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Oscillation of nonlinear neutral delay differential equations with applications

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Abstract. We consider nonlinear neutral delay differential equations with variable coefficients. Finite and infinite integral conditions for oscillation are obtained. As an example, the neutral delay logistic differential equation is discussed.

1. Introduction. A neutral delay differential equation is a differential equation in which the highest order derivative of the unknown function appears both with and without delays. The study of the asymptotic and oscillatory behavior of solutions of neutral differential equations is of importance in applications. This is due to the fact that such equations appear in various phenomena including networks containing lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equations for the minimization of functionals involving a time delay in some variational problems, in the theory of automatic control and in neuromechanical systems in which inertia plays an important role (see Hale [11], Driver [3], Brayton and Willoughby [2], Popov [26] and Boe and Chang [1] and references cited therein). The construction of these models using delays has been paralleled by mathematical investigation of nonlinear equations.

Many authors have considered linear neutral delay differential equations and established sufficient conditions for oscillation of all solutions. We refer to the articles [6, 7, 28, 17, 20, 22, 27, 31] and the references cited therein.

In recent years there has been much research activity concerning the linearized oscillation theory for nonlinear neutral delay differential equations,

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which is in some sense parallel to the so-called linearized stability theory (see [5, 8, 19, 25, 29]).

In this paper we consider the first-order nonlinear neutral delay differential equation with variable coefficients

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + Q(t)f(x(t-\sigma)) = 0, \tag{1}$$

where

$$0 < P(t) < 1, \ \sigma, \tau \text{ are positive constants}, Q \in C[[t_0, \infty), \mathbb{R}^+], \tag{2}$$

$$f \in C[\mathbb{R}, \mathbb{R}], \qquad uf(u) > 0 \quad \text{ for } u \neq 0, \tag{3}$$

and there exists a positive number δ such that

$$\begin{cases} f(u) \leq u & \text{for } u \in [0, \delta], \\ f(u) \geq u & \text{for } u \in [-\delta, 0], \end{cases} \tag{4}$$

and

$$\lim_{u \to 0} \frac{u}{f(u)} = \beta > 0, \tag{5}$$

or the more general one,

$$\frac{d}{dt}\Big[x(t) - \sum_{i=1}^{n} P_i(t)x(t-\tau_i)\Big] + \sum_{j=1}^{m} Q_j(t)f(x(t-\sigma_j)) = 0, \tag{6}$$

where $\tau_i, \sigma_j, P_i, Q_j$ and f satisfy the same assumptions as τ, σ, P, Q and f for $i = 1, \dots, n$ and $j = 1, \dots, m$.

Our aim in this paper is to give some finite and infinite integral sufficient conditions for oscillation of solutions of (1) and (6). In Section 2, we present some lemmas which will be used in the proofs of our main results. In Section 3, we give oscillation criteria for (1). In Section 4 the neutral delay logistic differential equation is considered to illustrate our results.

Let $\gamma = \max\{\sigma, \tau\}$ and let $t_1 \geq t_0$. By a solution of (1) on $[t_1, \infty)$ we mean a function $x \in C[[t_1 - \gamma, \infty), \mathbb{R}]$ such that $x(t) - P(t)x(t-\tau)$ is continuously differentiable for $t_1 \geq t_0$ and (1) is satisfied.

Let $t_1 \geq t_0$ be a given initial point and let $\phi \in C[[t_1 - \gamma, \infty), \mathbb{R}]$ be a given initial function. Then by the step-by-step method one can see that (1) has a unique solution on $[t_1, \infty)$ satisfying the initial condition

$$x(t) = \phi(t) \quad \text{ for } -\gamma \leq t \leq t_1. \tag{7}$$

As usual, when we say that *every solution* of (1) *oscillates* we mean that for every initial point $t_1 \geq t_0$ and for every initial function $\phi \in C[[t_1 - \gamma, \infty), \mathbb{R}]$, the unique solution of (1) with (7) has arbitrarily large zeros. Otherwise the solution is called *nonoscillatory*.

In what follows, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

2. Some lemmas. In this section we present some lemmas that are used in the proofs of our main results.

LEMMA 2.1 [10, Lemma 1.5.4]. *Let* $a \in (-\infty, 0)$, $\tau \in (0, \infty)$, $t_0 \in \mathbb{R}$ *and suppose that* $x \in C[[t_0, \infty), \mathbb{R}]$ *satisfies the inequality*

$$x(t) \leq a + \max_{t-\tau \leq s \leq t} x(s) \quad \textit{for } t \geq t_0.$$

Then $x(t)$ *cannot be a nonnegative function.*

LEMMA 2.2 [10, Lemma 1.5.5]. *Let* $F, G, P \in C[[t_0, \infty), \mathbb{R}]$, *and* $c \in (0, \infty)$ *be such that*

$$F(t) = G(t) - P(t)G(t-c), \quad t \geq t_0 + c.$$

Assume that

$$F(t) > 0 \quad \textit{and} \quad G(t) > 0 \quad \textit{for} \quad t \geq t_0, \quad \lim_{t\to\infty} F(t) = 0,$$

and $0 < P(t) < 1$. *Then* $\lim_{t\to\infty} G(t) = 0$.

LEMMA 2.3 *Assume that* (2) *and* (3) *are satisfied. Set*

$$z(t) = x(t) - P(t)x(t-\tau), \tag{8}$$

where $x(t)$ *is an eventually positive solution of* (1). *Then* $z(t)$ *is an eventually nonincreasing positive function*, $\lim_{t\to\infty} z(t) = 0$, *and* $\lim_{t\to\infty} x(t) = 0$.

Proof. Since $x(t)$ is an eventually positive solution of (1), we have

$$z'(t) \leq -Q(t)f(x(t-\sigma)).$$

Then from (2) and (3) we have

$$z'(t) \leq -Q(t)f(x(t-\sigma) \leq 0, \tag{9}$$

where $z'(t) = dz(t)/dt$. This shows that $z(t)$ is nonincreasing. Now we show that $z(t)$ is positive. For otherwise there exists a $t_2 \geq t$ such that $z(t_2) < 0$. Because $\dot{z}(t) \leq 0$ for $t \geq t_1 + \varrho$ and $\dot{z}(t) \neq 0$ on $[t_1 + \varrho, \infty)$ there exists $t_3 \geq t_2$ such that $z(t) \leq z(t_3)$ for $t \geq t_3$. Then from (8) it follows that for $t \geq t_3$,

$$x(t) = z(t) + P(t)x(t-\tau) \leq z(t_3) + P(t)x(t-\tau).$$

Hence

$$x(t) \leq z(t_2) + \max_{t-\varrho \leq s \leq t} x(s)P(t).$$

From (2) we have

$$x(t) \leq z(t_2) + \max_{t-\varrho \leq s \leq t} x(s) \quad \text{for } t \geq t_3.$$

Thus by Lemma 2.1, $x(t)$ cannot be a non-negative function on $[t_3, \infty)$, and this is a contradiction. Hence $z(t)$ is non-increasing and positive.

Now we show that $x(t)$ is bounded. Otherwise there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} x(t_n) = \infty$ and $x(t_n) = \max_{s\le t_n} x(s)$. From (8) we have

$$z(t_n) = x(t_n) - P(t_n)x(t_n - \tau).$$

Hence by (2) we obtain

$$z(t_n) \ge x(t_n)[1 - P(t_n)] \to \infty \quad \text{ as } n \to \infty,$$

which contradicts (9). Then by (8) and (9) we see that $z(t)$ is also bounded, and

$$\lim_{t\to\infty} z(t) = k \in [-\infty, \infty). \tag{10}$$

If $k = -\infty$, then there is a $t_1 \ge t_0$ such that $z(t) < 0$ for $t \ge t_1$. From (8) we have $\limsup_{t\to\infty} x(t) = +\infty$, which contradicts the fact that $x(t)$ is bounded. Thus $k \ne -\infty$.

From (9) and (10) it follows that the function $z(t)$ is monotonic. By integrating both sides of (9) from t_1 to ∞ for t_1 sufficiently large, we obtain

$$k - z(t_1) = -\int_{t_1}^{\infty} Q(t) f(x(t-\sigma))\, dt. \tag{11}$$

We claim that

$$\liminf_{t\to\infty} x(t) = 0. \tag{12}$$

Otherwise there exist a positive constant c and a $t_2 \ge t_1$ such that $x(t) \ge c$ for $t \ge t_2$. Then from (2) and (3) and for t sufficiently large, $Q(t)f(x(t-\sigma))$ is bounded from below by a positive constant. This contradicts (11) and so (12) holds.

Let $\{t_k\}$ be a sequence of points such that

$$\lim_{k\to\infty} t_k = \infty \quad \text{and} \quad \lim_{k\to\infty} x(t_k) = 0.$$

From (8) we see that

$$z(t_k) \le x(t_k) \to 0 \quad \text{ as } k \to \infty$$

and also

$$z(t_k + \tau) \ge -P(t_k + \tau)x(t_k) \to 0 \quad \text{ as } k \to \infty.$$

As $z(t)$ is positive and monotonic, it follows that $\lim_{t\to\infty} z(t) = 0$. Then by Lemma 2.2 we see that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. ∎

3. Oscillation criteria. In this section we give finite and infinite integral conditions for oscillation of solutions of (1).

THEOREM 3.1. *Assume that* (2)–(5) *hold. Then the conditions*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} [Q_1(s)P(t-\sigma) + Q_1(s)]\,ds > \frac{1}{e} \tag{13}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} [Q_1(s)P(t-\sigma) + Q_1(s)]\,ds > 1 \tag{14}$$

each imply that every solution of (1) *oscillates, where*

$$Q_1(t) = \frac{1}{\beta}Q(t), \quad \textit{and} \quad \tau \geq \sigma.$$

Proof. Without loss of generality we assume that (1) has an eventually positive solution $x(t)$ (the case that $x(t)$ is negative is similar and will be omitted). Suppose $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_0$. Then from Lemma 2.3 we have $\lim_{t\to\infty} x(t) = 0.$ By (5) we have

$$\lim_{u\to 0} \frac{u}{f(u)} = \beta > 0. \tag{15}$$

Let $\varepsilon \in (0,\beta)$. Then there exists a T_ε such that for $t \geq T_\varepsilon$, $x(t-\sigma) > 0$ and $f(x(t-\sigma)) \geq x(t-\sigma)/(\beta-\varepsilon)$.

Set $z(t)$ as in (8). Then from (1) and Lemma 2.3, $z(t)$ is a positive function, $x(t) > z(t)$ and

$$z'(t) + Q(t)\frac{x(t-\sigma)}{\beta-\varepsilon} \leq 0, \qquad t \geq T_\varepsilon. \tag{16}$$

Hence,

$$z'(t) + Q_1(t)x(t-\sigma) \leq 0, \qquad t \geq T_\varepsilon, \tag{17}$$

where $Q_1(t) = Q(t)/\beta$. Then from (8),

$$\begin{aligned} z'(t) &\leq -Q_1(t)z(t-\sigma) - Q_1(t)P(t-\sigma)x(t-\tau-\sigma), \qquad t \geq T_\varepsilon, \\ &\leq -Q_1(t)z(t-\sigma) + \frac{Q_1(t)}{Q_1(t-\tau)}P(t-\sigma)z^{'}(t-\tau). \end{aligned} \tag{18}$$

Hence $z(t)$ is positive and satisfies the delay differential inequality

$$z'(t) - \frac{Q_1(t)}{Q_1(t-\tau)}P(t-\sigma)z'(t-\tau) + Q_1(t)z(t-\sigma) \leq 0. \tag{19}$$

Set

$$\lambda(t) = -\frac{z'(t)}{z(t)}. \tag{20}$$

Then (19) becomes

$$\lambda(t) \geq \lambda(t-\tau)\frac{Q_1(t)}{Q_1(t-\tau)}P(t-\sigma) \times \exp\Big(\int_{t-\tau}^{t} \lambda(s)\,ds\Big) + Q_1(t)\exp\Big(\int_{t-\sigma_1}^{t} \lambda(s)\,ds\Big). \tag{21}$$

It is obvious that $\lambda(t) > 0$ for $t \geq t_0$. From (21) we have $\lambda(t) \geq Q_1(t)$, thus

$$\lambda(t) \geq Q_1(t)P(t-\sigma)\exp\Big(\int_{t-\tau}^{t} \lambda(s)\,ds\Big) + Q_1(t)\exp\Big(\int_{t-\sigma_1}^{t} \lambda(s)\,ds\Big). \tag{22}$$

Then from (22) and (20) one can see that $z(t)$ is a positive solution of the delay differential inequality

$$z'(t) + Q_1(t)P(t-\sigma)z(t-\tau) + Q_1(t)z(t-\sigma) \leq 0.$$

As $z'(t) \leq 0$ and $\tau \geq \sigma$, we see that

$$z'(t) + [Q_1(t)P(t-\sigma) + Q_1(t)]z(t-\sigma) \leq 0.$$

Then by Corollary 3.2.2 of [10] the delay differential equation

$$z'(t) + [Q_1(t)P(t-\sigma) + Q_1(t)]z(t-\sigma) = 0 \tag{23}$$

has an eventually positive solution as well. However, it is well known that (14) and (15) each imply that (23) has no eventually positive solution (see, for example, [10, p. 46, Theorem 2.3.3] and [10, p. 78, Theorem 3.4.3]). This is a contradiction and so the proof is complete. ∎

Remark 1. It is clear that every solution of (1) oscillates when (23) has no eventually positive solution. The problem how to fill the gap between the conditions (13) and (14) for the equation (23) has recently been investigated by several authors. Taking the results of [4, 14, 18, 23, 24, 30] respectively into account and the fact that every solution of (1) oscillates when (23) has no eventually positive solution we obtain the following applications, where we set

$$\overline{Q}(t) = Q_1(t)P(t-\sigma) + Q_1(t).$$

Corollary 3.1. *Assume that* (2)–(5) *hold*,

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \overline{Q}(s)\,ds = k \leq \frac{1}{e}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} \overline{Q}(s)\,ds > \frac{\ln(\lambda)+1}{\lambda},$$

where λ *is the smaller solution of the transcendental equation*

$$\lambda = e^{\lambda k}. \tag{24}$$

Then every solution of (1) *oscillates.*

Corollary 3.2. *Assume that* (2)–(5) *hold,*

$$k = \liminf_{t\to\infty} \int_{t-\sigma}^{t} \overline{Q}(s)\,ds, \qquad L = \limsup_{t\to\infty} \int_{t-\sigma}^{t} \overline{Q}(s)\,ds,$$

$L < 1$ *and* $0 < k \le 1/e$. *Then every solution of* (1) *oscillates if*

$$L > \frac{\ln(\lambda)+1}{\lambda} - \frac{1-k-\sqrt{1-k-k^2}}{2},$$

where λ *is the smaller root of* (24).

Corollary 3.3. *Assume that* (2)–(5) *hold and*

$$\sum_{i=1}^{\infty} \left[\int_{t_{i-1}}^{t_i} \overline{Q}(s) - \frac{1}{e} \right] ds = \infty.$$

Then every solution of (1) *oscillates.*

Corollary 3.4. *Assume that* (2)–(5) *hold,*

$$\int_{t-\sigma}^{t} \overline{Q}(s)\,ds \ge \frac{1}{e},$$

and

$$\int_{t_0+\sigma}^{\infty} \overline{Q}(t) \left[\exp\left(\int_{t-\sigma}^{t} \overline{Q}(s)\,ds - \frac{1}{e} \right) - 1 \right] ds = \infty.$$

Then every solution of (1) *oscillates.*

Corollary 3.5. *Assume that* (2)–(5) *hold,*

$$\int_{t-\sigma}^{t} \overline{Q}(s)\,ds \ge \frac{1}{e},$$

and

$$\int_{t_0+n\sigma}^{\infty} \overline{Q}(t) \left[e^{n-1}\overline{Q}_n(t) - \frac{1}{e} \right] dt = \infty.$$

Then every solution of (1) *oscillates. Here*

$$\overline{Q}_1(t) = \int_{t-\sigma}^{t} \overline{Q}(s)ds, \qquad t \ge t_0+\sigma,$$

$$\overline{Q}_{k+1}(t) = \int_{t-\sigma}^{t} \overline{Q}(s)\overline{Q}_k(s)\,ds, \qquad t \ge t_0+(k+1)\sigma.$$

COROLLARY 3.6. *Assume that* (2)–(5) *hold*,

$$\limsup_{t\to\infty} \int_t^{t+\sigma} \overline{Q}(s)\,ds > 0 \qquad \textit{for } t \ge t_0 \textit{ for some } t_0 > 0,$$

and

$$\int_{t_0}^{\infty} \overline{Q}(t) \ln\Big[e \int_t^{t+\sigma} \overline{Q}(s)\,ds\Big]\,dt = \infty.$$

Then every solution of (1) *oscillates.*

THEOREM 3.2. *Assume that* (2)–(5) *hold. Then the conditions*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} Q_1(t)P(t-\sigma) > \frac{1}{e} \tag{25}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} Q_1(t)P(t-\sigma)\,ds > 1 \tag{26}$$

each imply that every solution of (1) *oscillates.*

Proof. Without loss of generality, assume that (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (8). Then from (23) one can see that $z(t)$ is a positive solution of the inequality

$$z'(t) + Q_1(t)P(t-\sigma)z(t-\sigma) \le 0. \tag{27}$$

Therefore by Corollary 3.2.2 of [10] the delay differential equation

$$z'(t) + Q_1(t)P(t-\sigma)z(t-\sigma) = 0 \tag{28}$$

has an eventually positive solution as well. But it is well known that (25) and (26) each imply that (28) has no eventually positive solution. This is a contradiction and so the proof is complete. ∎

THEOREM 3.3. *Assume that* (2)–(5) *hold. Then the conditions*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} Q_1(s)\,ds > \frac{1}{e} \tag{29}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} Q_1(s)\,ds > 1 \tag{30}$$

each imply that every solution of (1) *oscillates.*

Proof. Assume that (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (8). Then from (23) one can see that

$$z'(t) + Q_1(t)z(t-\sigma) \le 0. \tag{31}$$

Therefore by Corollary 3.2.2 of [10] the delay differential equation

$$z'(t) + Q_1(t)z(t-\sigma) = 0 \tag{32}$$

has an eventually positive solution as well. But it is well known that (29) and (30) each imply that (32) has no eventually positive solution. This is a contradiction and so the proof is complete. ∎

THEOREM 3.4. *Assume that* (2)–(5) *hold. Then the conditions*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} [Q_1(s)P(s-\sigma)P(s-\tau-\sigma) + Q_1(s)]\,ds > \frac{1}{e} \tag{33}$$

and

$$\limsup_{t\to\infty} \int_{t-\sigma}^{t} [Q_1(s)P(s-\sigma)P(s-\tau-\sigma) + Q_1(s)]\,ds > 1 \tag{34}$$

each imply that every solution of (1) *oscillates.*

Proof. Assume that (1) has an eventually positive solution $x(t)$. From (21) it is obvious that $\lambda(t) > 0$ for $t \ge t_0$, and $\lambda(t) \ge Q_1(t)$. Then $\lambda(t-\tau) \ge Q_1(t-\tau)$, and so

$$\lambda(t) \ge Q_1(t)P(t-\sigma)\exp\Big(\int_{t-\tau}^{t} \lambda(s)\,ds\Big) + Q_1(t)\exp\Big(\int_{t-\sigma}^{t} \lambda(s)\,ds\Big), \tag{35}$$

which guarantees that $\lambda(t) \ge Q_1(t)P(t-\sigma)$. Thus

$$\lambda(t-\tau) \ge Q_1(t-\tau)P(t-\tau-\sigma).$$

From (32) we have

$$\begin{aligned}\lambda(t) \ge{}& Q_1(t)P(t-\sigma)P(t-\tau-\sigma)\\ &\times \exp\Big(\int_{t-\tau}^{t} \lambda(s)\,ds\Big) + Q_1(t)\exp\Big(\int_{t-\sigma}^{t} \lambda(s)\,ds\Big)\end{aligned} \tag{36}$$

and therefore $z(t)$ satisfies the inequality

$$z'(t) + [Q_1(t)P(t-\sigma)P(t-\tau-\sigma) + Q_1(t)]z(t-\sigma) \le 0.$$

Then by Corollary 3.2.2 of [10] the delay differential equation

$$z'(t) + [Q_1(t)P(t-\sigma)P(t-\tau-\sigma) + Q_1(t)]z(t-\sigma) = 0 \tag{37}$$

has an eventually positive solution as well. But it is well known that (33) and (34) each imply that (37) has no eventually positive solution. Thus every solution of (1) oscillates. ∎

By applying the above corollaries to the equations (37) we can obtain sufficient conditions for oscillation of all solutions of (1). Their statements are omitted here.

REMARK 2. For the general equation

$$\frac{d}{dt}\Big[x(t) - P\sum_{i=1}^{n} P_i(t)x(t-\tau_i)\Big] + \sum_{i=1}^{n} Q_i(t)x(t-\sigma_i) = 0, \qquad t \geq t_0,$$

the analogues of the above results are also true. We omit the details.

4. Oscillation in a nonautonomous neutral delay logistic equation. The scalar autonomous ordinary differential equation

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t)}{K}\right]$$

is known as the *logistic equation* in mathematical ecology and it is a prototype in the modelling of the dynamics of single-species population systems whose biomass or density is described by a differentiable function $N(t)$. The constant r is called the growth rate and K is called the carrying capacity of the habitat. Hutchinson [13] suggested the following modification:

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right]. \tag{38}$$

Equation (38) is commonly known as the "delay equation" and has been extensively investigated by numerous authors (see for example Wright [31], Kakutani and Markus [16] and Jones [15]). Hegazi and Saker [12] considered the non-autonomous delay logistic and food limited equations and presented infinite integral conditions for oscillations.

Györi [9] considered the neutral delay logistic equation with constant coefficients of the form

$$\dot{N}(t) = N(t)\left[r\left(1 - \frac{N(t-\tau)}{K}\right) + c\dot{N}(t-\tau)\right], \tag{39}$$

and established oscillation criteria for all positive solutions.

The effects of varying environment are often important in dynamical nature of populations. Then we consider the nonautonomous neutral delay equation

$$\dot{N}(t) = N(t)\left[r(t)\left(1 - \frac{N(t-\sigma)}{K}\right) + c(t)\dot{N}(t-\tau)\right], \tag{40}$$

where

$$r, c \in C[[t_0,\infty), \mathbb{R}^+], \qquad K, \tau, \sigma \in (0,\infty), \tag{41}$$

$r(t)$ is the growth rate function, K is the carrying capacity of the environment, and $c(t)$ is the growth rate function associated with the growth rate at time $t-\tau$.

With (38) one associates an initial condition of the form

$$N(t)=\phi(t) \quad \text{for } -\gamma\le t\le 0, \quad \phi\in C[[-\tau,0],\mathbb{R}^+], \quad \phi(0)>0, \tag{42}$$

where $\gamma=\max\{\tau,\sigma)$. Then by the step-by-step method, the initial value problem (40) and (42) has a unique solution $N(t)$ for $t\ge 0$. We will only consider those solutions $N(t)$ which are positive. Note that such solutions exist because if $\phi(0)>0$, then $N(t)>0$ for $t\ge 0$.

THEOREM 4.1. *Assume that* (41) *holds*, $0<c(t)<1$, *and*

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} r(s)\,ds>\frac{1}{e}. \tag{43}$$

Then every solution of (40) *oscillates.*

Proof. The change of variables $N(t)=Ke^{x(t)}$ reduces (40) to the delay equation

$$\frac{d}{dt}[x(t)-c(t)x(t-\tau)]+r(t)[e^{x(t-\sigma)}-1]=0. \tag{44}$$

Clearly, $N(t)$ oscillates about K if and only if $x(t)$ oscillates about zero. From (44) we have

$$\frac{d}{dt}[x(t)-c(t)x(t-\tau)]+r(t)f(x(t-\sigma))=0 \tag{45}$$

with

$$f(u)=e^u-1. \tag{46}$$

It is clear that

$$f\in C[\mathbb{R},\mathbb{R}], \qquad uf(u)>0 \quad \text{for } u\ne 0, \tag{47}$$

$$\lim_{u\to 0}\frac{u}{f(u)}=1. \tag{48}$$

Then by Theorem 3.2 and the condition (43) every solution of (44) oscillates. Thus every positive solution of (40) oscillates about K. ■

REMARK 3. One can apply the above theorems and corollaries to obtain many sufficient conditions for oscillations. Also one can extend these results to the generalized neutral logistic equation

$$\dot N(t)=\sum_{i=1}^{n}N(t)\Big[r_i(t)\Big(1-\frac{N(t-\sigma_i)}{K}\Big)+c_i(t)N(t-\tau_i)\Big].$$

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