R^{b+1}\times R^{b+1}$). In particular, the Zariski closure $\overline{\mathcal{T}}$ of $\mathcal{T}$ in $R^{b+1}\times R^{b+1}$ has dimension $b+1$ and hence Δ is an irreducible component of $\overline{\mathcal{T}}$. Let $\mathcal{T}'$ be the union of the irreducible components of $\overline{\mathcal{T}}$ different from Δ. Since $\dim(\Delta\cap\mathcal{T}') < b+1$, there exist $u_0\in R^{b+1}$ and an open neighborhood B of u_0 in R^{b+1} such that $(\mathrm{B}\times\mathrm{B})\cap\mathcal{T}' = \emptyset$ and hence

$$(\mathrm{B}\times\mathrm{B})\cap\mathcal{S}_\pi \subset \Delta. \tag{2.9}$$

Step III. Let $K\in R[U,X]$ be a polynomial whose zero set in $R^{b+1}\times R^n$ is equal to $\{u_0\}\times V$. Define the map $\Sigma : R^{b+1}\times R^n\times R^k \to R^{b+1}\times R^n \times R^k$, the subset $\mathbf{V}$ of $R^{b+1}\times V\times R^k$ and the map $\mathbf{p} : \mathbf{V}\to R^{b+1}$ by setting $\Sigma(u,x,z) := (u,x,z\cdot K(u,x))$, $\mathbf{V} := \Sigma(\mathcal{V})$ and $\mathbf{p}(u,x,z) := u$, respectively. By Theorem $\mathcal{F}^+$, (iii), there exists a Nash map $\Phi : R^{b+1}\times V \to R^k$ such that $\mathcal{V}$ is the graph of Φ. It follows that $\mathbf{V}$ is the graph of the Nash map from $R^{b+1}\times V$ to R^k sending (u,x) into $\Phi(u,x)K(u,x)$. Observe that $\Sigma(\pi^{-1}(u_0)) = \{u_0\}\times V\times\{0\}$, $\mathbf{p}^{-1}(u) = \Sigma(\pi^{-1}(u))$ for each $u\in R^{b+1}$ and the restriction of Σ from $(R^{b+1}\setminus\{u_0\})\times V\times R^k$ into itself is a biregular isomorphism. In particular, $\mathbf{p}^{-1}(u_0)$ is biregularly isomorphic to V and $\mathbf{p}^{-1}(u)$ is biregularly isomorphic to $\pi^{-1}(u)$ for each $u\in R^{b+1}\setminus\{u_0\}$. Since $\mathcal{V}$ is projectively closed with respect to $\{0\}\times\{0\}\times R^k$

(see (2.8)), Lemma 2.1 applies. It follows that $\mathbf{V}$ is a real algebro-Nash submanifold of $R^{b+1} \times V \times R^k$ and $\mathbf{p}$ is an algebro-Nash real-deformation of V. Furthermore, there exists a polynomial $L' \in R[U, X, Z]$ whose degree with respect to Z is $< 2e$ such that, defining $\mathcal{L} := L_{2e} + KL'$, we have

$$\mathbf{V} = \{(u, x, z) \in R^{b+1} \times R^n \times R^k \mid \mathcal{L}(u, x, z) = 0\}.$$

Step IV. Let H be the subset of $\mathrm{B} \setminus \{u_0\}$ consisting of points u such that $V \sim \mathbf{p}^{-1}(u)$ (or, equivalently, $V \sim \pi^{-1}(u)$). By (2.9), H contains at most one element. In this way, it is possible to find a b-dimensional sphere S of R^{b+1} contained in $\mathrm{B} \setminus \mathrm{H}$ and containing u_0, a point $u_1 \in \mathrm{S} \setminus \{u_0\}$ and a biregular isomorphism $\lambda : R^b \to \mathrm{S} \setminus \{u_1\}$ such that $\lambda(0) = u_0$. Let $i : \mathrm{S} \setminus \{u_1\} \hookrightarrow R^{b+1}$ be the inclusion map. By using Proposition 2.1.1 of [2] again, there exist polynomials $\mu_1, \ldots, \mu_{b+1}, \nu \in R[Y] = R[Y_1, \ldots, Y_b]$ such that ν is nowhere null on R^b and $\lambda(y) = \mu(y)/\nu(y)$ for each $y \in R^b$, where $\mu : R^b \to R^{b+1}$ denotes the map sending y into $(\mu_1(y), \ldots, \mu_{b+1}(y))$. Define the real algebro-Nash submanifold $\mathcal{N}'$ of $R^b \times R^{b+1} \times V \times R^k$ as the fiber product of $i \circ \lambda$ and $\mathbf{p}$. By rearranging coordinates, $\mathcal{N}'$ is the subset of $R^b \times R^n \times R^k \times R^{b+1}$ consisting of points (y, x, z, u) such that

$$\mathcal{L}(\mu(y)/\nu(y), x, z) = 0 \quad \text{and} \quad \nu(y)u = \mu(y). \tag{2.10}$$

Furthermore, $\mathcal{N}'$ is equal to the graph of the Nash map from $R^b \times V$ to $R^k \times R^{b+1}$ sending (y, x) into $(\Phi(\lambda(y), x)K(\lambda(y), x), \lambda(y))$. Denote by $\varpi' : \mathcal{N}' \to R^b$ the projection sending (y, x, z, u) into y. Inclusion (2.9) and the choice of S imply at once that ϖ' is an algebro-Nash real-deformation of V perfectly parametrized by R^b. We have just proved that $\mathcal{N}'$ and ϖ' satisfy conditions (i), (iii) and (iv). It remains to show that $\mathcal{N}'$ can be modified in order to ensure the truth of (ii) too. We will do it in the next step.

Step V. Let us prove that $\mathcal{N}'$ is projectively closed with respect to $\{0\} \times \{0\} \times R^k \times R^{b+1}$. Let $h_1 \in \mathbb{N}^*$ and $\phi_1''', \phi_2''' \in R[Y]$ be such that

$$(\phi_1''(\mu(y)/\nu(y)))^{2e} = \phi_1'''(y)/(\nu(y))^{2eh_1},$$
$$(\phi_2''(\mu(y)/\nu(y)))^{2e} = \phi_2'''(y)/(\nu(y))^{2eh_1}$$

for each $y \in R^b$. Evidently, ϕ_1''' and ϕ_2''' assume only positive values on R^b. Define $L_{2e}' \in R[Y, X, Z]$ homogeneous of degree $2e$ with respect to Z by setting

$$\begin{aligned} L_{2e}'(Y, X, Z) = {} & \phi_1'''(Y)(\textstyle\sum_{j=1}^{k-5} Z_j^{2e}) + \phi_2'''(Y) Z_{k-4}^{2e} \\ & + (\nu(Y))^{2eh_1}(Z_{k-3}^2 + Z_{k-2}^2)^e + (\nu(Y))^{2eh_1}(Z_{k-1}^2 + Z_k^2)^e. \end{aligned}$$

By (2.7), for each $(y, x, z) \in R^b \times R^n \times R^k$,

$$(\nu(y))^{2eh_1} L_{2e}(\mu(y)/\nu(y), x, z) = L_{2e}'(y, x, z).$$

The latter equality implies the existence of a nonnegative integer h_2 and a polynomial $L'' \in R[Y,X,Z]$ whose degree with respect to Z is $< 2e$ and such that

$$(2.11)\quad (\nu(y))^{2eh_1+h_2}\mathcal{L}(\mu(y)/\nu(y),x,z) = (\nu(y))^{h_2}L'_{2e}(y,x,z) + L''(y,x,z)$$

for each $(y,x,z) \in R^b \times R^n \times R^k$. Denote by $L''_{2e} \in R[Y,X,Z]$ the polynomial $\nu^{h_2}L'_{2e}$. By (2.10) and (2.11), it follows that $\mathcal{N}'$ consists of points (y,x,z,u) in $R^b \times R^n \times R^k \times R^{b+1}$ such that

$$(2.12)\qquad L''_{2e}(y,x,z) + L''(y,x,z) = 0 \quad\text{and}\quad \nu(y)u = \mu(y).$$

Moreover, by definition of L''_{2e}, it follows that

$$(2.13)\qquad L''_{2e} \text{ vanishes only on } R^b \times R^n \times \{0\}.$$

Define $N' \in R[Y,X,Z,U]$ by setting

$$N'(Y,X,Z,U) := (L''_{2e}(Y,X,Z) + L''(Y,X,Z))^2 + \sum_{i=1}^{b+1}(\nu(Y)U_i - \mu_i(Y))^{4e}.$$

Bearing in mind (2.12), we infer that the zero set of N' is equal to $\mathcal{N}'$. Furthermore, if $N'_{Z,U}$ denotes the principal part of N' with respect to (Z,U), then

$$N'_{Z,U}(Y,X,Z,U) = (L''_{2e}(Y,X,Z))^2 + (\nu(Y))^{4e}\sum_{j=1}^{b+1}U_j^{4e}.$$

By (2.13), it follows that $N'_{Z,U}$ vanishes only on $R^b \times R^n \times \{0\} \times \{0\}$. We have just proved that $\mathcal{N}'$ is projectively closed with respect to $\{0\} \times \{0\} \times R^k \times R^{b+1} = \{0\} \times \{0\} \times R^{k+b+1}$. In this way, if θ denotes the integer $k+b+1$ and $\xi : R^k \times R^{b+1} = R^\theta \to \mathbb{P}^\theta(R)$ is the coordinate chart sending (z,u) into $[1,z,u]$, then $\mathcal{N}'' := (\mathrm{id}_{R^b} \times \mathrm{id}_V \times \xi)(\mathcal{N}')$ is Zariski closed in $R^b \times V \times \mathbb{P}^\theta(R)$.

Let us complete the proof. Let $m := (\theta+1)^2$, let $\Theta : \mathbb{P}^\theta(R) \to S^{m-1}$ be the biregular embedding sending $[q_0,q_1,\ldots,q_\theta]$ to $(q_iq_j/\sum_{a=0}^{\theta} q_a^2)_{i,j}$ (see [2, p. 38]) and let $\mathcal{N} := (\mathrm{id}_{R^b} \times \mathrm{id}_V \times \Theta)(\mathcal{N}'')$. The real algebro-Nash manifold $\mathcal{N}$ and the projection $\varpi : \mathcal{N} \to R^b$ sending (y,x,p) to y have all the required properties. ∎

Let us give the proof of Theorem 1.1 in the unbounded case.

Let V be a real algebraic submanifold of some R^n of positive dimension. Suppose that V is unbounded, that is, not bounded. We recall that the Alexandrov compactification of V can be made algebraic (see [2, Lemma 2.6.2] and [5, pp. 76–77]). More precisely, there exist a bounded real algebraic subset $\dot{V}$ of R^{n+1}, a point $p \in \dot{V}$ and a biregular isomorphism from V to $\dot{V} \setminus \{p\}$. Identify V with $\dot{V} \setminus \{p\}$ via such a biregular isomorphism. Observe that $V \subset \mathrm{Reg}(\dot{V})$. By Hironaka's desingularization theorem, there exist a bounded real algebraic submanifold V^* of some R^m and a regular map

$\varrho : V^* \to \dot{V}$ such that the restriction of ϱ from $\varrho^{-1}(V)$ to V is a biregular isomorphism. Identify V with $\varrho^{-1}(V)$ via ϱ.

Let $b \in \mathbb{N}^*$. By Theorem 2.2, there exists an algebro-Nash real-deformation $\pi^* : \mathcal{V}^* \to R^b$ of V^* perfectly parametrized by R^b. Let $\pi' : \mathcal{V}^* \to V^*$ be a regular map such that $\pi^* \times \pi' : \mathcal{V}^* \to R^b \times V^*$ is a Nash isomorphism and the restriction of π' to $(\pi^*)^{-1}(0)$ is a biregular isomorphism. Define $\mathcal{V} := (\pi^* \times \pi')^{-1}(R^b \times V)$ and $\pi : \mathcal{V} \to R^b$ as the restriction of π^* to $\mathcal{V}$. It is evident that π is an algebro-Nash real-deformation of V perfectly parametrized by R^b.

The proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2. We organize the proof into four steps.

Step I. Let $\dot{W}$ be a bounded real algebraic subset of some R^n obtained from W by adding a point p at infinity. Identify W with $\dot{W} \setminus \{p\}$. By Hironaka's desingularization theorem, there exist $k \in \mathbb{N}$ and a nonsingular real algebraic subset W' of $\dot{W} \times \mathbb{P}^k(R)$ such that, denoting by $\eta : W' \to \dot{W}$ the projection sending (x, q) to x, the restriction of η from $\eta^{-1}(\mathrm{Reg}(\dot{W}))$ to $\mathrm{Reg}(\dot{W})$ is a biregular isomorphism.

Let $\ell := (k+1)^2$ and let $\Theta : \mathbb{P}^k(R) \to S^{\ell-1}$ be the biregular embedding sending $[q_0, q_1, \ldots, q_k]$ to $(q_i q_j / \sum_{a=0}^k q_a^2)_{i,j}$. Define $W^* := (\mathrm{id}_{\dot{W}} \times \Theta)(W') \subset R^n \times R^\ell$ and $\varrho : W^* \to \dot{W}$ by setting $\varrho(x, x') = x$. Evidently, ϱ has the same property of η: its restriction from $\varrho^{-1}(\mathrm{Reg}(\dot{W}))$ to $\mathrm{Reg}(\dot{W})$ is a biregular isomorphism. Clearly,

$$W^* \subset \dot{W} \times S^{\ell-1}. \tag{3.1}$$

Let $W_1^*, \ldots, W_h^*$ be all the irreducible components of W^* and define $\sigma : \mathrm{Reg}(\dot{W}) \to R^\ell$ to be the regular map such that

$$(x, \sigma(x)) \in W^* \quad \text{for each } x \in \mathrm{Reg}(\dot{W}). \tag{3.2}$$

By applying Theorem 2.2 to W_j^*, with $b+2$ in place of b, inductively on $j \in \{1, \ldots, h\}$, we find $c_1, \ldots, c_h \in \mathbb{N}^*$ with $c_1 < \cdots < c_h$ and, for each $j \in \{1, \ldots, h\}$, $m_j \in \mathbb{N}^*$, a real algebro-Nash submanifold $\mathcal{W}_j^*$ of $R^{b+2} \times W_j^* \times S^{m_j-1} \subset R^{b+2} \times (R^n \times R^\ell) \times R^{m_j}$ and a Nash map $\Phi_j^* : R^{b+2} \times W_j^* \to R^{m_j}$ such that: $\mathcal{W}_j^*$ is the graph of Φ_j^*, the projection $\varpi_j^* : \mathcal{W}_j^* \to R^{b+2}$ sending $(u, x, x', y_j) = (u, (x, x'), y_j)$ to u is an algebro-Nash real-deformation of W_j^* perfectly parametrized by R^{b+2} and, for each $u \in R^{b+2} \setminus \{0\}$, the real algebraic manifold $W^*(u, j) := (\varpi_j^*)^{-1}(u)$ admits a nonsingular complexification $Z(u, j)$ with ample canonical complex line bundle $\omega_{Z(u,j)}$ and $\omega^r_{Z(u,j)} = c_j$, where $r = \dim W = \dim W_j^*$. Observe that, thanks to (3.1),

$$\mathcal{W}_j^* \subset R^{b+2} \times \dot{W} \times S^{\ell-1} \times S^{m_j-1} \quad \text{for each } j \in \{1, \ldots, h\}. \tag{3.3}$$

Let $m := \sum_{j=1}^h m_j$. Identify each Euclidean space $R^{b+2} \times R^n \times R^\ell \times R^{m_j}$ with the real vector subspace $R^{b+2} \times R^n \times R^\ell \times \{0_{<j}\} \times R^{m_j} \times \{0_{>j}\}$ of $R^{b+2} \times R^n \times R^\ell \times \prod_{0 \le i < j} R^{m_i} \times R^{m_j} \times \prod_{j < i \le h} R^{m_i} = R^{b+2} \times R^n \times R^\ell \times R^m$, where $0_{<j}$ and $0_{>j}$ denote the origins of $\prod_{0 \le i < j} R^{m_i}$ and of $\prod_{j < i \le h} R^{m_i}$, respectively. Define the projectively closed real algebraic subset O of R^m by setting $O := \bigcup_{j=1}^h (\{0_{<j}\} \times S^{m_j - 1} \times \{0_{>j}\})$. Denote by $\mathcal{W}^*$ the real algebraic subset of $R^{b+2} \times R^n \times R^\ell \times R^m$ equal to the disjoint union of the $\mathcal{W}_j^*$'s, by $\varpi^* : \mathcal{W}^* \to R^{b+2}$ the projection sending (u, x, x', z) to u, by $W^*(u)$ the fiber $(\varpi^*)^{-1}(u)$ of ϖ^* over u for each $u \in R^{b+2}$ and by $\Phi^* : R^{b+2} \times W^* \to R^m$ the Nash map defined as follows: $\Phi^*(u, x, x') := (0_{<j}, \Phi_j^*(u, x, x'), 0_{>j})$ if $(u, x, x') \in R^{b+2} \times W_j^*$. Observe that, for each $u \in R^{b+2}$, $W^*(u, 1), \ldots, W^*(u, h)$ are the irreducible components of $W^*(u)$. Moreover, by (3.3),

$$\mathcal{W}^* \subset R^{b+2} \times \dot{W} \times S^{\ell-1} \times O. \tag{3.4}$$

Step II. Fix $u, v \in R^{b+2} \setminus \{0\}$ with $u \neq v$. Suppose $W^*(u) \sim W^*(v)$. Then there exists $j \in \{1, \ldots, h\}$ such that $W^*(u, 1) \sim W^*(v, j)$. In particular, $Z(u, 1)$ is complex birationally isomorphic to $Z(v, j)$. Bearing in mind that $\omega_{Z(u,1)}$ and $\omega_{Z(v,j)}$ are ample, we find that $Z(u, 1)$ and $Z(v, j)$ are also complex biregularly isomorphic (see [8, p. 170]) and hence $c_1 = \omega^r_{Z(u,1)} = \omega^r_{Z(v,j)} = c_j$. It follows that $j = 1$. This is impossible, because ϖ_1^* is perfectly parametrized by R^{b+2}. We have just proved that

$$W^*(u) \nsim W^*(v) \quad \text{for each } u, v \in R^{b+2} \setminus \{0\} \text{ with } u \neq v.$$

The latter fact implies that the subset H_1 of $R^{b+2} \setminus \{0\}$ consisting of points u such that $W^*(0) \sim W^*(u)$ contains at most one element. Let S_1 be a sphere of R^{b+2} containing $\{0\}$ and disjoint from H_1, let $u_0 \in \mathrm{S}_1 \setminus \{0\}$, let $\lambda_1 : R^{b+1} \to \mathrm{S}_1 \setminus \{u_0\}$ be a biregular isomorphism such that $\lambda_1(0) = 0$, let $i_1 : \mathrm{S}_1 \setminus \{u_0\} \hookrightarrow R^{b+2}$ be the inclusion map, let $\mathcal{W}_1$ be the fiber product of $i_1 \circ \lambda_1$ and ϖ^*, and let $\Phi_1 : R^{b+1} \times W^* \to R^m \times R^{b+2}$ be the Nash map sending (w, x, x') to $(\Phi^*(\lambda_1(w), x, x'), \lambda_1(w))$. By rearranging coordinates, we have

$$\mathcal{W}_1 = \{(w, (x, x'), z, u) \in R^{b+1} \times W^* \times R^m \times R^{b+2} \mid (z, u) = \Phi_1(w, x, x')\}.$$

By (3.4), we infer that

$$\mathcal{W}_1 \subset R^{b+1} \times \dot{W} \times S^{\ell-1} \times O \times \mathrm{S}_1. \tag{3.5}$$

Denote by $\varpi_1 : \mathcal{W}_1 \to R^{b+1}$ the projection sending (w, x, x', z, u) to w. By definition of $\mathcal{W}_1$, $(\varpi_1)^{-1}(w)$ is biregularly isomorphic to $W^*(\lambda_1(w))$ for each $w \in R^{b+1}$ and hence

$$(\varpi_1)^{-1}(w) \nsim (\varpi_1)^{-1}(w') \quad \text{for each } w, w' \in R^{b+1} \text{ with } w \neq w'. \tag{3.6}$$

Step III. Set $\theta := \ell + m + b + 2$ and denote by W and X the indeterminates $(\mathrm{W}_1, \ldots, \mathrm{W}_{b+1})$ and $(X_1, \ldots, X_n)$, respectively. Since $S^{\ell-1} \times O \times \mathrm{S}_1$

is projectively closed in $R^\theta = R^\ell \times R^m \times R^{b+2}$, (3.5) implies that $\mathcal{W}_1$ is projectively closed in $R^{b+1} \times R^n \times R^\theta$ with respect to $\{0\} \times \{0\} \times R^\theta$.

Let $W^\bullet := (\{0\} \times \dot{W}) \cup (R^{b+1} \times (\dot{W} \setminus \mathrm{Reg}(\dot{W})))$, let $K \in R[\mathrm{W}, X]$ be a polynomial whose zero set in $R^{b+1} \times R^n$ is equal to $W^\bullet$, let $\Sigma : R^{b+1} \times \dot{W} \times R^\theta \to R^{b+1} \times \dot{W} \times R^\theta$ be the regular map sending (w, x, ξ) to $(w, x, \xi \cdot K(w, x))$ and let $\mathcal{W}_2 := (W^\bullet \times \{0\}) \cup \Sigma(\mathcal{W}_1)$. By Lemma 2.1, $\mathcal{W}_2$ is Zariski closed in $R^{b+1} \times \dot{W} \times R^\theta$. Let $\varpi_2 : \mathcal{W}_2 \to R^{b+1}$ be the projection sending (w, x, ξ) to w. Since the restriction of Σ from $(R^{b+1} \times \dot{W} \times R^\theta) \setminus (W^\bullet \times R^\theta)$ into itself is a biregular isomorphism, we see that $(\varpi_2)^{-1}(w) \sim (\varpi_1)^{-1}(w)$ for each $w \in R^{b+1} \setminus \{0\}$. In this way, by (3.6),

$$(3.7)\quad (\varpi_2)^{-1}(w) \nsim (\varpi_2)^{-1}(w') \quad \text{for each } w, w' \in R^{b+1} \setminus \{0\} \text{ with } w \neq w'.$$

Moreover, $(\varpi_2)^{-1}(0)$ is equal to $\{0\} \times \dot{W} \times \{0\}$ and hence it is biregularly isomorphic to $\dot{W}$. Define the semialgebraic map $\Phi_2 : R^{b+1} \times \dot{W} \to R^\ell \times R^{m+b+2} = R^\theta$ as follows: $\Phi_2(w, x) = 0$ if $(w, x) \in W^\bullet$ and $\Phi_2(w, x) := (\sigma(x) \cdot K(w, x), \Phi_1(w, x, \sigma(x)) \cdot K(w, x))$ if $(w, x) \notin W^\bullet$ (see (3.2) for the definition of σ). By (3.1), the image of σ is contained in $S^{\ell-1}$ and hence is bounded. This ensures that Φ_2 is continuous. Moreover, it is evident that the graph of Φ_2 is equal to $\mathcal{W}_2$ and the restriction of Φ_2 to $(R^{b+1} \times \dot{W}) \setminus W^\bullet$ is a Nash map.

Step IV. Let us repeat the argument used in the second half of Step II. Let H_2 be the subset of $R^{b+1} \setminus \{0\}$ consisting of points w such that $(\varpi_2)^{-1}(0) \sim (\varpi_2)^{-1}(w)$. By (3.7), H_2 contains at most one element. Let S_2 be a sphere of R^{b+1} containing $\{0\}$ and disjoint from H_2, let $w_0 \in \mathrm{S}_2 \setminus \{0\}$, let $\lambda_2 : R^b \to \mathrm{S}_2 \setminus \{w_0\}$ be a biregular isomorphism such that $\lambda_2(0) = 0$, and let $i_2 : \mathrm{S}_2 \setminus \{0\} \hookrightarrow R^{b+1}$ be the inclusion map, let $\mathcal{W}_3$ be the fiber product of $i_2 \circ \lambda_2$ and ϖ_2. Upon rearranging coordinates, $\mathcal{W}_3$ is equal to the subset of $R^b \times \dot{W} \times R^\theta \times R^{b+1}$ consisting of points (y, x, ξ, w) such that $(w, x, \xi) \in \mathcal{W}_2$ and $w = \lambda_2(y)$. Let $\mathrm{P} : \mathcal{W}_3 \to R^b \times \dot{W}$ be the projection sending (y, x, ξ, w) to (y, x), let $\mathcal{W} := \mathrm{P}^{-1}(R^b \times W)$ and let $\Pi : \mathcal{W} \to R^b$ and $\Pi' : \mathcal{W} \to W$ be the projections sending (y, x, ξ, w) to y and to x, respectively. By construction, $\mathcal{W}$, Π and Π' have all the desired properties, that is, Π is a good semialgebraic real-deformation of W perfectly parametrized by R^b.

Remark 3.1. As an immediate consequence of the preceding proof, we can add the following property of Π to the statement of Theorem 1.2:

(iv) *For every $y \in R^b$, $\Pi^{-1}(y)$ and W have the same number of irreducible components of maximum dimension, that is, of dimension* $\dim W$.

Acknowledgements. We wish to thank the referee for his precious advice.

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Received 24.4.2013
and in final form 16.10.2013 (3245)