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A predator-prey model with state dependent impulsive effects

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Abstract. We investigate a Lotka–Volterra predator-prey model with state dependent impulsive effects, in which the control strategies by releasing natural enemies and spraying pesticide at different thresholds are considered. We present some sufficient conditions to guarantee the existence and asymptotical stability of semi-trivial periodic solutions and positive periodic solutions.

1. Introduction. Many systems in physics, chemistry, biology, and information science have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes. Those systems are often modeled by impulsive differential equations. For Lotka–Volterra type systems, the effect of pest control in population dynamics has been studied extensively (see, e.g., [JL, NP, TS] and the references therein).

The following system is investigated in [NP]:

$$(1.1)\qquad \begin{cases} \left.\begin{array}{l} \frac{dx(t)}{dt} = x(t)[b_1 - a_{11}x(t) - a_{12}y(t)] \\ \frac{dy(t)}{dt} = y(t)[-b_2 + a_{21}x(t)] \end{array}\right\} & x \neq h_1, h_2, \\ \left.\begin{array}{l} \triangle x = 0 \\ \triangle y = y(t^+) - y(t) = \alpha \end{array}\right\} & x = h_1, \\ \left.\begin{array}{l} \triangle x = x(t^+) - x(t) = -px(t) \\ \triangle y = y(t^+) - y(t) = -qy(t) \end{array}\right\} & x = h_2, \end{cases}$$

where $x(t)$ and $y(t)$ represent the population densities at time t; b_1, b_2, a_{12} and a_{21} are positive constants, $a_{11}, \alpha \geq 0$, $p, q \in (0,1)$ and $h_2 > h_1 > (1-p)h_2 > 0$. When the amount $x(t)$ of prey reaches the threshold h_1 at time t_{h_1}, we release natural enemies (the predator) and the amount $y(t)$ of predator abruptly turns to $y(t_{h_1}) + \alpha$. Further, when the amount of prey

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reaches the threshold h_2 at time t_{h_2}, we spray pesticide and the amounts of prey and predator abruptly turn to $(1-p)x(t_{h_2})$ and $(1-q)y(t_{h_2})$, respectively. In [NP], the authors present several sufficient conditions for the existence and stability of a semi-trivial solution and a positive periodic solution.

Tian et al. [TS] deny most results of [NP] by an example (see [TS, Example 1]). In fact, when the prey $x(t)$ reaches the threshold h_1, the predator $y(t)$ jumps to $y(t)+\alpha$ and $x(t)$ remains unchanged. Since the amount of prey is still h_1, the idea in [TS, Example 1] is that the impulsive effect works instantaneously, thus $x(t)$ remains equal to h_1 and $y(t)$ turns to $y(t)+2\alpha$. Inductively, this process leads to $y(t)+k\alpha \to \infty$ as $k \to \infty$.

However, we think the idea in [TS, Example 1] is inappropriate. First, practically, it is impossible to apply an infinite number of impulsive effects at the same time. Second, in the mathematical theory of impulsive dynamical systems, after an impulsive action occurs, the trajectory of the impulsive system goes according to the original system without impulsive effects (see [K1] and [LB, Sec. 4.7], or the next section). So, Theorems 2–5 in [NP] are valid. However, in [NP, Theorems 3–5] the authors give some complicated conditions to guarantee the existence of a positive periodic orbit, whereafter they present a still more complicated condition relating to the unknown periodic orbit to get its asymptotic stability. Clearly, such results are hard to use. On the other hand, our conclusions in this paper just depend on the given parameters and are easy to test. Finally, indeed there is a mistake in [NP, Theorem 1] (see Section 3).

Our goal in this paper is to use some easy analysis and simple computations to get strong results. The paper is organized as follows. In the next section, we present some basic definitions and notations. In Section 3, first we show that Theorem 1 in [NP] is not correct; next we state and prove new criteria for the existence and asymptotic stability of a semi-trivial periodic solution of system (1.1). Sufficient conditions for the existence and stability of positive periodic solutions of (1.1) are established in Section 4.

2. Preliminaries. For convenience, we first fix some terminology. Consider the following autonomous system that models the interaction of prey and predator:

$$\dot{x} = x(b_1 - a_{11}x - a_{12}y), \quad \dot{y} = y(-b_2 + a_{21}x). \tag{2.1}$$

In the qualitative analysis of system (2.1), the isoclines $b_1 - a_{11}x - a_{12}y = 0$ and $-b_2 + a_{21}x = 0$ play important roles. Throughout this paper, we assume that (2.1) has a unique positive equilibrium, i.e., the condition $b_1a_{21} > b_2a_{11}$ holds. Then it is easy to obtain the phase portrait of (2.1) in Figure 1. Clearly, $\Omega = \{(x,y) \mid x, y \geq 0\}$ is an invariant region of (2.1). We will consider (2.1) only in the region Ω.

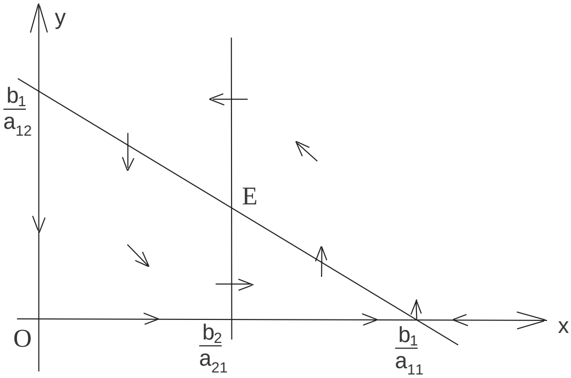

Fig. 1. The phase portrait of system (2.1)

It is easy to see that the solutions of (2.1) define a dynamical system in Ω, i.e., a continuous function $\pi : \Omega \times \mathbb{R} \to \Omega$ satisfying:

(1) $\pi(P, 0) = P$ for each $P \in \Omega$,
(2) $\pi(\pi(P,t), s) = \pi(P, t+s)$ for each $P \in \Omega$ and $t, s \in \mathbb{R}$.

For brevity, we write $P \cdot t = \pi(P,t)$, and also let $S \cdot J = \{P \cdot t \mid P \in S,\ t \in J\}$ for $S \subset \Omega$ and $J \subset \mathbb{R}$. If either S or J is a singleton, i.e., $S = \{P\}$ or $J = \{t\}$, then we simply write $P \cdot J$ and $S \cdot t$ for $\{P\} \cdot J$ and $S \cdot \{t\}$, respectively. For any $P \in \Omega$, the function $\pi_P : \mathbb{R} \to \Omega$ defined by $\pi_P(t) = \pi(P,t)$ is clearly continuous, and we call π_P the *trajectory* (or *motion*) through P. The set $P \cdot \mathbb{R}$ is said to be the *orbit* of P, and is sometimes denoted by $\gamma(P)$. Similarly, denote $\gamma^+(P) = P \cdot \mathbb{R}^+$, the *positive orbit* of P. Replacing $\mathbb{R}$ by $\mathbb{R}^+$ in the definition of a dynamical system, we get the definition of a semi-dynamical system. For the elementary properties of dynamical systems and semi-dynamical systems, we refer to [BH, BS, NS].

Let $\Sigma = \{(x,y) \mid x = (1-p)h_1 \text{ and } y \geq 0\}$, $\Sigma^1 = \{(x,y) \mid x = h_1 \text{ and } y \geq 0\}$ and $\Sigma^2 = \{(x,y) \mid x = h_2 \text{ and } y \geq 0\}$. Denote $M = \Sigma^1 \cup \Sigma^2$, which is the *impulsive set* of (1.1). We define the *impulsive function* $I : M \to \Omega$ to be the following continuous map:

$$I(x,y) = \begin{cases} (x, y+\alpha) & \text{if } (x,y) \in \Sigma^1, \\ ((1-p)x, (1-q)y) & \text{if } (x,y) \in \Sigma^2. \end{cases}$$

If $P = (x,y) \in M$, we shall denote $I(P)$ by P^+ and say that P jumps to P^+. For each $P \in \Omega$, we set $M^+(P) = (P \cdot \mathbb{R}^+ \cap M) \setminus \{P\}$. Obviously, we can define a continuous map $\phi : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\phi(P) = \begin{cases} s & \text{if } P \cdot s \in M \text{ and } P \cdot t \notin M \text{ for } t \in (0,s), \\ +\infty & \text{if } M^+(P) = \emptyset. \end{cases}$$

For the continuity of ϕ, we refer to [C].

Now, following Kaul [K2], we define an impulsive semi-dynamical system $\tilde{\pi}(P,t)$ by specifying the positive orbit of each point in Ω. The impulsive trajectory of $P \in \Omega$ is an Ω-valued function $\tilde{\pi}_P$ defined on $\mathbb{R}^+$. If $M^+(P) = \emptyset$, then we set $\tilde{\pi}_P(t) = P \cdot t$ for all $t \in \mathbb{R}^+$. If $M^+(P) \neq \emptyset$, it is easy to see that there is a smallest positive number t_0 such that $P \cdot t_0 = P_1 \in M$ and $P \cdot t \notin M$ for $0 < t < t_0$. Thus, we define $\tilde{\pi}_P$ on $[0, t_0]$ by

$$\tilde{\pi}_P(t) = \begin{cases} P \cdot t, & 0 \le t < t_0, \\ P_1^+, & t = t_0, \end{cases}$$

where $P_1^+ = I(P_1)$ and $\phi(P) = t_0$.

Since $t_0 < +\infty$, we continue the process by starting with P_1^+. Similarly, if $M^+(P_1^+) = \emptyset$, i.e., $\phi(P_1^+) = +\infty$, we define $\tilde{\pi}_P(t) = P_1^+ \cdot (t - t_0)$ for $t_0 < t < +\infty$. Otherwise, let $\phi(P_1^+) = t_1$ and $P_1^+ \cdot t_1 = P_2 \in M$, then we define $\tilde{\pi}_P(t)$ on $[t_0, t_0 + t_1]$ by

$$\tilde{\pi}_P(t) = \begin{cases} P_1^+ \cdot (t - t_0), & t_0 \le t < t_0 + t_1, \\ P_2^+, & t = t_1, \end{cases}$$

where $P_2^+ = I(P_2)$.

Thus, continuing inductively, the process above either ends after a finite number of steps, when $M^+(P_n^+) = \emptyset$ for some n, or it continues indefinitely, if $M^+(P_n^+) \neq \emptyset$ for $n = 1, 2, \dots$, and $\tilde{\pi}_P$ is well defined on $\mathbb{R}^+$. We call $\{t^n = \sum_{i=0}^{n} t_i \mid n = 0, 1, 2, \dots\}$ the *impulsive times* of $\tilde{\pi}_P$. Obviously, this gives rise to either a finite or an infinite number of jumps at points $\{P_n\}$ for the trajectory $\tilde{\pi}_P$. Having the trajectory $\tilde{\pi}_P$ for every point P in Ω, we let $\tilde{\pi}(P,t) = \tilde{\pi}_P(t)$ for $P \in \Omega$ and $t \in \mathbb{R}^+$, and then we get a discontinuous system with the following properties:

(i) $\tilde{\pi}(P, 0) = P$ for $P \in \Omega$,
(ii) $\tilde{\pi}(\tilde{\pi}(P,t), s) = \tilde{\pi}(P, t+s)$ for $P \in \Omega$ and t, $s \in \mathbb{R}^+$.

We call $\tilde{\pi}(P,t)$, with $\tilde{\pi}$ as defined above, the *impulsive semi-dynamical system* associated with (1.1). Also, we denote $P * t = \tilde{\pi}(P,t)$ for brevity. Then (ii) reads $(P*t)*s = P*(t+s)$. For the theory of impulsive semi-dynamical systems, we refer to [A, HC, K1, K2, LB].

Throughout the paper, for a point P in Ω, let $B(P,\delta) = \{Q \in \Omega \mid d(P,Q) < \delta\}$ be the open disk in Ω with center P and radius $\delta > 0$, where d is the ordinary metric on $\mathbb{R}^2$. In addition, for $S \subset \Omega$, the r-neighborhood of S in Ω is denoted by $N(S,r) = \{P \in \Omega \mid d(P,S) < r\}$ for $r > 0$, where $d(P,S) = \inf\{d(P,Q) \mid Q \in S\}$. Here, with no confusion, we also use d for the distance between a point and a set. Now, we recall several definitions.

Definition 2.1 ([SB]). Let $P_0 \in \Omega$. The positive orbit $P_0 * \mathbb{R}^+$ is said to be *orbitally stable* if, given an $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for any $P \in B(P_0, \delta)$, we have $P * \mathbb{R}^+ \subset N(P_0 * \mathbb{R}^+, \epsilon)$.

DEFINITION 2.2 ([SB]). Let $P_0 \in \Omega$. The positive orbit $P_0 * \mathbb{R}^+$ is said to be *asymptotically orbitally stable* if it is orbitally stable and there exists an $\eta > 0$ such that if $P \in B(P_0, \eta)$, then $\lim_{t\to\infty} d(P * t, P_0 * \mathbb{R}^+) = 0$.

DEFINITION 2.3 ([K1]). An orbit $P_0 * \mathbb{R}^+$ for $P_0 \in \Omega$ is *periodic* if there exists a $t > 0$ such that $P_0 = P_0 * t$; then $\tilde{\pi}_{P_0}$ is called a *periodic trajectory* with period t. In particular, if a periodic orbit $P_0 * \mathbb{R}^+$ lies on the boundary $\partial\Omega$ of Ω, i.e., $P_0 * \mathbb{R}^+$ lies on the positive x-axis or y-axis, then we call it a *semi-trivial periodic orbit*.

For a periodic orbit, the concept of *order* has been defined in [K1]. However, in this paper we only deal with 1-order periodic orbits (see [NP]), so we just call it a periodic orbit with period t.

Finally, we end this section by recalling the Lambert W function (see [CG]), defined to be a multivalued inverse of the function $f : z \mapsto ze^z$ satisfying $W(z)\exp(W(z)) = z$. For $z \geq -1/e$, the Lambert function $W(z)$ has two branches; the branch satisfying $-1 \leq W(z)$ is denoted by $W_0(x)$, and the branch satisfying $W(x) \leq -1$ by $W_{-1}(x)$.

3. Semi-trivial periodic orbits. In this section, we consider the existence and stability of semi-trivial periodic orbits. By definition, the existence of such an orbit for (1.1) implies $\alpha = 0$, since otherwise, at the point $(h_1, 0)$ the orbit jumps to a point in the interior of Ω. Thus, in this section we always assume $\alpha = 0$.

Clearly, in the x-axis, $y(t) \equiv 0$ holds for $t \in [0, +\infty)$, and we have the subsystem

$$\begin{cases} \frac{dx(t)}{dt} = x(t)[b_1 - a_{11}x(t)], & x \neq h_2, \\ \triangle x = x(t^+) - x(t) = -px(t), & x = h_2. \end{cases} \tag{3.1}$$

Note that there exists an equilibrium on the positive x-axis. Without the impulsive effect, each orbit on the positive x-axis tends to the equilibrium $(b_1/a_{11}, 0)$. Let

$$\lambda = (1-q)(1-p)^{b_2/b_1}\left[\frac{b_1 - (1-p)a_{11}h_2}{b_1 - a_{11}h_2}\right]^{a_{21}/a_{11} - b_2/b_1}. \tag{3.2}$$

In [NP, Theorem 1], the authors assert that if $0 < \lambda < 1$, then system (1.1) with $\alpha = 0$ has a semi-trivial periodic orbit. However, if $h_2 > b_1/a_{11}$, we can choose a sufficiently small p and a q close to 1 such that both $b_1 - (1-p)a_{11}h_2 < 0$ and $0 < \lambda < 1$; then (1.1) with $\alpha = 0$ has no semi-trivial periodic orbits. In fact, for $h_2 > b_1/a_{11}$, the orbit through $((1-p)h_2, 0)$ does not reach the section Σ_2, so the system has no semi-trivial periodic orbits. Hence, Theorem 1 in [NP] is not true.

In the following, we only deal with the cases $h_2 \leq b_2/a_{21}$ and $b_2/a_{21} < h_2 < b_1/a_{11}$. From the phrase portrait of (1.1) with $\alpha = 0$ (see Figure 2), it is easy to deduce the following result.

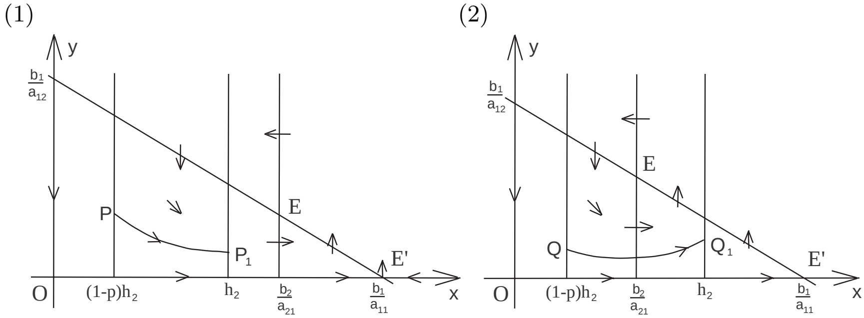

Fig. 2. The existence and stability of semi-trivial periodic orbits: (1) the case $h_2 \leq b_2/a_{21}$; (2) the case $b_2/a_{21} < h_2 < b_1/a_{11}$

Theorem 3.1. *If* $\alpha = 0$ *and* $h_2 \leq b_2/a_{21}$*, then system* (1.1) *has an asymptotically orbitally stable semi-trivial periodic orbit.*

Proof. Let $P_0 = ((1-p)h_2, 0)$. Clearly, $\tilde{\pi}_{P_0}$ is a semi-trivial periodic trajectory, and its orbit is $P_0 * \mathbb{R}^+ = \{(x, 0) \mid (1-p)h_2 \leq x < h_2\}$. For a point $P = ((1-p)h_2, y_0) \in \Sigma$ close to P_0, its trajectory meets Σ^2 at $P_1 = (h_2, y_1)$ with $y_1 < y_0$ (see Figure 2(1)). Now, P_1 jumps to $P_1^+ = ((1-p)h_2, y_1^+)$, where $y_1^+ = (1-q)y_1$. Then, the trajectory goes ahead and reaches Σ^2 again at the point $P_2 = (h_2, y_2)$ with $y_2 < y_1^+$. By induction, there exist two sequences of points $\{P_n = (h_2, y_n)\} \subset \Sigma^2$ and $\{P_n^+ = ((1-p)h_2, y_n^+)\} \subset \Sigma$ $(n = 1, 2, \dots)$ satisfying $y_{n+1} < y_n^+$ and $y_n^+ = (1-q)y_n$. Thus, $y_{n+1} < (1-q)^n y_1$, and so $y_n \to 0$ and $y_n^+ \to 0$ as $n \to \infty$. Hence, $P_n^+ \to P_0$ as $n \to \infty$. By the continuous dependence on the initial conditions (see [NS, p. 327]), we have $d(P * t, P_0 * \mathbb{R}^+) \to 0$ as $t \to +\infty$. Clearly, for each point $Q \in \Omega$ close to P_0, from the phase portrait (see Figure 2(1)) it is easy to see that the trajectory $\tilde{\pi}_Q$ has a similar dynamical behavior to $\tilde{\pi}_P$. Thus, the orbit $P_0 * \mathbb{R}^+$ is asymptotically orbitally stable. ■

Theorem 3.2. *If* $\alpha = 0$, $b_2/a_{21} < h_2 < b_1/a_{11}$ *and*

$$0 < \lambda = (1-q)(1-p)^{b_2/b_1} \left[\frac{b_1 - (1-p)a_{11}h_2}{b_1 - a_{11}h_2} \right]^{a_{21}/a_{11} - b_2/b_1} < 1,$$

then system (1.1) *has an asymptotically orbitally stable semi-trivial periodic orbit.*

Proof. Let $P_0 = ((1-p)h_2, 0)$. Obviously, $\tilde{\pi}_{P_0}$ is a semi-trivial periodic trajectory. Choose a point $Q = ((1-p)h_2, n_0) \in \Sigma$ close to P_0, where $n_0 > 0$ is sufficiently small. Let $Q_1 = Q \cdot \phi(Q) = (h_2, n)$. By the continuous dependence on initial conditions, $n = n(n_0)$ is also small. Thus, by a simple estimate of the integral

$$\int\limits_{(1-p)h_2}^{h_2} \frac{-b_2 + a_{21}x}{x(b_1 - a_{11}x - a_{12}y)}\,dx,$$

where $y = y(x, n_0)$ is small and $\{(x, y(x)) \mid (1-p)h_2 \le x \le h_2\}$ is an orbit segment of system (2.1) through the point Q, we have

$$n = n_0(1-p)^{b_2/b_1}\left[\frac{b_1 - (1-p)a_{11}h_2}{b_1 - a_{11}h_2}\right]^{a_{21}/a_{11} - b_2/b_1} + o(n_0). \tag{3.3}$$

This can also be obtained immediately from formula (2.13) in [Y, p. 29]. Now, let $Q * \phi(Q) = Q_1^+ = ((1-p)h_2, n_1)$, where $n_1 = n(1-q)$. Hence, if $0 < \lambda < 1$, it follows that $n_1 < n_0$ for any small n_0, i.e., Q_1^+ lies below Q in Σ. Then, the trajectory $\tilde{\pi}_Q$ goes ahead and meets Σ^2 at a point Q_2 below Q_1. Again, Q_2 jumps to a point Q_2^+ in Σ, where Q_2^+ lies below Q_1^+. By induction, the sequence $\{Q_k^+\}$ goes down in Σ as $k \to \infty$. If $Q_k^+ \to P_0$, it is easy to see that $P_0 * \mathbb{R}^+$ is asymptotically orbitally stable, and we are done. Otherwise, if $Q_k^+ \to Q' \in \Sigma$, it follows from [K1, Theorem 2] that $\tilde{\pi}_{Q'}$ is a periodic trajectory. However, by the argument above, $Q' * \phi(Q')$ lies below Q' in Σ, which means that $\tilde{\pi}_{Q'}$ is not periodic. This contradiction completes the proof. ∎

4. Positive periodic orbits. In this section, we consider the existence and stability of positive periodic orbits of (1.1) with $a_{11} = 0$. In this case the function

$$H(x, y) = a_{21}x + a_{12}y - b_2 \ln x - b_1 \ln y \tag{4.1}$$

is constant on each orbit of (2.1) in the interior of Ω (see [HS, p. 261]). In the following, we deal separately with the cases $h_2 \le b_2/a_{21}$ and $h_1 \le b_2/a_{21} < h_2$.

4.1. The case $h_2 \le b_2/a_{21}$. Let $U = \{(x, y) \mid 0 < x < b_2/a_{21},\ 0 < y < b_1/a_{12}\}$ in Ω. Clearly, by (2.1) we have $\dot{x} > 0$ and $\dot{y} < 0$ in U. It is easy to obtain the phase portrait of system (1.1) with $a_{11} = 0$ (see Figure 3). Now, we define the Poincaré map in Σ as follows. Let $A = ((1-p)h_2, b_1/a_{12})$ and $A_1 = A \cdot \phi(A) = (h_1, y_b)$, where y_b is defined by $H(h_1, y_b) = H(A)$, i.e.,

$$y_b = -\frac{b_1}{a_{12}} W_0\left(-\exp\left\{\frac{1}{b_1}[H(A) - H(h_1, b_1) + b_1(a_{12} - \ln a_{12})]\right\}\right). \tag{4.2}$$

Assume that $y_b + \alpha \le b_1/a_{12}$, which means that $C = A * \phi(A)$ lies on Σ^1 and below the line $y = b_1/a_{12}$. Thus, $C_1 = C \cdot \phi(C)$ lies on Σ^2 and also below the line $y = b_1/a_{12}$ (see Figure 3). Let $D = ((1-p)h_2, 0)$. On the segment $\overline{AD} = \{(x, y) \mid x = (1-p)h_2 \text{ and } 0 \le y \le b_1/a_{12}\}$ in Σ, we define

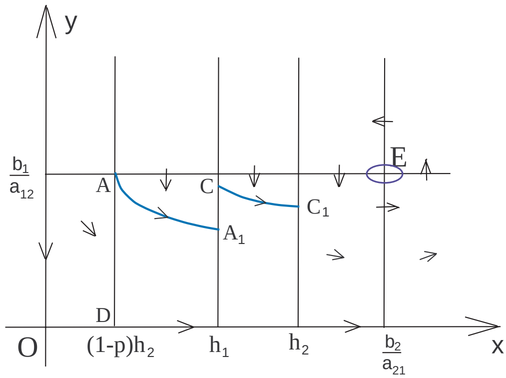

Fig. 3. The case $h_2 \le b_2/a_{21}$

the Poincaré map $F : \overline{AD} \to \overline{AD}$ by $F(P) = (P * \phi(P)) * \phi(P * \phi(P))$ for each point $P \in \overline{AD}$. It is easy to verify that F is a continuous map.

LEMMA 4.1. *Assume that* $h_2 \le b_2/a_{21}$, $y_b + \alpha \le b_1/a_{12}$ *and* $\alpha(1-q) \le 1$. *Then the map* $F : \overline{AD} \to \overline{AD}$ *defined as above is contractive, i.e., for any different points* P *and* Q *in* $\overline{AD}$, *we have* $d(F(P), F(Q)) < d(P, Q)$.

Proof. Let P and Q be different points in $\overline{AD}$, and suppose their trajectories meet Σ^1 at P_1 and Q_1 respectively. Assume that the orbit segments PP_1 and QQ_1 are the graphs of two functions $y = f(x)$ and $y = g(x)$ for $(1-p)h_2 \le x \le h_1$, respectively. Clearly,

$$\frac{df(x)}{dx} = \frac{f(x)(-b_2 + a_{21}x)}{x(b_1 - a_{12}f(x))} \quad \text{and} \quad \frac{dg(x)}{dx} = \frac{g(x)(-b_2 + a_{21}x)}{x(b_1 - a_{12}g(x))}. \tag{4.3}$$

If $f((1-p)h_2) > g((1-p)h_2)$, then $f(x) > g(x)$ for $(1-p)h_2 \le x \le h_1$. Let $H(x) = f(x) - g(x)$. By (4.3) we have

$$\frac{dH(x)}{dx} = \frac{b_1(a_{21}x - b_2)H(x)}{x(b_1 - a_{12}f(x))(b_1 - a_{12}g(x))}. \tag{4.4}$$

Since the orbit segments PP_1 and QQ_1 are below the line $y = b_1/a_{12}$, it means that $g(x) < f(x) < b_1/a_{12}$. From $h_2 \le b_2/a_{21}$, it follows that $dH(x)/dx < 0$ for $(1-p)h_2 < x < h_1$. Thus, $d(P_1, Q_1) < d(P, Q)$.

Now, assume that the trajectories of P_1^+ and Q_1^+ meet Σ^2 at P_2 and Q_2, respectively. By a similar argument, $d(P_2, Q_2) < d(P_1^+, Q_1^+)$. Note that $F(P) = P_2^+$ and $F(Q) = Q_2^+$. So,

$$\begin{aligned} d(F(P), F(Q)) &= (1-q)d(P_2, Q_2) < (1-q)d(P_1^+, Q_1^+) \\ &= \alpha(1-q)d(P_1, Q_1) < d(P, Q). \ \blacksquare \end{aligned}$$

THEOREM 4.2. *If* $h_2 \le b_2/a_{21}$ *and* $y_b + \alpha \le b_1/a_{12}$, *then system* (1.1) *with* $a_{11} = 0$ *has a positive periodic orbit in* U. *Further, if* $\alpha(1-q) \le 1$, *then system* (1.1) *with* $a_{11} = 0$ *has a unique periodic orbit in* U *that is asymptotically orbitally stable.*

Proof. For $h_2 \le b_2/a_{21}$ and $y_b + \alpha \le b_1/a_{12}$, we can define the continuous Poincaré map $F : \overline{AD} \to \overline{AD}$ as above. Since $\overline{AD}$ is homeomorphic to the interval $[0,1]$, it has the fixed point property. It follows that there exists a fixed point $P_0 \in \overline{AD}$ of F, thus $\tilde{\pi}_{P_0}$ is a positive periodic orbit in U of (1.1) with $a_{11} = 0$. Further, if $\alpha(1-q) \le 1$, then Lemma 4.1 states that F is contractive, so it has a unique fixed P_0. Clearly, for each $P \in \overline{AD}$ we have $F^n(P) \to P_0$ as $n \to \infty$. Hence, $\tilde{\pi}_{P_0}$ is asymptotically orbitally stable. ∎

4.2. The case $h_1 \le b_2/a_{21} < h_2$. Note that in the interior of Ω, all orbits of (2.1) with $\alpha = 0$ are periodic, and surround the equilibrium $E = (b_2/a_{21}, b_1/a_{12})$. For two such periodic orbits γ_1 and γ_2, if γ_1 lies in the region surrounded by γ_2, it is easy to see that H has a greater value on γ_2 than on γ_1. Of course, H has a minimum at E.

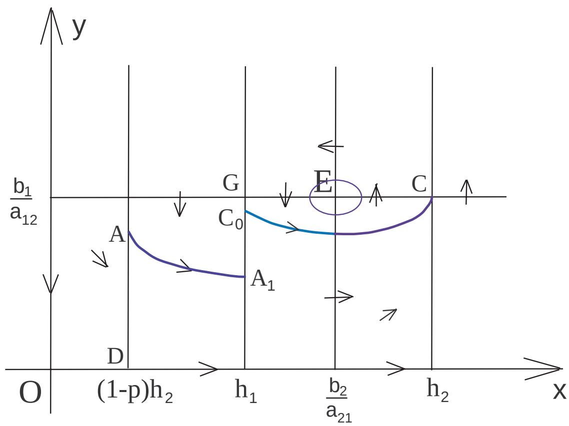

Fig. 4. The case $h_1 \le b_2/a_{21} < h_2$

Let $C = (h_2, b_1/a_{12})$ and $G = (h_1, b_1/a_{12})$ (see Figure 4). Assume $H(C) \ge H(G)$. Then the negative orbit $C \cdot \mathbb{R}^-$ of (2.1) meets Σ^1 at a point $C_0 = (h_1, y_c)$, which lies in Σ^1 and below G. Note that y_c is determined by $H(h_1, y_c) = H(C)$, i.e.,

$$(4.3) \quad y_c = -\frac{b_1}{a_{12}} W_0\left(-\exp\left\{ \frac{1}{b_1}[H(C) - H(h_1, b_1) + b_1(a_{12} - \ln a_{12})] \right\} \right).$$

Let $A = C^+ = ((1-p)h_2, (1-q)b_1/a_{12})$. Then $A_1 = A \cdot \phi(A) = (h_1, y_a)$ lies in Σ^1 and below G, where y_a is determined by $H(h_1, y_a) = H(A)$, i.e.,

$$(4.4) \quad y_c = -\frac{b_1}{a_{12}} W_0\left(-\exp\left\{ \frac{1}{b_1}[H(C^+) - H(h_1, b_1) + b_1(a_{12} - \ln a_{12})] \right\} \right).$$

Suppose that $y_a + \alpha \le y_c$. Then, as in Section 4.1, we define the Poincaré map $F : \overline{AD} \to \overline{AD}$ by $F(P) = (P * \phi(P)) * \phi(P * \phi(P))$ for each $P \in \overline{AD}$. Clearly, F is a continuous map and has a fixed point in $\overline{AD}$. Thus, we deduce the following result.

THEOREM 4.3. *Assume that* $h_1 \le b_2/a_{21} < h_2$ *and* $H(h_2, b_1/a_{12}) \ge H(h_1, b_1/a_{12})$. *If* $y_a + \alpha \le y_c$, *where* y_a *and* y_c *are defined by* (4.3) *and* (4.4) *respectively, then system* (1.1) *with* $a_{11} = 0$ *has a positive periodic orbit.*

Now, in order to get asymptotic stability, we are going to find a condition ensuring that F is contractive. Choose $P = (h_1, y_0)$ and $Q = (h_1, y_0 + \eta)$ in Σ^1 below the line $y = b_1/a_{12}$, where $\eta \in (0, b_1/a_{12})$. Next, denote $P_1 = P \cdot \phi(P) = (h_2, y_1)$ and $Q_1 = Q \cdot \phi(Q) = (h_2, y_2)$. Then we have the following lemma.

LEMMA 4.4. *Assume that* $h_1 \le b_2/a_{21} < h_2$, $H(h_2, b_1/a_{12}) \ge H(h_1, b_1/a_{12})$ *and* $y_a + \alpha \le y_c$. *If* $a_{21}(h_2 - h_1) \le b_2 \ln(h_2/h_1)$, *then* $d(P_1, Q_1) \le d(P, Q)$ *for each* $\eta \in (0, b_1/a_{12})$.

Proof. Since $a_{21}(h_2 - h_1) \le b_2 \ln(h_2/h_1)$, it follows that

$$a_{21}h_1 + a_{12}\lambda - b_2 \ln h_1 - b_1 \ln \lambda \ge a_{21}h_2 + a_{12}\lambda - b_2 \ln h_2 - b_1 \ln \lambda$$

for each $\lambda \in (0, b_1/a_{12})$, i.e., $H(h_1, \lambda) \ge H(h_2, \lambda)$. Thus, $y_0 \ge y_1$: in fact, let $\lambda = y_0$; since $H(h_2, y_1) = H(h_1, y_0) \ge H(h_2, y_0)$, it follows that $y_0 \ge y_1$. Then, $y_0/(y_0 + \eta) \ge y_1/(y_1 + \eta)$ for each $\eta \in (0, b_1/a_{12})$, which leads to

$$\ln y_0 - \ln(y_0 + \eta) \ge \ln y_1 - \ln(y_1 + \eta). \tag{4.5}$$

On the other hand, from $H(Q_1) = H(Q) = H(h_1, y_0 + \eta)$ it follows that

$$\begin{aligned} H(Q_1) - H(h_2, y_1 + \eta) &= H(h_1, y_0 + \eta) - H(h_2, y_1 + \eta) \\ &= H(P) + a_{12}\eta + b_1 \ln y_0 - b_1 \ln(y_0 + \eta) \\ &\quad - [H(P_1) + a_{12}\eta + b_1 \ln y_1 - b_1 \ln(y_1 + \eta)] \\ &= b_1[\ln y_0 - \ln y_1 + \ln(y_1 + \eta) - \ln(y_0 + \eta)]. \end{aligned}$$

By (4.5), we have $H(Q_1) - H(h_2, y_1 + \eta) \ge 0$, which means $y_2 \le y_1 + \eta$, and so $d(P_1, Q_1) \le \eta = d(P, Q)$. The proof is complete. ∎

THEOREM 4.5. *Assume that* $h_1 \le b_2/a_{21} < h_2$, $H(h_2, b_1/a_{12}) \ge H(h_1, b_1/a_{12})$ *and* $y_a + \alpha \le y_c$, *where* y_a *and* y_c *are determined by* (4.3) *and* (4.4) *respectively. If* $a_{21}(h_2 - h_1) \le b_2 \ln(h_2/h_1)$ *and* $\alpha(1 - q) \le 1$, *then system* (1.1) *with* $a_{11} = 0$ *has a positive periodic orbit that is asymptotically orbitally stable.*

Proof. Consider the Poincaré map F defined as above. Let P and Q be different points in $\overline{AD}$, and suppose their trajectories meet Σ^1 at P_1 and Q_1 respectively. Since $h_1 \le b_2/a_{21}$, by a similar argument to the proof of Lemma 4.1, we have $d(P_1, Q_1) < d(P, Q)$. Next, assume that the trajectories of P_1^+ and Q_1^+ meet Σ^2 at P_2 and Q_2, respectively. It follows from Lemma 4.4 that $d(P_2, Q_2) \leq d(P_1^+, Q_1^+)$. Thus, we obtain

$$\begin{aligned} d(F(P), F(Q)) &= d(P_2^+, Q_2^+) = (1-q)d(P_2, Q_2) \\ &\leq (1-q)d(P_1^+, Q_1^+) = \alpha(1-q)d(P_1, Q_1) < d(P, Q), \end{aligned}$$

i.e., F is contractive. So, F has a unique fixed point in $\overline{AD}$, which corresponds to an asymptotically orbitally stable periodic orbit of (1.1) with $a_{11} = 0$. ■

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