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Existence and upper semicontinuity of uniform attractors in $H^1(\mathbb{R}^N)$ for nonautonomous nonclassical diffusion equations

by Cung The Anh (Hanoi) and Nguyen Duong Toan (Haiphong)

Abstract. We prove the existence of uniform attractors $\mathcal{A}_\varepsilon$ in the space $H^1(\mathbb{R}^N)$ for the nonautonomous nonclassical diffusion equation

$$u_t - \varepsilon \Delta u_t - \Delta u + f(x,u) + \lambda u = g(x,t), \quad \varepsilon \in [0,1].$$

The upper semicontinuity of the uniform attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ at $\varepsilon = 0$ is also studied.

1. Introduction. In this paper we consider the nonautonomous equation

$$(1.1)\quad \begin{cases} u_t-\varepsilon\Delta u_t-\Delta u+f(x,u)+\lambda u = g(t,x), & x\in\mathbb{R}^N\ (N\ge 3),\ t>\tau,\\ u|_{t=\tau}=u_\tau, \end{cases}$$

where $\varepsilon \in [0,1]$, and the nonlinearity f and the external force g satisfy some certain conditions specified later. This equation is known as the nonclassical diffusion equation when $\varepsilon > 0$, and the classical reaction-diffusion equation when $\varepsilon = 0$.

The nonclassical diffusion equation arises as a model to describe physical phenomena, such as non-Newtonian flows, solid mechanics, and heat conduction (see, e.g., [1, 10, 15, 16]); and if one considers viscoelasticity of the conductive medium, we obtain the so-called nonclassical diffusion equations with fading memory (see [20, 19]). The existence and long-time behavior of solutions to problem (1.1) has been studied extensively in recent years, for both the autonomous case [8, 12, 13, 18, 21, 22] and the nonautonomous case [2, 3, 4, 13]. However, to the best of our knowledge, most existing results related to this problem are valid in bounded domains, except some recent works [3, 9] where the existence of global/pullback attractors for problem (1.1) on $\mathbb{R}^N$ with the nonlinearity of polynomial type was proved.

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In this paper we will study the existence and upper semicontinuity of uniform attractors for a family of processes associated to problem (1.1) in the case of unbounded domains, the nonlinearity of Sobolev type, and the external force g depending on time t. To study the existence of weak solutions to problem (1.1), we assume the following conditions:

(H1) The function $f(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous in both variables, differentiable in the second variable, and satisfies

$$f(x,u)u \geq -\mu u^2 - \phi_1(x), \tag{1.2}$$

$$f'_u(x,u) \geq -\ell, \tag{1.3}$$

$$|f(x,u)| \leq C(\phi_2(x) + |u|^\rho), \tag{1.4}$$

$$\liminf_{|u|\to\infty} \frac{uf(x,u) - \kappa F(x,u)}{u^2} \geq 0, \tag{1.5}$$

$$\liminf_{|u|\to\infty} \frac{F(x,u)}{u^2} \geq 0, \tag{1.6}$$

where $\phi_1 \in L^1(\mathbb{R}^N)$, $\phi_2 \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ are nonnegative functions, $1 \leq \rho \leq \min\{\frac{N+2}{N-2}, 2 + \frac{4}{N}\}$ if $\varepsilon > 0$ and $1 \leq \rho \leq \frac{N}{N-2}$ if $\varepsilon = 0$, $0 < \mu < \lambda$, and $F(x,u) = \int_0^u f(x,s)\,ds$.

(H2) The external force g is in $L^2_b(\mathbb{R}; L^2(\mathbb{R}^N))$, the space of translation bounded functions in $L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^N))$, that is, $g \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^N))$ satisfies

$$\|g\|_b^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(s)\|^2_{L^2(\mathbb{R}^N)}\,ds < \infty.$$

The aim of this paper is to prove the existence of uniform attractors $\mathcal{A}_\varepsilon$, $\varepsilon \in [0,1]$, for problem (1.1) on the whole space $\mathbb{R}^N$ and the upper semicontinuity of $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ at $\varepsilon = 0$. It is known that there are two main difficulties in studying the nonclassical diffusion equation (1.1) in unbounded domains. The first one is the unboundedness of the domain $\mathbb{R}^N$, so the Sobolev embeddings are no longer compact. The second one is the appearance of the term $-\varepsilon \Delta u_t$, thus if the initial datum u_τ belongs to $H^1(\mathbb{R}^N)$, then the solution with initial condition $u(\tau) = u_\tau$ is always in $H^1(\mathbb{R}^N)$ and has no higher regularity, which is similar to hyperbolic equations. These bring some essential difficulties in proving the existence of solutions and uniform attractors for problem (1.1). On the other hand, since uniform attractors are not "strictly invariant" like global attractors, this introduces some significant difficulty when one wants to show the upper semicontinuity of a family of uniform attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ with respect to the parameter ε.

Let us describe the method used in this paper. First, the existence of a unique weak solution is proved by the Galerkin approximation and the compactness method. However, because the domain considered is unbounded, the proof is more complicated than that in the case of bounded domains since we cannot use the Compactness Lemma of Aubin and Lions directly. Next, we show the existence of uniform attractors in $L^2(\mathbb{R}^N)$, $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ under some stronger conditions on the external force g, namely conditions (H2′) in Section 3. Using the so-called "tail estimates" method, introduced by B. Wang [17], we first prove the existence of a uniform attractor in $L^2(\mathbb{R}^N)$. After that, using the asymptotic a priori estimate method of [11], we prove the existence of uniform attractors in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and then in $H^1(\mathbb{R}^N)$. Finally, by using the structure of uniform attractors, that is, a uniform attractor can be viewed as a union of kernel sections (see Definition 2.4), and the continuous dependence of solutions to problem (1.1) on ε as $\varepsilon \to 0^+$ established in Lemma 5.1, we prove the upper semicontinuity in $H^1(\mathbb{R}^N)$ of the family of uniform attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ at $\varepsilon = 0$. It is worth noticing that the results obtained are also valid for problem (1.1) in an arbitrary (bounded or unbounded) domain Ω (we then need to add the homogeneous Dirichlet condition on $\partial\Omega$), and when the external force is time-independent, the uniform attractor becomes the classical global attractor and we formally get the corresponding results for the global attractor. Up to the best of our knowledge, these results are new even for autonomous nonclassical diffusion equations in unbounded domains.

The paper is organized as follows. In Section 2, for the convenience of the reader, we recall some results on uniform attractors and the space of translation bounded functions. Section 3 proves the existence and weak continuity of the family of processes associated to the problem. In Section 4, we prove the existence of uniform attractors $\mathcal{A}_\varepsilon$ for this family of processes in various spaces under some additional conditions of the external force. The upper semicontinuity of the uniform attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ at $\varepsilon = 0$ is investigated in Section 5.

Throughout this paper, we denote by $\|\cdot\|$, $(\cdot,\cdot)$ the norm and scalar product in $L^2(\mathbb{R}^N)$, respectively. We also denote by C a generic constant, which can vary from line to line, even in the same line.

2. Preliminaries

2.1. Uniform attractors. Let Σ be a parameter set, and X, Y be Banach spaces. A family $\{U_\sigma(t,\tau),\ t \geq \tau,\ \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, is said to be a *family of processes* in X if for each $\sigma \in \Sigma$, $\{U_\sigma(t,\tau)\}$ is a *process*, that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from X to X satisfies

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \qquad \forall t \geq s \geq \tau,\ \tau \in \mathbb{R},$$
$$U_\sigma(\tau,\tau) = \mathrm{Id}, \qquad \tau \in \mathbb{R},$$

where Id is the identity operator and Σ is called the symbol space, $\sigma \in \Sigma$ is the symbol. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X.

We first recall concepts of absorbing sets and uniform asymptotic compactness which are related to the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$.

DEFINITION 2.1. A set $B_0 \in \mathcal{B}(Y)$ is said to be an (X,Y)-*uniform* (with respect to $\sigma \in \Sigma$) *absorbing set* for $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if for any $\tau \in \mathbb{R}$, and $B \in \mathcal{B}(X)$, there exists $T_0 \geq \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq T_0$.

DEFINITION 2.2. A family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is called (X,Y)-*uniformly* (with respect to $\sigma \in \Sigma$) *asymptotically compact* if for any $\tau \in \mathbb{R}$, any $B \in \mathcal{B}(X)$, the set $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is relatively compact in Y, where $\{x_n\} \subset B$, $\{t_n\} \subset [\tau,\infty)$, $t_n \to \infty$ and $\{\sigma_n\} \subset \Sigma$ are arbitrary.

We now introduce definitions of uniform attractor and kernel.

DEFINITION 2.3. A subset $\mathcal{A}_\Sigma \subset Y$ is said to be an (X,Y)-*uniform attractor* of the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if

(1) $\mathcal{A}_\Sigma$ is compact in Y;
(2) for each fixed $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(X)$ we have
$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}_Y(U_\sigma(t,\tau)B, \mathcal{A}_\Sigma) = 0,$$
where $\operatorname{dist}_Y(\cdot,\cdot)$ denotes the Hausdorff semidistance in the Banach space Y,
$$\operatorname{dist}_Y(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_Y;$$
(3) if $\mathcal{A}'_\Sigma$ is a closed subset of Y satisfying (2), then $\mathcal{A}_\Sigma \subset \mathcal{A}'_\Sigma$.

DEFINITION 2.4. The *kernel* $\mathcal{K}$ of a process $\{U(t,\tau)\}$ acting on X consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:
$$\mathcal{K} = \{u(\cdot) \mid U(t,\tau)u(\tau) = u(t),\ \operatorname{dist}(u(t),u(0)) \leq C_u,\ \forall t \geq \tau,\ \tau \in \mathbb{R}\}.$$
For $s \in \mathbb{R}$, the set $\mathcal{K}(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}\}$ is said to be the *kernel section* at time s.

We will use the following result on the existence and structure of uniform attractors.

THEOREM 2.5 ([5]). *Assume that the family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *satisfies the following conditions:*

(1) Σ *is weakly compact, and* $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *is* $(X\times\Sigma,Y)$*-weakly continuous, that is, for any fixed* $t \geq \tau$*, the mapping* $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ *is weakly continuous in* Y*. Moreover, there is a weakly continuous semigroup* $\{T(h)\}_{h\geq 0}$ *acting on* Σ *satisfying*
$$T(h)\Sigma = \Sigma, \quad U_\sigma(t+h,\tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma,\ t \geq \tau,\ h \geq 0.$$

(2) $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *has an* (X,Y)*-uniform (with respect to* $\sigma\in\Sigma$*) absorbing set* B_0.
(3) $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *is* (X,Y)*-uniformly (with respect to* $\sigma\in\Sigma$*) asymptotically compact.*

Then it possesses an (X,Y)*-uniform attractor* $\mathcal{A}_\Sigma$*, and*

$$\mathcal{A}_\Sigma=\bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(s),\quad \forall s\in\mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ *is the kernel section at time* s *of the process* $U_\sigma(t,\tau)$.

2.2. Translation bounded functions

DEFINITION 2.6. Let $\mathcal{E}$ be a reflexive Banach space. A function φ in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is said to be *translation bounded* if

$$\|\varphi\|_b^2=\|\varphi\|^2_{L^2_b(\mathbb{R};\mathcal{E})}=\sup_{t\in\mathbb{R}}\int_t^{t+1}\|\varphi\|^2_{\mathcal{E}}\,ds<\infty.$$

For $g\in L^2_b(\mathbb{R};L^2(\mathbb{R}^N))$, we denote by $\mathcal{H}_w(g)$ the closure of the set $\{g(\cdot+h)\mid h\in\mathbb{R}\}$ in $L^2_b(\mathbb{R};L^2(\mathbb{R}^N))$ with the weak topology. We have the following results.

LEMMA 2.7 ([6, Chapter 5, Proposition 4.2]).

(1) *For all* $\sigma\in\mathcal{H}_w(g)$*, we have* $\|\sigma\|_b^2\le\|g\|_b^2$.
(2) *The translation group* $\{T(h)\}$ *is weakly continuous on* $\mathcal{H}_w(g)$.
(3) $T(h)\mathcal{H}_w(g)=\mathcal{H}_w(g)$.
(4) $\mathcal{H}_w(g)$ *is weakly compact.*

3. Existence and weak continuity of the associated family of processes

DEFINITION 3.1. A function $u(t,x)$ is called a *weak solution* of equation (1.1) on the interval $[\tau,T]$ if

$$u\in C([\tau,T];H^1(\mathbb{R}^N)),\quad \frac{du}{dt}\in L^2(\tau,T;H^1(\mathbb{R}^N)),$$
$$u(x,\tau)=u_\tau(x)\quad \text{for a.e. } x\in\mathbb{R}^N,$$

and

$$\int_\tau^T\int_{\mathbb{R}^N}u_t\varphi\,dx\,dt+\varepsilon\int_\tau^T\int_{\mathbb{R}^N}\nabla u_t\nabla\varphi\,dx\,dt+\int_\tau^T\int_{\mathbb{R}^N}\nabla u\nabla\varphi\,dx\,dt$$
$$+\lambda\int_\tau^T\int_{\mathbb{R}^N}u\varphi\,dx\,dt+\int_\tau^T\int_{\mathbb{R}^N}f(x,u)\varphi\,dx\,dt=\int_\tau^T\int_{\mathbb{R}^N}g(s)\varphi\,dx\,dt$$

for all test functions $\varphi\in C^\infty([\tau,T]\times\mathbb{R}^N)$.

THEOREM 3.2. *Assume that f satisfies condition* (H1), *and g satisfies condition* (H2). *Then for any $u_\tau \in H^1(\mathbb{R}^N)$, any $\sigma \in \mathcal{H}_w(g)$ and any given $T > \tau$, $\tau \in \mathbb{R}$, problem* (1.1) (*with σ in place of g*) *has a unique weak solution u on $[\tau, T]$, and the weak solution depends continuously on the initial data. Moreover, for any $t > \tau$,*

$$\begin{aligned}(3.1)\qquad &\|u(t)\|^2 + \varepsilon\|\nabla u(t)\|^2 \\ &\qquad \le C\left(e^{-\delta(t-\tau)}(\|u_\tau\|^2 + \varepsilon\|\nabla u_\tau\|^2) + 1 + \frac{1}{1-e^{-\delta}}\|g\|_b^2\right)\end{aligned}$$

for some positive number $\delta < \min\{2, 2(\lambda - \mu)\}$.

Proof. (i) *Existence.* For each integer $n \ge 1$, we denote by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j$$

a solution of

$$(3.2)\qquad \begin{cases} \dfrac{d}{dt}(u_n, w_j) + \varepsilon(\nabla\partial_t u_n, \nabla w_j) + (\nabla u_n, \nabla w_j) + \langle f(x, u_n), w_j\rangle \\ \qquad\qquad\qquad\qquad + \lambda(u_n, w_j) = (\sigma(t), w_j), \\ (u_n(\tau), w_j) = (u_\tau, w_j), \quad j = 1, \ldots, n, \end{cases}$$

where $\{w_j \mid j \ge 1\} \subset H^1(\mathbb{R}^N)$ is a Hilbert basis of $L^2(\mathbb{R}^N)$ such that $\operatorname{span}\{w_j\}_{j\ge1}$ is dense in $H^1(\mathbb{R}^N)$. Replacing w_j by u_n in (3.2), we get

$$\begin{aligned}(3.3)\qquad \frac{1}{2}\frac{d}{dt}(\|u_n\|^2 + \varepsilon\|\nabla u_n\|^2) + \|\nabla u_n\|^2 + \int_{\mathbb{R}^N} f(x, u_n) u_n\,dx + \lambda\|u_n\|^2 \\ = \int_{\mathbb{R}^N} \sigma(t) u_n\,dx.\end{aligned}$$

Using (1.2), we have

$$(3.4)\qquad \int_{\mathbb{R}^N} f(x, u_n) u_n\,dx \ge -\mu\|u_n\|^2 - \|\phi_1\|_{L^1(\mathbb{R}^N)}.$$

By the Cauchy inequality, we have

$$(3.5)\qquad \int_{\mathbb{R}^N} \sigma(t) u_n\,dx \le \|\sigma(t)\|\,\|u_n\| \le \frac{1}{2\eta}\|\sigma(t)\|^2 + \frac{\eta}{2}\|u_n\|^2.$$

Combining (3.3)–(3.5), we get

$$\begin{aligned}(3.6)\qquad \frac{d}{dt}(\|u_n\|^2 + \varepsilon\|\nabla u_n\|^2) + 2\|\nabla u_n\|^2 + 2\left(\lambda - \mu - \frac{\eta}{2}\right)\|u_n\|^2 \\ \le \frac{1}{\eta}\|\sigma(t)\|^2 + 2\|\phi_1\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

Integrating (3.6) from τ to t, $t \in [\tau, T]$, we obtain

$$\|u_n(t)\|^2 + \varepsilon\|\nabla u_n(t)\|^2 + \delta\int_\tau^t (\|u_n(s)\|^2 + \varepsilon\|\nabla u_n(s)\|^2)\,ds$$
$$\le (\|u_n(\tau)\|^2 + \varepsilon\|\nabla u_n(\tau)\|^2) + \frac{1}{\eta}\int_\tau^T \|\sigma(t)\|^2\,ds + 2\|\phi_1\|_{L^1(\mathbb{R}^N)}(T-\tau),$$

where $0 < \delta < \min\{2, 2(\lambda-\mu)\}$. Let $P_n : H^1(\mathbb{R}^N) \to \operatorname{span}\{w_1, \dots, w_n\}$ be the canonical projector. Since $u_n(\tau) = P_n u_\tau$, we have $\|u_n(\tau)\|_{H^1(\mathbb{R})} \le \|u_\tau\|_{H^1(\mathbb{R}^N)}$. Therefore, from the inequality above we obtain

$$\|u_n(t)\|^2 + \varepsilon\|\nabla u_n(t)\|^2 + \delta\int_\tau^t (\|u_n(s)\|^2 + \varepsilon\|\nabla u_n(s)\|^2)\,ds$$
$$\le (\|u_\tau\|^2 + \varepsilon\|\nabla u_\tau\|^2) + \frac{1}{\eta}\int_\tau^T \|\sigma(t)\|^2\,ds + 2\|\phi_1\|_{L^1(\mathbb{R}^N)}(T-\tau).$$

This inequality implies that

$$\{u_n\} \text{ is bounded in } L^\infty(\tau, T; H^1(\mathbb{R}^N)). \tag{3.7}$$

Hence, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup u \quad \text{weakly-star in } L^\infty(\tau, T; H^1(\mathbb{R}^N)). \tag{3.8}$$

We deduce from (3.7) that $\{\Delta u_n\}$ is bounded in $L^2(\tau, T; H^{-1}(\mathbb{R}^N))$, thus

$$\Delta u_n \rightharpoonup \Delta u \quad \text{in } L^2(\tau, T; H^{-1}(\mathbb{R}^N)). \tag{3.9}$$

On the other hand, replacing w_j by $\partial_t u_n$ in (3.2), we get

$$\|\partial_t u_n\|^2 + \varepsilon\|\nabla\partial_t u_n\|^2 + \frac{1}{2}\frac{d}{dt}\Big(\lambda\|u_n\|^2 + \|\nabla u_n\|^2 + 2\int_{\mathbb{R}^N} F(x, u_n)\,dx\Big)$$
$$= \int_{\mathbb{R}^N} \sigma(t)\partial_t u_n\,dx \le \frac{1}{2}\|\sigma(t)\|^2 + \frac{1}{2}\|\partial_t u_n\|^2. \tag{3.10}$$

Integrating (3.10) from τ to t, we obtain

$$\{\partial_t u_n\} \text{ is bounded in } L^2(\tau, T; H^1(\mathbb{R}^N)), \tag{3.11}$$

thus

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{in } L^2(\tau, T; H^1(\mathbb{R}^N)), \tag{3.12}$$

and

$$\Delta\partial_t u_n \rightharpoonup \Delta u_t \quad \text{in } L^2(\tau, T; H^{-1}(\mathbb{R}^N)), \text{ up to a subsequence.} \tag{3.13}$$

By (1.4), we have

$$\|f(\cdot,u_n)\|^{\frac{2N}{N+2}}_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} \le C\Big(\int\limits_{\mathbb{R}^N} |\phi_2(x)|^{\frac{2N}{N+2}}\,dx + \int\limits_{\mathbb{R}^N} |u_n|^{\rho\frac{2N}{N+2}}\,dx\Big)$$
$$= C\Big(\|\phi_2\|^{\frac{2N}{N+2}}_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} + \|u_n\|^{\rho\frac{2N}{N+2}}_{L^{\rho\frac{2N}{N+2}}(\mathbb{R}^N)}\Big).$$

Since $\rho\frac{2N}{N+2} \le \frac{2N}{N-2}$ and $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ continuously, we get

$$\|f(\cdot,u_n)\|^{\frac{2N}{N+2}}_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} \le C\Big(\|\phi_2\|^{\frac{2N}{N+2}}_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} + \|u_n\|^{\rho\frac{2N}{N+2}}_{H^1(\mathbb{R}^N)}\Big). \tag{3.14}$$

From (3.7) and (3.14) we deduce that

$$\{f(\cdot,u_n)\} \text{ is bounded in } L^{\frac{2N}{N+2}}(\tau,T;L^{\frac{2N}{N+2}}(\mathbb{R}^N)), \tag{3.15}$$

thus

$$f(\cdot,u_n) \rightharpoonup \chi \quad \text{in } L^{\frac{2N}{N+2}}(\tau,T;L^{\frac{2N}{N+2}}(\mathbb{R}^N)) \text{ up to a subsequence.} \tag{3.16}$$

We now prove that $\chi = f(x,u)$. For each $m \ge 1$, we denote $B_m = \{x \in \mathbb{R}^N \mid |x| \le m\}$. Let $\theta \in C^1([0,\infty))$ be a function such that $0 \le \theta \le 1$, $\theta|_{[0,1]} = 1$ and $\theta(s) = 0$ for all $s \ge 2$. For each n and m we define

$$v_{n,m}(x,t) = \theta\Big(\frac{|x|^2}{m^2}\Big)u_n(x,t).$$

We infer from (3.7) that, for all $m \ge 1$, the sequence $\{v_{n,m}\}_{n\ge1}$ is bounded $L^\infty(\tau,T;H_0^1(B_{2m}))$. As B_{2m} is a bounded set, $H_0^1(B_{2m}) \hookrightarrow L^2(B_{2m})$ compactly. Then, by [14, Theorem 13.3 and Remark 13.1] we deduce that

$$\{v_{n,m}\} \text{ is precompact in } L^2(\tau,T;L^2(B_{2m})),$$

and thus

$$\{u_n|_{B_m}\} \text{ is precompact in } L^2(\tau,T;L^2(B_m)).$$

By a diagonal procedure, using (3.8), we deduce that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$u_n \to u \quad \text{a.e. in } B_m \times (\tau,\infty) \text{ as } n \to \infty,\ \forall m \ge 1.$$

Then, since $f(\cdot,\cdot)$ is continuous,

$$f(x,u_n) \to f(x,u) \quad \text{a.e. in } B_m \times (\tau,\infty),$$

and since $\{f(x,u_n)\}$ is bounded in $L^{\frac{2N}{N+2}}(\tau,T;L^{\frac{2N}{N+2}}(B_m))$, by [7, Chapter 1, Lemma 3.1], we have

$$f(\cdot,u_n) \rightharpoonup f(\cdot,u) \quad \text{in } L^{\frac{2N}{N+2}}(\tau,T;L^{\frac{2N}{N+2}}(B_m)).$$

From (3.16),

$$f(x,u_n) \rightharpoonup \chi|_{B_m\times(\tau,T)} \quad \text{in } L^{\frac{2N}{N+2}}(\tau,T;L^{\frac{2N}{N+2}}(B_m)).$$

Hence

$$\chi = f(x,u) \quad \text{a.e. in } B_m \times (\tau,T),\ \forall m \geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^{\infty} B_m = \mathbb{R}^N$, we obtain

$$\chi = f(x,u) \quad \text{a.e. in } \mathbb{R}^N \times (\tau,T). \tag{3.17}$$

Now, combining (3.8), (3.9), (3.12), (3.13), (3.16) and (3.17), we see that

$$u_t - \varepsilon \Delta u_t - \Delta u + f(x,u) + \lambda u = \sigma(t) \text{ in } H^{-1}(\mathbb{R}^N) \quad \text{for a.e. } t \in [\tau,T].$$

We now prove that $u \in C([\tau,T];H^1(\mathbb{R}^N))$. Putting $w_n = u_n - u$, we have

$$\partial_t w_n - \varepsilon \Delta(\partial_t w_n) - \Delta w_n + \lambda w_n + f(x,u_n) - f(x,u) = 0. \tag{3.18}$$

Multiplying (3.18) by w_n, then integrating over $\mathbb{R}^N$, we obtain

$$\frac{d}{dt}(\|w_n\|^2 + \varepsilon\|\nabla w_n\|^2) + 2\|\nabla w_n\|^2 + 2\lambda\|w_n\|^2 + 2(f(x,u_n) - f(x,u), w_n) = 0.$$

Using Lagrange's theorem and (1.3), we get

$$\frac{d}{dt}(\|w_n\|^2 + \varepsilon\|\nabla w_n\|^2) + 2\|\nabla w_n\|^2 + 2\lambda\|w_n\|^2 \leq 2\ell(\|w_n\|^2 + \varepsilon\|\nabla w_n\|^2).$$

Applying the Gronwall inequality, we obtain

$$\|w_n(t)\|^2 + \varepsilon\|\nabla w_n(t)\|^2 \leq e^{2\ell(T-\tau)}(\|w_n(\tau)\|^2 + \varepsilon\|\nabla w_n(\tau)\|^2). \tag{3.19}$$

Since $u_n(\tau) \to u_\tau$ in $H^1(\mathbb{R}^N)$, we conclude from (3.19) that $u_n \to u$ uniformly in $C([\tau,T];H^1(\mathbb{R}^N))$. This implies the desired result.

It remains to show that $u(\tau) = u_\tau$. To do this, we choose some test function $\varphi \in C^1([\tau,T];H^1(\mathbb{R}^N))$ with $\varphi(T) = 0$. Multiplying (1.1) by φ, then integrating from τ to t, we obtain

$$\int_\tau^T -(u,\varphi')\,dx\,dt + \int_\tau^T\int_{\mathbb{R}^N} \varepsilon\nabla u_t \nabla\varphi\,dx\,dt + \int_\tau^T\int_{\mathbb{R}^N} \nabla u\nabla\varphi\,dx\,dt$$
$$+ \lambda\int_\tau^T\int_{\mathbb{R}^N} u\varphi\,dx\,dt + \int_\tau^T\int_{\mathbb{R}^N} (f(x,u)-\sigma)\varphi\,dx\,dt = (u(\tau),\varphi(\tau)).$$

Doing the same in the Galerkin approximations yields

$$\int_\tau^T -(u_n,\varphi')\,dx\,dt + \int_\tau^T\int_{\mathbb{R}^N} \varepsilon\nabla \partial_t u_n \nabla\varphi\,dx\,dt + \int_\tau^T\int_{\mathbb{R}^N} \nabla u_n\nabla\varphi\,dx\,dt$$
$$+ \lambda\int_\tau^T\int_{\mathbb{R}^N} u_n\varphi\,dx\,dt + \int_\tau^T\int_{\mathbb{R}^N} (f(x,u_n)-\sigma)\varphi\,dx\,dt = (u_n(\tau),\varphi(\tau)).$$

Taking limits as $n \to \infty$, we conclude that

$$\int_\tau^T -(u,\varphi')\,dx\,dt + \int_\tau^T \int_{\mathbb{R}^N} \varepsilon \nabla u_t \nabla\varphi\,dx\,dt + \int_\tau^T \int_{\mathbb{R}^N} \nabla u \nabla\varphi\,dx\,dt$$
$$+ \lambda \int_\tau^T \int_{\mathbb{R}^N} u\varphi\,dx\,dt + \int_\tau^T \int_{\mathbb{R}^N} (f(x,u)-\sigma)\varphi\,dx\,dt = (u_\tau, \varphi(\tau))$$

since $u_n(\tau) \to u_\tau$. Thus, $u(\tau) = u_\tau$.

(ii) *Uniqueness and continuous dependence.* Let u_1 and u_2 be two solutions with the initial data $u_1(\tau)$ and $u_2(\tau)$, respectively. Denoting $w = u_1 - u_2$, we have

$$w_t - \varepsilon \Delta w_t - \Delta w + f(x,u_1) - f(x,u_2) + \lambda w = 0. \tag{3.20}$$

Taking the inner product of (3.20) in $L^2(\mathbb{R}^N)$ with w, then repeating the argument as in the proof of the fact that $u \in C([\tau,T]; H^1(\mathbb{R}^N))$ and using assumption (1.3), we see that

$$\frac{d}{dt}(\|w\|^2 + \varepsilon\|\nabla w\|^2) + 2\|\nabla w\|^2 + 2\lambda\|w\|^2 \le 2\ell\|w\|^2 \le 2\ell(\|w\|^2 + \varepsilon\|\nabla w\|^2).$$

By the Gronwall inequality, we obtain

$$\|w(t)\|^2 + \varepsilon\|\nabla w(t)\|^2 \le e^{2\ell(T-\tau)}(\|w(\tau)\|^2 + \varepsilon\|\nabla w(\tau)\|^2).$$

This proves the uniqueness (when $u_1(\tau) = u_2(\tau)$) of the weak solution and its continuous dependence on the initial data.

(iii) *The a priori estimate* (3.1). Multiplying (1.1) by $u(t)$ and integrating over $\mathbb{R}^N$, we get

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \varepsilon\|\nabla u(t)\|^2) + \|\nabla u(t)\|^2 + \lambda\|u(t)\|^2 + \int_{\mathbb{R}^N} f(x,u(t))u(t)\,dx$$
$$= \int_{\mathbb{R}^N} \sigma(t)u(t)\,dx.$$

Arguing as in derivation of (3.6), we have

$$\frac{d}{dt}(\|u_n\|^2 + \varepsilon\|\nabla u_n\|^2) + \delta(\|u_n\|^2 + \varepsilon\|\nabla u_n\|^2) \le \frac{1}{\eta}\|\sigma(t)\|^2 + 2\|\phi_1\|_{L^1(\mathbb{R}^N)},$$

where $\delta \le \min\{2, 2(\lambda-\mu)\}$. Hence, by the Gronwall inequality,

$$\|u(t)\|^2 + \varepsilon\|\nabla u(t)\|^2 \le e^{-\delta(t-\tau)}(\|u_\tau\|^2 + \varepsilon\|\nabla u_\tau\|^2) + 2\|\phi_1\|_{L^1(\mathbb{R}^N)} + \frac{1}{\eta}\int_\tau^t e^{-\delta(t-s)}\|\sigma(s)\|^2\,ds. \tag{3.21}$$

On the other hand, we have

$$
\begin{aligned}
(3.22)\qquad & \int_\tau^t e^{-\delta(t-s)}\|\sigma(s)\|^2\,ds \\
&\le \Big(\int_{t-1}^{t} e^{-\delta(t-s)}\|\sigma(s)\|^2\,ds + \int_{t-2}^{t-1} e^{-\delta(t-s)}\|\sigma(s)\|^2\,ds + \cdots\Big) \\
&\le (1+e^{-\delta}+e^{-2\delta}+\cdots)\|\sigma\|_b^2 \le \frac{1}{1-e^{-\delta}}\|g\|_b^2,
\end{aligned}
$$

where we have used the fact that $\|\sigma\|_b^2 \le \|g\|_b^2$ for all $\sigma \in \mathcal{H}_w(g)$. Combining (3.21) and (3.22), we get the desired estimate (3.1). ∎

Theorem 3.2 allows us to define a family of continuous processes

$$U_\sigma(t,\tau) : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N), \quad \sigma \in \mathcal{H}_w(g),$$

where $U_\sigma(t,\tau)u_\tau$ is the unique weak solution of (1.1) (with σ in place of g) at time t with initial datum u_τ at τ.

We now prove the weak continuity of the family $\{U_\sigma(t,\tau)\}$.

Lemma 3.3. *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to problem* (1.1) *is $(H^1(\mathbb{R}^N)\times\mathcal{H}_w(g), H^1(\mathbb{R}^N))$-weakly continuous, that is, for any $u_\tau^{(n)} \rightharpoonup u_\tau$ in $H^1(\mathbb{R}^N)$ and $\sigma_n \rightharpoonup \sigma_0$ in $\mathcal{H}_w(g)$, we have*

$$U_{\sigma_n}(t,\tau)u_\tau^{(n)} \rightharpoonup U_{\sigma_0}(t,\tau)u_\tau \quad \text{in } H^1(\mathbb{R}^N),\ t\ge\tau.$$

Proof. Denoting $u_n(t) = U_{\sigma_n}(t,\tau)u_\tau^{(n)}$, we easily see that all estimates for approximate solutions in Theorem 3.2 are still valid for $u_n(t)$. Thus, there is $w(t)$ such that

$$(3.23)\qquad u_n \rightharpoonup w \quad \text{weakly-star in } L^\infty(\tau,t;H^1(\mathbb{R}^N)),$$

and

$$\{u_n(s)\},\ \tau\le s\le t, \text{ is bounded in } H^1(\mathbb{R}^N).$$

Using arguments in Theorem 3.2, we see that

$$(3.24)\qquad \partial_t u_n \rightharpoonup w_t \quad \text{in } L^2(\tau,t;H^1(\mathbb{R}^N)),$$

and

$$(3.25)\qquad f(x,u_n) \rightharpoonup f(x,w) \quad \text{in } L^{\frac{2N}{N+2}}(\tau,t;L^{\frac{2N}{N+2}}(\mathbb{R}^N)).$$

Hence, by combining (3.23)–(3.25), we find that w solves the problem

$$w_t - \varepsilon\Delta w_t - \Delta w + f(x,w) + \lambda w = \sigma_0, \quad w(\tau) = u_\tau,$$

and therefore $w = U_{\sigma_0}(t,\tau)u_\tau$ due to the uniqueness of the weak solution. ∎

4. Existence of uniform attractors

4.1. Existence of an $(H^1(\mathbb{R}^N), L^2(\mathbb{R}^N))$-uniform attractor. First, we prove the existence of an $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$-uniform absorbing set for the family $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$.

PROPOSITION 4.1. *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ associated to problem* (1.1) *has an $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$-uniform absorbing set, which is independent of ε.*

Proof. Multiplying (1.1) by $u+u_t$ in $L^2(\mathbb{R}^N)$, and after some standard computations, we get

$$\frac{1}{2}\frac{d}{dt}\big((\lambda+1)\|u\|^2+(\varepsilon+1)\|\nabla u\|^2\big)+\|u_t\|^2+\varepsilon\|\nabla u_t\|^2+\|\nabla u\|^2 + \lambda\|u\|^2+(f(x,u),u+u_t)=(\sigma(t),u+u_t). \tag{4.1}$$

Applying the Cauchy inequality, we see that

$$(\sigma(t),u+u_t)\le\Big(\frac{1}{2\lambda}+\frac{1}{4}\Big)\|\sigma(t)\|^2+\frac{\lambda}{2}\|u\|^2+\|u_t\|^2. \tag{4.2}$$

By (1.5), for any $\eta_1>0$, there exists $C_1>0$ such that

$$\int_{\mathbb{R}^N} f(x,u)u\,dx-\kappa\int_{\mathbb{R}^N}F(x,u)+\eta_1\|u\|^2+C_1\ge 0,\quad \forall u\in H^1(\mathbb{R}^N). \tag{4.3}$$

Combining (4.2), (4.3) and the fact that

$$\int_{\mathbb{R}^N} f(x,u)(u+u_t)\,dx=\int_{\mathbb{R}^N} f(x,u)u\,dx+\frac{d}{dt}\int_{\mathbb{R}^N}F(x,u)\,dx,$$

we deduce from (4.1) that

$$\frac{d}{dt}\Big((\lambda+1)\|u\|^2+(\varepsilon+1)\|\nabla u\|^2+2\int_{\mathbb{R}^N}F(x,u)\,dx\Big) + 2\lambda\|u\|^2+2\|\nabla u\|^2+2\|u_t\|^2+2\varepsilon\|\nabla u_t\|^2+2\kappa\int_{\mathbb{R}^N}F(x,u) \le\Big(\frac{1}{\lambda}+\frac{1}{2}\Big)\|\sigma(t)\|^2+C_1. \tag{4.4}$$

Thus, from (4.4) we have

$$\frac{d}{dt}y(t)+\zeta y(t)\le\frac{\lambda+2}{2\lambda}\|\sigma(t)\|^2+C_1,$$

where

$$y(t)=(\lambda+1)\|u\|^2+(\varepsilon+1)\|\nabla u\|^2+2\int_{\mathbb{R}^N}F(x,u)\,dx, \tag{4.5}$$

and $\zeta = \min\{\frac{2\lambda}{\lambda+1}, \kappa\}$. Hence, by the Gronwall inequality, we obtain

$$\begin{aligned}(4.6)\quad y(t) &\le e^{-\zeta(t-\tau)}y(\tau) + \frac{\lambda+2}{2\lambda}e^{-\zeta t}\int_\tau^t e^{\zeta s}\|\sigma(s)\|^2\,ds + C_1\\ &\le e^{-\zeta(t-\tau)}\Big((\lambda+1)\|u_\tau\|^2 + (\varepsilon+1)\|\nabla u_\tau\|^2 + 2\int_{\mathbb{R}^N} F(x,u_\tau)\,dx\Big)\\ &\quad + \frac{\lambda+2}{2\lambda}\,\frac{1}{1-e^{-\zeta}}\|g\|_b^2 + C_1.\end{aligned}$$

On the other hand, using (1.4) we have

$$(4.7)\qquad \int_{\mathbb{R}^N} F(x,u)\,dx \le \int_{\mathbb{R}^N} (\phi_2(x)u + |u|^{\rho+1})\,dx.$$

By the Hölder inequality, and the fact that $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ continuously, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \phi_2(x)u\,dx &\le \Big(\int_{\mathbb{R}^N} |\phi_2(x)|^{\frac{2N}{N+2}}\,dx\Big)^{\frac{N+2}{2N}}\Big(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{2N}}\\ &\le \|\phi_2\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}\|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \le C\|\phi_2\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}\|u\|_{H^1(\mathbb{R}^N)},\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |u|^{\rho+1}\,dx = \|u\|^{\rho+1}_{L^{\rho+1}(\mathbb{R}^N)} \le C\|u\|^{\rho+1}_{H^1(\mathbb{R}^N)},$$

where we have used the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\rho+1}(\mathbb{R}^N)$ since $2 \le \rho+1 \le \frac{2N}{N-2}$. This embedding follows from the facts that $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, and the interpolation inequality between Lebesgue spaces. Thus, from (4.7), we have

$$(4.8)\qquad \int_{\mathbb{R}^N} F(x,u_\tau)\,dx \le C.$$

Combining (4.6) and (4.8), we have

$$(4.9)\quad y(t) \le C\Big(e^{-\zeta(t-\tau)}(\|u_\tau\|^2 + \|\nabla u_\tau\|^2 + 1) + 1 + \frac{\lambda+2}{2\lambda}\,\frac{1}{1-e^{-\zeta}}\|g\|_b^2\Big),$$

where C is independent of t, τ, ε. On the other hand, by (1.6), for any $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$(4.10)\qquad \int_{\mathbb{R}^N} F(x,u)\,dx + \alpha\|u\|^2 + C_\alpha \ge 0, \qquad \forall u \in H^1(\mathbb{R}^N).$$

Combining (4.5) and (4.10), we have

$$\begin{aligned}(4.11)\qquad y(t) &\ge (\lambda+1-2\alpha)\|u(t)\|^2 + (\varepsilon+1)\|\nabla u(t)\|^2 + 2C_\alpha\\ &\ge \beta(\|u(t)\|^2 + \|\nabla u(t)\|^2).\end{aligned}$$

From (4.9) and (4.11), we see that there exists $\rho_0 > 0$ such that

$$\|u(t)\|^2 + \|\nabla u(t)\|^2 \le \rho_0^2 \tag{4.12}$$

for all $t \ge T_1$, $u_\tau \in B$ and $\sigma \in \mathcal{H}_w(g)$. This completes the proof. ■

Remark 4.2. The $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$-uniform absorbing set obtained in Proposition 4.1 is also an $(H^1(\mathbb{R}^N), L^2(\mathbb{R}^N))$- and $(H^1(\mathbb{R}^N), L^{\frac{2N}{N-2}}(\mathbb{R}^N))$-uniform absorbing set. Thus, in order to prove the existence of uniform attractors for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$, it remains to check the uniform asymptotic compactness of $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ in the corresponding function spaces.

To prove the uniform asymptotic compactness of $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$, we assume that the external force g satisfies a stronger hypothesis:

(H2$'$) $g, \partial_t g \in L_b^2(\mathbb{R}; L^2(\mathbb{R}^N))$ and

$$\lim_{k\to\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{|x|\ge k} |g(s,x)|^2\, dx\, ds = 0. \tag{4.13}$$

Lemma 4.3. *Let assumptions* (H1) *and* (H2$'$) *hold. Then for any $\tau \in \mathbb{R}$ and any bounded subset $B \subset H^1(\mathbb{R}^N)$, there exist $\rho_1 > 0$ and $T_2 > \tau$ such that*

$$\|u_t(s)\|^2 + 2\varepsilon\|\nabla u_t(s)\|^2 \le \rho_1^2, \quad \forall u_\tau \in B,\ s \ge T_2,\ \text{and } \sigma \in \mathcal{H}_w(g). \tag{4.14}$$

Proof. Multiplying (1.1) by u_t and applying the Cauchy inequality, we get

$$\|u_t\|^2 + 2\varepsilon\|\nabla u_t\|^2 + \frac{d}{dt}\Big(\|\nabla u\|^2 + 2\int_{\mathbb{R}^N} F(x,u)\, dx + \lambda\|u\|^2\Big) \le \|\sigma(t)\|^2. \tag{4.15}$$

Integrating (4.15) from t to $t+1$, $t \ge T_1$, and using (4.12) and (4.9), we have

$$\begin{aligned} &\int_t^{t+1} (\|u_t(s)\|^2 + 2\varepsilon\|\nabla u_t(s)\|^2)\, ds \\ &\quad\le \lambda\|u(t)\|^2 + \|\nabla u(t)\|^2 + 2\int_{\mathbb{R}^N} F(x,u(t))\, dx + \|\sigma\|_b^2 \le C(\lambda, \phi_2, \|g\|_b^2). \end{aligned} \tag{4.16}$$

On the other hand, differentiating (1.1) with respect to t, then denoting $v = u_t$ and multiplying by v in $L^2(\mathbb{R}^N)$, we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|\nabla v\|^2 + \|\nabla v\|^2 + \int_{\mathbb{R}^N} f_u'(x,u)v^2\, dx + \lambda\|v\|^2 = \int_{\mathbb{R}^N} \partial_t\sigma(t) v\, dx.$$

Using the facts that $f_u'(x,u) \ge -\ell$ and $\int_{\mathbb{R}^N} \partial_t\sigma(t)v \le \frac{1}{2\lambda}\|\partial_t\sigma(t)\|^2 + \frac{\lambda}{2}\|v\|^2$, we obtain

$$\frac{d}{dt}(\|v\|^2+\varepsilon\|\nabla v\|^2)+2\|\nabla v\|^2+\lambda\|v\|^2\le 2\ell\|v\|^2+\frac{1}{\lambda}\|\partial_t\sigma(t)\|^2,$$

thus

$$\frac{d}{dt}(\|v\|^2+\varepsilon\|\nabla v\|^2)\le 2\ell(\|v\|^2+\varepsilon\|\nabla v\|^2)+\frac{1}{\lambda}\|\partial_t\sigma(t)\|^2. \tag{4.17}$$

From (4.16) and (4.17), using the uniform Gronwall inequality, we get

$$\|v(t)\|^2+\varepsilon\|\nabla v(t)\|^2\le\rho_1^2$$

for all $t\ge T+1$. This completes the proof. ∎

Lemma 4.4. *Let B be a bounded subset in $H^1(\mathbb{R}^N)$. Then for any $\eta>0$, there exist $T_\eta>0$ and $K_\eta>0$ such that*

$$\int\limits_{|x|\ge K_\eta}|U_\sigma(t,\tau)u_\tau|^2\,dx<\eta,\qquad \forall t\ge T_\eta,\ \forall u_\tau\in B,\ \forall\sigma\in\mathcal{H}_w(g).$$

Proof. Let $\theta:\mathbb{R}_+\to\mathbb{R}$ be a smooth function satisfying $\theta(s)=1$, $0\le s\le 1$; $0\le\theta(s)\le 1$, $1\le s\le 2$; and $\theta(s)=0$, $s\ge 2$. It is easy to see that $\theta'(s)\le C$ for all $s\in[0,\infty)$ and $\theta'(s)=0$ for $s\ge 2$. Multiplying (1.1) by $\theta(|x|^2/k^2)u$ and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}&\bigg(\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx+\varepsilon\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2\,dx\bigg)\\&+\lambda\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx+\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|\nabla u|^2\,dx+\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)f(x,u)u\,dx\\&+\int\limits_{\mathbb{R}^N}\frac{2x}{k^2}\theta'\Big(\frac{|x|^2}{k^2}\Big)u\nabla u\,dx+\varepsilon\int\limits_{\mathbb{R}^N}\frac{2x}{k^2}\theta'\Big(\frac{|x|^2}{k^2}\Big)u\nabla u_t\,dx\\&=\int\limits_{\mathbb{R}^N}\sigma(t)\theta\Big(\frac{|x|^2}{k^2}\Big)u\,dx.\end{aligned}\tag{4.18}$$

By (1.2), we have

$$\begin{aligned}\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)f(x,u)u&\ge-\mu\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2-\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)\phi_1(x)\\&\ge-\mu\int\limits_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2-\int\limits_{|x|\ge\sqrt2 k}\phi_1(x).\end{aligned}\tag{4.19}$$

Using $\theta'(s)=0$ for all $s>2$, we get

$$\bigg|\int\limits_{\mathbb{R}^N}\frac{2x}{k^2}\theta'\Big(\frac{|x|^2}{k^2}\Big)u\nabla u\bigg|\le C\int\limits_{|x|\le\sqrt2 k}\frac{2|x|}{k^2}|u|\,|\nabla u|\le\frac{C}{k}(\|u\|^2+\|\nabla u\|^2),\tag{4.20}$$

and similarly,

$$\left|\varepsilon \int_{\mathbb{R}^N} \frac{2x}{k^2}\theta'\left(\frac{|x|^2}{k^2}\right)u\nabla u_t\right| \le \frac{C\varepsilon}{k}(\|u\|^2+\varepsilon^2\|\nabla u_t\|^2). \tag{4.21}$$

By the Cauchy inequality,

$$\left|\int_{\mathbb{R}^N} \sigma(t)\theta\left(\frac{|x|^2}{k^2}\right)u\right| \le \frac{\lambda}{2}\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2+\frac{1}{2\lambda}\int_{|x|\ge\sqrt{2}k}|\sigma(t)|^2. \tag{4.22}$$

Combining (4.18)–(4.22), we deduce that

$$\begin{aligned}\frac{d}{dt}&\left(\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2+\varepsilon\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\right)\\ &\qquad+\delta\left(\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|u|^2+\varepsilon\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\right)\\ &\le\frac{1}{\lambda}\int_{|x|\ge\sqrt{2}k}|\sigma(t)|^2+2\int_{|x|\ge\sqrt{2}k}\phi_1(x)+\frac{C}{k}(\|u\|^2+\|\nabla u\|^2+\varepsilon^2\|\nabla u_t\|^2).\end{aligned} \tag{4.23}$$

Multiplying (4.23) by $e^{\delta t}$ and integrating from T^* to t, where $T^*=\max\{T_1,T_2\}$, we find that

$$\begin{aligned}&\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|u(t)|^2+\varepsilon\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|\nabla u(t)|^2\\ &\le e^{-\delta(t-T^*)}(\|u(T^*)\|^2+\varepsilon\|\nabla u(T^*)\|^2)+\frac{e^{-\delta t}}{\lambda}\int_{T^*}^t\int_{|x|\ge\sqrt{2}k}e^{\delta s}|\sigma(s)|^2\\ &\quad+2e^{-\delta t}\int_{T^*}^t\int_{|x|\ge\sqrt{2}k}e^{\delta s}\phi_1(x)+\frac{Ce^{-\delta t}}{k}\int_{T^*}^t e^{\delta s}(\|u\|^2+\|\nabla u\|^2+\varepsilon^2\|\nabla u_t\|^2).\end{aligned} \tag{4.24}$$

Since $\phi_1\in L^1(\mathbb{R}^N)$, we have

$$\limsup_{t\to\infty}\limsup_{k\to\infty} e^{-\delta t}\int_\tau^t\int_{|x|\ge k}e^{\delta s}\phi_1(x)=0. \tag{4.25}$$

Using the arguments in (3.22) and assumption (4.13), we get

$$\limsup_{t\to\infty}\limsup_{k\to\infty} e^{-\delta t}\int_\tau^t\int_{|x|\ge k}e^{\delta s}|\sigma(s)|^2=0. \tag{4.26}$$

Using (4.12) and (4.14), we see that

$$e^{-\delta(t-T^*)}(\|u(T^*)\|^2+\varepsilon\|\nabla u(T^*)\|^2)\le e^{-\delta(t-T^*)}\rho_0^2\to 0 \quad \text{as } t\to\infty, \tag{4.27}$$

and

$$\text{(4.28)}\quad \frac{Ce^{-\delta t}}{k}\int_{T^*}^{t} e^{\delta s}(\|u\|^2+\|\nabla u\|^2+\varepsilon^2\|\nabla u_t\|^2) \le \frac{Ce^{-\delta t}}{k}\int_{T^*}^{t} e^{\delta s}(\rho_0^2+\rho_1^2) \le \frac{C(\rho_0,\rho_1)}{k}\to 0 \quad \text{as } k\to\infty.$$

From (4.24)–(4.28), we can take T_η and $K_\eta>0$ large enough such that

$$\int_{|x|\ge K_\eta}(|u(t)|^2+\varepsilon|\nabla u(t)|^2)<\eta \quad \text{for all } t\ge T_\eta,\ u_\tau\in B \text{ and } \sigma\in\mathcal{H}_w(g).$$

The proof is complete. ∎

THEOREM 4.5. *Assume that f satisfies* (H1) *and g satisfies* (H2′). *Then the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ possesses an $(H^1(\mathbb{R}^N),L^2(\mathbb{R}^N))$-uniform attractor $\mathcal{A}_2$. Moreover,*

$$\mathcal{A}_2=\bigcup_{\sigma\in\mathcal{H}_w(g)}\mathcal{K}_\sigma(s),\qquad \forall s\in\mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ is the kernel section at time s of the process $U_\sigma(t,\tau)$.

Proof. By Proposition 4.1, $\{U_\sigma(t,\tau)\}$ has an $(H^1(\mathbb{R}^N),L^2(\mathbb{R}^N))$-uniform absorbing set. It remains to show that $\{U_\sigma(t,\tau)\}$ is $(H^1(\mathbb{R}^N),L^2(\mathbb{R}^N))$-uniform asymptotically compact. Fix $\tau\in\mathbb{R}$, let $\{x_n\}\subset H^1(\mathbb{R}^N)$ be a bounded sequence, $\{t_n\}$ be a sequence such that $\lim_{n\to\infty}t_n=\infty$, and $\{\sigma_n\}\subset\mathcal{H}_w(g)$; we have to show that $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is precompact in $L^2(\mathbb{R}^N)$.

We will prove that, for any $\eta>0$, there exists a finite covering by balls with radii η for $\{U_{\sigma_n}(t_n,\tau)x_n\}$. Since $t_n\to\infty$, we can choose N large enough such that $t_n\ge T_\eta$ for all $n\ge N$, where T_η is stated in Lemma 4.4. From Lemma 4.4, there exists $K_\eta>0$ satisfying

$$\text{(4.29)}\qquad \|U_{\sigma_n}(t_n,\tau)x_n\|^2_{L^2(B^c_{K_\eta})}<\eta/4,\qquad \forall n\ge N,$$

where $B^c_{K_\eta}=\{x\in\mathbb{R}^N \mid |x|>K_\eta\}$. On the other hand, from Proposition 4.1, the sequence $\{U_{\sigma_n}(t_n,\tau)x_n\}$ is bounded in $H^1(B_{k_\eta})$; taking into account that $H^1(B_{K_\eta})\hookrightarrow L^2(B_{K_\eta})$ compactly, we see that

(4.30) $\{U_{\sigma_n}(t_n,\tau)x_n\}$ has a finite covering by balls with radii less than $\eta/4$ in $L^2(B_{K_\eta})$.

Combining (4.29) and (4.30), there is a finite covering by balls with radii η for $\{U_{\sigma_n}(t_n,\tau)x_n\}$. Thus, we get the existence of an $(H^1(\mathbb{R}^N),L^2(\mathbb{R}^N))$-uniform attractor $\mathcal{A}_2$. The structure of $\mathcal{A}_2$ follows directly from Theorem 2.5. ∎

4.2. Existence of an $(H^1(\mathbb{R}^N), L^{\frac{2N}{N-2}}(\mathbb{R}^N))$-uniform attractor

THEOREM 4.6. *The family $\{U_\sigma(t,\tau)\}$ has an $(H^1(\mathbb{R}^N), L^{\frac{2N}{N-2}}(\mathbb{R}^N))$-uniform attractor.*

Proof. By [5, Corollary 3.12], Proposition 4.1 and Theorem 2.5, we only need to prove that, for any $\eta > 0$, and any bounded set B in $H^1(\mathbb{R}^N)$, there exist constants $M = M(\eta, B)$ and $T = T(\eta, B)$ such that

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau| \ge M)} |U_\sigma(t,\tau)u_\tau|^{\frac{2N}{N-2}}\,dx \le C\eta \quad \text{for all } u_\tau \in B,\ t \ge T, \tag{4.31}$$

where $\Omega(|U_\sigma(t,\tau)u_\tau| \ge M) = \{x \in \mathbb{R}^N \mid |U_\sigma(t,\tau)u_\tau(x)| \ge M\}$.

Multiplying (1.1) by $2u_M^+$, where $u_M^+ = u - M$ if $u \ge M$ and $u_M^+ = 0$ otherwise, and using (1.2) and the Cauchy inequality, we obtain

$$\frac{d}{dt}(\|u_M^+\|^2 + \varepsilon\|\nabla u_M^+\|^2) + 2\|\nabla u_M^+\|^2 + 2(\lambda - \mu - \eta_1)\|u_M^+\|^2 \le \frac{1}{\eta_1}\int_{\Omega(u \ge M)} |\sigma(t)|^2\,dx$$

for $\eta_1 > 0$ small enough. Hence for $0 < \delta < \min\{2, 2(\lambda - \mu)\}$, noting that $\varepsilon \in (0,1]$, we have

$$\frac{d}{dt}(\|u_M^+\|^2 + \varepsilon\|\nabla u_M^+\|^2) + \delta(\|\nabla u_M^+\|^2 + \varepsilon\|u_M^+\|^2) \le \frac{1}{\eta_1}\int_{\Omega(u \ge M)} |\sigma(t)|^2\,dx. \tag{4.32}$$

Thus, from (4.32), we find that

$$y(t) \le e^{-\delta(t-\tau)}y(\tau) + Ce^{-\delta t}\int_\tau^t e^{\delta s}\int_{\Omega(u \ge M)} |\sigma(s)|^2\,dx\,ds, \tag{4.33}$$

where $y(t) = \|u_M^+(t)\|^2 + \varepsilon\|\nabla u_M^+(t)\|^2$. Noting that $\int_\tau^t e^{-\delta(t-s)}\|\sigma(s)\|^2\,ds < \infty$, we have

$$Ce^{-\delta t}\int_\tau^t e^{\delta s}\int_{\Omega(u \ge M)} |\sigma(s)|^2\,dx\,ds \le C\eta$$

for M large enough. Combining this with (4.33), we obtain

$$\|u_M^+\|^2 + \|\nabla u_M^+\|^2 \le C\eta$$

for M large enough and $t \ge T$. Since $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ continuously, we also have

$$\|u_M^+\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \le C\eta. \tag{4.34}$$

Repeating the above arguments, just replacing u_M^+ by u_M^-, where $u_M^- = u + M$ if $u \le -M$ and $u_M^- = 0$ otherwise, we get

$$\|u_M^-\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \le C\eta. \tag{4.35}$$

Combining (4.34) and (4.35) yields (4.31), which ends the proof. ∎

4.3. Existence of an $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$-uniform attractor. We have the following result, which is very useful in verifying the uniform asymptotic compactness of the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$.

LEMMA 4.7 ([11, Theorem 2.2]). *Suppose a family $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ satisfies the uniform (X,Y)-Condition* (C), *that is, for any fixed $\tau\in\mathbb{R}$, $B\in\mathcal{B}(X)$, and any $\eta>0$, there exist $T\ge\tau$ and a finite-dimensional subspace Y_1 of Y such that:*

(i) *$P(\bigcup_{\sigma\in\Sigma}\bigcup_{t\ge T} U_\sigma(t,\tau)B)$ is bounded in Y_1,*
(ii) *$\|(\mathrm{Id}_Y - P)y\|_Y \le \eta$ for all $y\in\bigcup_{\sigma\in\Sigma}\bigcup_{t\ge T}U_\sigma(t,\tau)B$,*

where $P: Y\to Y_1$ is a bounded projector and Id_Y is the identity in Y. Then $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is (X,Y)-uniformly (*with respect to $\sigma\in\Sigma$*) *asymptotically compact.*

We prove the following lemma.

LEMMA 4.8. *Assume that $2\le q<\infty$ and that $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ has an $(H^1(\mathbb{R}^N), L^q(\mathbb{R}^N))$-uniform attractor. Then, for any $\eta>0$, any $\tau\in\mathbb{R}$ and any bounded subset $B\subset H^1(\mathbb{R}^N)$, there exist $T\ge\tau$ and $m_0\in\mathbb{N}$ such that*

$$\int_{\mathbb{R}^N} |(I-P_m)U_\sigma(t,\tau)u_\tau|^q \le \eta \quad \text{for any } t\ge T,\ u_\tau\in B,\ m\ge m_0,\ \sigma\in\Sigma,$$

where P_m is the projection of $L^q(\mathbb{R}^N)$ onto an m-dimensional subspace generated by the first m elements of a Schauder basis of $L^q(\mathbb{R}^N)$.

Proof. Let $\mathcal{A}$ be the $(H^1(\mathbb{R}^N), L^q(\mathbb{R}^N))$-uniform attractor of the family $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$. Then for any $\eta>0$, any $\tau\in\mathbb{R}$ and any bounded subset $B\subset H^1(\mathbb{R}^N)$, there exists T_0 such that

$$\bigcup_{t\ge T_0}\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset \mathcal{N}_{L^q}(\mathcal{A},\eta),$$

where $\mathcal{N}_{L^q}(\mathcal{A},\eta)$ is the η-neighborhood of $\mathcal{A}$ in $L^q(\mathbb{R}^N)$. Since $\mathcal{A}$ is compact in $L^q(\mathbb{R}^N)$, there exist $n\in\mathbb{N}$ and $v_i\in L^q(\mathbb{R}^N)$, $i=1,\dots,n$, such that

$$\bigcup_{t\ge T_0}\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset \bigcup_{i=1}^n \mathcal{N}_{L^q}(v_i,\eta).$$

For each v_i there is an m_i such that

$$\int_{\mathbb{R}^N} |(I - P_m)v_i|^q \leq \eta \quad \text{for all } m \geq m_i.$$

Take $m_0 = \max\{m_1, \ldots, m_n\}$ and denote $Q_{m_0} = I - P_{m_0}$. For any $t \geq T_0$, any $u_\tau \in D$ and any $\sigma \in \mathcal{H}_w(g)$, there exists some v_i such that

$$\begin{aligned}
\int_{\mathbb{R}^N} |Q_{m_0}U_\sigma(t,\tau)u_\tau|^q &= \int_{\mathbb{R}^N} |Q_{m_0}U_\sigma(t,\tau)u_\tau - Q_{m_0}v_i + Q_{m_0}v_i|^q \\
&\leq 2^q \int_{\mathbb{R}^N} |Q_{m_0}U_\sigma(t,\tau)u_\tau - Q_{m_0}v_i|^q + 2^q \int_{\mathbb{R}^N} |Q_{m_0}v_i|^q \\
&\leq 2^q C_q \int_{\mathbb{R}^N} |U_\sigma(t,\tau)u_\tau - v_i|^q + 2^q \int_{\mathbb{R}^N} |Q_{m_0}v_i|^q \\
&\leq 2^q(C_q + 1)\eta,
\end{aligned}$$

where C_q depends only on q. ■

We are now ready to prove the main result of this section.

Theorem 4.9. *Let conditions* (H1) *and* (H2′) *hold. Then the family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ *generated by* (1.1) *has an* $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$*-uniform attractor* $\mathcal{A}_{\mathcal{H}_w(g)}$*. Moreover,*

$$\mathcal{A}_{\mathcal{H}_w(g)} = \bigcup_{\sigma\in\mathcal{H}_w(g)} \mathcal{K}_\sigma(s), \qquad \forall s \in \mathbb{R}.$$

Proof. Since $H^1(\mathbb{R}^N)$ is separable, we can choose a set $\{w_k\}_{k=1}^\infty$ which forms an orthogonal basis in both $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$. Let $H_m = \operatorname{span}\{w_1, \ldots, w_m\}$, P_m be the canonical projector on H_m and I be the identity. Then for any $u \in H^1(\mathbb{R}^N)$, u has a unique decomposition $u = P_m u + (I - P_m)u = u_1 + u_2$. Let $\eta > 0$ be arbitrary. Taking u_2 as a test function in (1.1), we obtain

$$\begin{aligned}
\text{(4.36)} \qquad \frac{d}{dt}(\|u_2\|^2 + \varepsilon\|\nabla u_2\|^2) + 2\|\nabla u_2\|^2 + 2\int_{\mathbb{R}^N} f(x,u)u_2 + 2\lambda\|u_2\|^2 \\
= \int_{\mathbb{R}^N} \sigma(t)u_2.
\end{aligned}$$

Using the Hölder inequality in (4.36), we obtain

$$\begin{aligned}
\text{(4.37)} \qquad \frac{d}{dt}(\|u_2\|^2 + \varepsilon\|\nabla u_2\|^2) + 2\|\nabla u_2\|^2 + 2\lambda\|u_2\|^2 \\
\leq 2\|f(x,u)\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)}\|u_2\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} + 2\|\sigma(t)\|\,\|u_2\|.
\end{aligned}$$

By (3.15), $\|f(x,u)\|_{L^q(\mathbb{R}^N)}$ is bounded when t is large enough in view of Proposition 4.1. Since $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has $(H^1(\mathbb{R}^N), L^2(\mathbb{R}^N))$- and

$(H^1(\mathbb{R}^N), L^{\frac{2N}{2N-2}}(\mathbb{R}^N))$-uniform attractors, for any $\eta > 0$, by Lemma 4.8, we get m^* such that

$$\|u_2\| < \eta \quad \text{and} \quad \|u_2\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} < \eta, \qquad \text{for all } m \geq m^*. \tag{4.38}$$

From (4.37) and (4.38) we get

$$\frac{d}{dt}(\|u_2\|^2 + \varepsilon\|\nabla u_2\|^2) + \delta(\|u_2\|^2 + \varepsilon\|\nabla u_2\|^2) \leq C\eta + \eta\|\sigma\|.$$

By the Gronwall inequality, we obtain

$$\begin{aligned}
&\|u_2\|^2 + \varepsilon\|\nabla u_2\|^2 \\
&\quad \leq e^{-\delta(t-\tau)}(\|u(\tau)\|^2 + \varepsilon\|\nabla u(\tau)\|^2) + C\eta + \eta\int_\tau^t e^{-\delta(t-s)}\|\sigma(s)\|\,ds \\
&\quad \leq e^{-\delta(t-\tau)}(\|u(\tau)\|^2 + \varepsilon\|\nabla u(\tau)\|^2) + C\eta \\
&\qquad + \eta\Big(\int_\tau^t e^{-\delta(t-s)}ds\Big)^{1/2}\Big(\int_\tau^t e^{-\delta(t-s)}\|\sigma(s)\|^2\,ds\Big)^{1/2} \\
&\quad \leq e^{-\delta(t-\tau)}(\|u(\tau)\|^2 + \varepsilon\|\nabla u(\tau)\|^2) + C\eta + \eta\frac{1}{\sqrt{\delta(1-e^{-\delta})}}\|g\|_b,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and an argument as in (3.22). Thus, we can find $t_0 \geq \tau$ and $m_0 \in \mathbb{N}$ such that

$$\|u_2\|^2 + \varepsilon\|\nabla u_2\|^2 \leq C\eta$$

for any $t \geq t_0$, $u_\tau \in B$ and $m \geq m_0$. This shows that $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ satisfies condition (i) in Lemma 4.7. The condition (ii) is obviously satisfied since $\bigcup_{\sigma\in\mathcal{H}_w(g)}\bigcup_{t\geq t^*} U_\sigma(t,\tau)D$ is bounded and P_m is a bounded projector for any m. Then, by Lemma 4.7, we see that $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ is $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$-uniformly asymptotically compact. Thus, we obtain the existence of an $(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N))$-uniform attractor $\mathcal{A}_{\mathcal{H}_w(g)}$. Finally, the structure of the uniform attractor $\mathcal{A}_{\mathcal{H}_w(g)}$ follows from Theorem 2.5. ∎

5. The upper semicontinuity of uniform attractors at $\varepsilon = 0$. Hereafter, we denote by $\{U_\sigma^\varepsilon(t,\tau)\}$ the process associated to equation (1.1) with $-\varepsilon\Delta u_t$ term and the external force σ, and by $\mathcal{A}_\varepsilon$ the corresponding uniform attractor in $H^1(\mathbb{R}^N)$. In the case $\varepsilon = 0$, using arguments as in Section 2, one can prove the existence of a uniform attractor $\mathcal{A}_0$ in $H^1(\mathbb{R}^N)$ for the family of processes $U_0^\varepsilon(t,\tau)$ associated to the reaction-diffusion equation. In this section, we will prove that $\mathcal{A}_\varepsilon$ tends to $\mathcal{A}_0$ (in the sense of Hausdorff semi-distance) in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0^+$. To do this, we need the following lemma.

LEMMA 5.1. *For any sequences* $\{\varepsilon_n\}_{n\geq 1} \subset [0,1]$, $\{\sigma_n\} \subset \mathcal{H}_w(g)$, *and* $\{\phi_n\} \subset H^1(\mathbb{R}^N)$ *such that* $\varepsilon_n \to 0$, $\sigma_n \rightharpoonup \sigma$ *in* $\mathcal{H}_w(g)$ *and* $\phi_n \to \phi$ *in* $L^2(\mathbb{R}^N)$ *as* $n \to \infty$, *we have, for all* $t \geq \tau$,

$$\lim_{n\to\infty} \|U^{\varepsilon_n}_{\sigma_n}(t,\tau)\phi_n - U^0_\sigma(t,\tau)\phi\|_{H^1(\mathbb{R}^N)} = 0.$$

Proof. Denoting $u_n(t) = U^{\varepsilon_n}_{\sigma_n}(t,\tau)\phi_n$, $v(t) = U^0_\sigma(t,\tau)\phi$ and $w_n(t) = u_n(t) - v(t)$, we have

$$\partial_t w_n - \varepsilon_n \Delta\partial_t u_n - \Delta w_n + f(x,u_n) - f(x,v) + \lambda w_n = 0 \tag{5.1}$$

and $w_n(\tau) = \phi_n - \phi$. Multiplying (5.1) by $2w_n$, then integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}\frac{d}{dt}\|w_n(t)\|^2 + 2\varepsilon_n \int_{\mathbb{R}^N} \nabla\partial_t u_n \nabla w_n + 2\|\nabla w_n(t)\|^2 \\ + 2\int_{\mathbb{R}^N} (f(x,u_n) - f(x,v))w_n\,dx + 2\lambda\|w_n(t)\|^2 = 0.\end{aligned} \tag{5.2}$$

By (1.3) we get

$$2\int_{\mathbb{R}^N} (f(x,u_n) - f(x,v))w_n\,dx \geq -2\ell\|w_n(t)\|^2.$$

Thus,

$$\frac{d}{dt}\|w_n(t)\|^2 \leq 2\ell\|w_n(t)\|^2 + 2\varepsilon_n\|\nabla\partial_t u_n(t)\|\,\|\nabla w_n(t)\|.$$

By the Gronwall inequality, we have

$$\begin{aligned}\|w_n(t)\|^2 &\leq e^{2\ell(t-\tau)}\|w_n(\tau)\|^2 + 2\varepsilon_n\int_\tau^t e^{2\ell(t-\tau)}\|\nabla\partial_t u_n(s)\|\,\|\nabla w_n(s)\| \\ &\leq e^{2\ell(t-\tau)}\|\phi_n-\phi\|^2 \\ &\quad + 2\sqrt{\varepsilon_n}e^{2\ell(t-\tau)}\Big(\varepsilon_n\int_\tau^t \|\nabla\partial_t u_n(s)\|^2\,ds\Big)^{1/2}\Big(\int_\tau^t \|\nabla w_n(s)\|^2\,ds\Big)^{1/2}.\end{aligned}$$

By (3.11) and the fact that $\phi_n \to \phi$ in $L^2(\mathbb{R}^N)$, we deduce

$$\|w_n(t)\|^2 \to 0 \quad \text{as } n\to\infty. \tag{5.3}$$

On the other hand, from (5.2) we have

$$\|\nabla w_n(t)\|^2 \leq \|\partial_t w_n(t)\|\,\|w_n(t)\| + 2\varepsilon_n\|\nabla\partial_t u_n(t)\|\,\|\nabla w_n(t)\| + 2\ell\|w_n(t)\|^2.$$

Using (3.11) once again and (5.3), we find that

$$\|w_n(t)\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as } n\to\infty.$$

This completes the proof. ∎

LEMMA 5.2. *The set* $\mathfrak{A} = \overline{\bigcup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon}$ *is a compact set in* $L^2(\mathbb{R}^N)$.

Proof. We will prove that, for any $\eta > 0$, there exist finite balls with radii less than η that cover $\mathfrak{A}(t)$. By the invariance of $\mathcal{A}_\varepsilon$, for any $u \in \mathcal{A}_\varepsilon$ and $t > \tau$ there is a $v \in \mathcal{A}_\varepsilon$ satisfying $u = U_\varepsilon(t,\tau)v$. Then, using Lemma 4.4 for $U_\varepsilon(t,\tau)v$, we can get $K > 0$ independent of u such that

$$\|u\|_{L^2(B_K^c)} \le \eta/4, \quad \forall u \in \mathfrak{A}. \tag{5.4}$$

On the one hand, by the existence of a family of uniform absorbing sets in $H^1(\mathbb{R}^N)$, which is independent of ε, there is a constant $C > 0$ such that

$$\|u\|_{H^1(B_K)} < C, \quad \forall u \in \mathfrak{A}. \tag{5.5}$$

Since $H^1(B_K) \hookrightarrow L^2(B_K)$ compactly, the set $\mathfrak{A}$ is precompact in $L^2(B_K)$; hence there is a finite covering in $L^2(B_K)$ of balls with radii less than $\eta/4$.

On the other hand, from Proposition 4.1 and Theorem 2.5 we see that the set $\mathfrak{A} = \bigcup_{\varepsilon\in(0,1]} \mathcal{A}_\varepsilon$ is bounded in $H^1(\mathbb{R}^N)$. Combining this with (5.4) and (5.5), we conclude that $\mathfrak{A}$ is precompact in $L^2(\mathbb{R}^N)$. ∎

THEOREM 5.3. *The family of uniform attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ is upper semicontinuous in $H^1(\mathbb{R}^N)$ at $\varepsilon = 0$, that is,*

$$\lim_{\varepsilon\to 0} \operatorname{dist}_{H^1(\mathbb{R}^N)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

Proof. For contradiction, assume that

$$\operatorname{dist}_{H^1(\mathbb{R}^N)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \nrightarrow 0 \quad \text{as } \varepsilon \to 0^+.$$

Then there exists $\delta > 0$ such that

$$\limsup_{\varepsilon\to 0} \operatorname{dist}_{H^1(\mathbb{R}^N)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \ge \delta.$$

Since $\mathcal{A}_\varepsilon$ is compact for any $\varepsilon \in [0,1]$, we can choose a sequence ε_n with $\varepsilon_n \to 0$ as $n \to \infty$, and $\psi_n \in \mathcal{A}_{\varepsilon_n}$ satisfying

$$\operatorname{dist}_{H^1(\mathbb{R}^N)}(\psi_n, \mathcal{A}_0) \ge \delta \quad \text{for all } n \ge 1. \tag{5.6}$$

By Proposition 4.1 and Theorem 2.5 we see that the set $\mathfrak{A} = \bigcup_{\varepsilon\in(0,1]} \mathcal{A}_\varepsilon$ is bounded, and by the uniform attracting property of $\mathcal{A}_0$, we can choose t large enough such that

$$\operatorname{dist}_{H^1(\mathbb{R}^N)}(U_\sigma^0(t,0)\mathfrak{A}, \mathcal{A}_0) \le \delta/2 \quad \text{for all } \sigma \in \mathcal{H}_w(g). \tag{5.7}$$

From Theorem 4.5, we know that

$$\mathcal{A}_{\varepsilon_n} = \bigcup_{\sigma\in\mathcal{H}_w(g)} \mathcal{K}_\sigma^{\varepsilon_n}(t),$$

thus, since $\psi_n \in \mathcal{A}_{\varepsilon_n}$, there exists a $\sigma_n \in \mathcal{H}_w(g)$ such that $\psi_n \in \mathcal{K}_{\sigma_n}^{\varepsilon_n}(t)$. By definition of $\mathcal{K}_{\sigma_n}^{\varepsilon_n}$, we get a $\phi_n \in \mathcal{K}_{\sigma_n}^{\varepsilon_n}(0)$ satisfying $\psi_n = U_{\sigma_n}^{\varepsilon_n}(t,0)\phi_n$.

By Lemma 5.2, $\mathfrak{A}$ is precompact in $L^2(\mathbb{R}^N)$ and as $\{\phi_n\} \subset \bigcup_{n\ge 1} \mathcal{K}_{\sigma_n}^{\varepsilon_n}(0)$, we can assume (without loss of generality) that $\phi_n \to \phi$ in $L^2(\mathbb{R}^N)$ for

$\phi \in L^2(\mathbb{R}^N)$. Since $\mathcal{H}_w(g)$ is weakly compact, we obtain

$$\sigma_n \rightharpoonup \sigma_0 \quad \text{in } L^2(\tau, t; L^2(\mathbb{R}^N)).$$

Now applying Lemma 5.1, we deduce that

$$\|\psi_n - U^0_{\sigma_0}(t,0)\phi\|_{H^1(\mathbb{R}^N)} = \|U^{\varepsilon_n}_{\sigma_n}(t,0)\phi_n - U^0_{\sigma_0}(t,0)\phi\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty,$$

which contradicts (5.6) and (5.7). This completes the proof. ∎

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Cung The Anh
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy, Cau Giay
Hanoi, Vietnam
E-mail: anhctmath@hnue.edu.vn

Nguyen Duong Toan
Department of Mathematics
Haiphong University
171 Phan Dang Luu, Kien An
Haiphong, Vietnam
E-mail: ngduongtoanhp@gmail.com

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