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The Bergman projection in spaces of entire functions

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Abstract. We establish L^p-estimates for the weighted Bergman projection on a nonsingular cone. We apply these results to the weighted Fock space with respect to the minimal norm in $\mathbb{C}^n$.

1. Introduction and main results. Let $n \geq 2$ and consider the nonsingular cone

$$\mathbb{H} := \{z \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 0,\ z \neq 0\}.$$

This is the orbit of the vector $(1, i, 0, \ldots, 0)$ under the $SO(n+1, \mathbb{C})$-action on $\mathbb{C}^{n+1}$. It is well-known that $\mathbb{H}$ can be identified with the cotangent bundle of the unit sphere $\mathbb{S}^n$ in the n-dimensional sphere in $\mathbb{R}^{n+1}$ minus its zero section. It was proved in [OPY] that there is a unique (up to a multiplicative constant) $SO(n+1,\ \mathbb{C})$-invariant holomorphic form α on $\mathbb{H}$. The restriction of this form to $\mathbb{H} \cap (\mathbb{C} \setminus \{0\})^{n+1}$ is given by

$$\alpha(z) = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1}.$$

For any $t > 0$ we consider the Gaussian volume form ω_t defined on $\mathbb{H}$ by

$$\omega_t(z) = \frac{2t^{n-1}}{(n-2)!m_n} e^{-t|z|^2} \alpha(z) \wedge \bar{\alpha}(z), \qquad z \in \mathbb{H},$$

where

$$m_n := 2(n-1) \int\limits_{\{z \in \mathbb{H}:\, |z|<1\}} \alpha(z) \wedge \bar{\alpha}(z).$$

For each $s > 0$ and $1 \leq p < \infty$, let $L^p(\mathbb{H}, \omega_s)$ denote the Banach space of all functions on $\mathbb{H}$ which are L^p-integrable with respect to the volume form

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ω_s equipped with the norm

$$\|f\|_{L^p(\mathbb{H},\omega_s)} = \Big(\int_{\mathbb{H}} |f(z)|^p \omega_s(z)\Big)^{1/p}, \qquad f \in L^p(\mathbb{H},\omega_s).$$

The weighted Bergman space $\mathcal{A}^p_s(\mathbb{H})$ is the closed subspace of $L^p(\mathbb{H},\omega_s)$ consisting of holomorphic functions. When $p = 2$, the orthogonal projection P_s from $L^2(\mathbb{H},\omega_s)$ onto $\mathcal{A}^2_s(\mathbb{H})$ is called the *weighted Bergman projection*. It is well-known that P_s is the integral operator on $L^2(\mathbb{H},\omega_s)$ given by the formula

$$P_s f(z) = \int_{\mathbb{H}} K_s(z,w) f(w) \omega_s(w),$$

where $K_s(\cdot,\cdot)$ is the reproducing kernel on $\mathcal{A}^2_s(\mathbb{H})$. This is the weighted Bergman kernel. In the following we denote by T_s the integral operator defined by

$$T_s f(z) = \int_{\mathbb{H}} |K_s(z,w)| f(w) \omega_s(w).$$

Next, let $\mathcal{F}_{p,s}(\mathbb{H})$ denote the linear span of the functions

$$f_{k,a}(z) := z_{n+1}^{2k+1} e^{-a|z|^2}, \qquad k \in \mathbb{N},\ a > 0,$$

equipped with the norm $\|\cdot\|_{L^p(\mathbb{H},\omega_s)}$.

Our first main result in this paper is the following:

Theorem A. *Suppose that $t, s > 0$ and $p \ge 1$. Then the following conditions are equivalent:*

(a) *T_t is bounded on $L^p(\mathbb{H},\omega_s)$.*
(b) *P_t is bounded on $L^p(\mathbb{H},\omega_s)$.*
(c) *P_t is bounded on $\mathcal{F}_{p,s}(\mathbb{H})$.*
(d) *$pt = 2s$.*

To give some applications, we recall that the minimal norm in $\mathbb{C}^n$ is given by

$$N_*(z) = \sqrt{|z|^2 + |z \bullet z|},$$

where $z \bullet w = z_1 w_1 + \cdots + z_n w_n$ for $z = (z_1,\ldots,z_n), w = (w_1,\ldots,w_n) \in \mathbb{C}^n$. This norm was shown to be of interest in the study of several problems related to proper holomorphic mappings and the Bergman kernel; see [HP], [OY], [OPY], [MY] and [M].

For each $s > 0$, let dV_s denote the measure on $\mathbb{C}^n$ with density $e^{-sN_*^2}$ with respect to the Lebesgue measure. Precisely,

$$dV_s(z) := e^{-s(|z|^2+|z\bullet z|)} dV(z)$$

where $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$ normalized so that the volume of the unit ball is equal to one. For any $1 \le p < \infty$, the *Fock* *space* $\mathcal{A}^p_s(\mathbb{C}^n)$ with respect to the minimal norm in $\mathbb{C}^n$ consists of all entire functions f with the following property:

$$\|f\|^p_{p,s} := \int_{\mathbb{C}^n} |f(z)|^p |z \bullet z|^{(p-2)/2}\, dV_s(z) < \infty. \tag{1.1}$$

We let $L^p_s(\mathbb{C}^n)$ denote the space of all measurable functions f in $\mathbb{C}^n$ satisfying (1.1). Using the technique developed in the proof of part (1) of Lemma 4.1 in [MY], it can be seen that the Fock space $\mathcal{A}^p_s(\mathbb{C}^n)$ is a closed subspace of $L^p_s(\mathbb{C}^n)$. In addition, the arguments used in the proof of part (2) of the latter lemma show that the linear operator U_0 defined from $L^p_s(\mathbb{C}^n)$ into $L^p(\mathbb{H}, \omega_s)$ by

$$U_0(f)(z) := z_{n+1} f(z_1, \dots, z_n), \qquad (z_1, \dots, z_{n+1}) \in \mathbb{H},$$

is an isometry. More precisely, we have

$$\int_{\mathbb{H}} |U_0 f(z)|^p \omega_s(z) = \frac{4(n+1)^2 s^{n-1}}{(n-2)! m_n} \int_{\mathbb{C}^n} |f(z)|^p |w \bullet w|^{(p-2)/2}\, dV_s(z).$$

In addition, the image $\mathcal{E}^p_s(\mathbb{H})$ of $\mathcal{A}^p_s(\mathbb{C}^n)$ under U_0 is a closed proper subspace of $\mathcal{A}^p_s(\mathbb{H})$, and $\big(\frac{(n-2)! m_n}{4(n+1)^2 s^{n-1}}\big)^{1/p} U_0$ is a unitary operator from $\mathcal{A}^p_s(\mathbb{C}^n)$ onto $\mathcal{E}^p_s(\mathbb{H})$. In particular, $\mathcal{A}^p_s(\mathbb{C}^n)$ is a Banach space.

When $p = 2$, the natural inner product turns $\mathcal{A}^2_s(\mathbb{C}^n)$ into a Hilbert space which has a reproducing kernel $\widetilde{K}_s(z, w)$. We denote by $\widetilde{P}_s$ the corresponding Bergman projection. We also let $\widetilde{T}_s$ be the integral operator on $L^p_s(\mathbb{C}^n)$ associated to the kernel $|\widetilde{K}_s(z, w)|$.

We also consider the vector space $\widetilde{\mathcal{F}}_{p,s}(\mathbb{C}^n)$ spanned by the functions

$$\widetilde{f}_{k,a}(z) := (z \bullet z)^k e^{-a(|z|^2 + |z \bullet z|)}, \qquad k \in \mathbb{N},\ a > 0,$$

and equipped with the norm $\|\cdot\|_{p,s}$.

Our second main result is the following:

THEOREM B. *Suppose that $t, s > 0$ and $p \ge 1$. Then the following are equivalent:*

(a) *$\widetilde{T}_t$ is bounded on $L^p_s(\mathbb{C}^n)$.*
(b) *$\widetilde{P}_t$ is bounded on $L^p_s(\mathbb{C}^n)$.*
(c) *$\widetilde{P}_t$ is bounded on $\widetilde{\mathcal{F}}_{p,s}(\mathbb{C}^n)$.*
(d) *$pt = 2s$.*

In that case the operators P_t and $\widetilde{P}_t$ have the same norm given by

$$\|\widetilde{P}_t\|_p = \|P_t\|_p = 2^{n-1}\sqrt{2e(n-1)!(n-1)} \qquad \textit{when } pt = 2s.$$

2. Preparatory results. The orthogonal group $O(n+1, \mathbb{R})$ acts transitively on the boundary $\mathbb{X}$ of the unit ball in $\mathbb{H}$. Thus there is a unique

$O(n+1,\mathbb{R})$-invariant probability measure μ on $\mathbb{X}$. This measure is induced by the Haar probability measure of $O(n+1,\mathbb{R})$. We will need the following lemma which was established in [MY, Lemma 2.1, p. 506].

Lemma 2.1. *For any C^∞-function f on $\mathbb{H}$, we have*

$$\int_{\mathbb{H}} f(z)\alpha(z)\wedge\bar{\alpha}(z) = m_n \int_0^\infty r^{2n-3} \int_{\mathbb{X}} f(r\xi)\,d\mu(\xi)\,dr$$

provided that the integrals make sense.

We also need the following proposition.

Proposition 2.2. *The Bergman kernel of the weighted Bergman space $\mathcal{A}_s^2(\mathbb{H})$ is given by the formula*

$$K_s(z,w) = \left(1 + \frac{2s}{n-1} z \bullet \bar{w}\right) e^{sz\bullet\bar{w}}$$

for all z and w in $\mathbb{H}$.

Proof. We only need to prove that the operator P_s induced by K_s reproduces the functions of $\mathcal{A}_s^2(\mathbb{H})$. Let $f \in \mathcal{A}_s^2(\mathbb{H})$. Then it follows from the proof of Theorem 3.2 in [MY] that any function $f \in \mathcal{A}_s^2(\mathbb{H})$ can be written in the form $f = \sum_{k=0}^n p_k$ where p_k is a member of the space $\mathcal{P}_k$ of homogeneous polynomials of degree k on $\mathbb{H}$. If we denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ the scalar product of $L^p(\mathbb{H},\omega_s)$ then by binomial series expansion and Lemma 2.1 it follows that, for all $z \in \mathbb{H}$,

$$\begin{aligned}
\langle f, K_s(\cdot,z)\rangle_{\mathbb{H}} &= \int_{\mathbb{H}} f(w)\left(1 + \frac{2s}{n-1} z\bullet\bar{w}\right) e^{sz\bullet\bar{w}}\omega_s(w) \\
&= \sum_{k,l=0}^\infty \frac{(2l+n-1)s^l}{l!(n-1)} \int_{\mathbb{H}} p_k(w)(z\bullet\bar{w})^l\omega_s(w) \\
&= \sum_{k,l=0}^\infty a_{k,l}(s) \int_0^\infty r^{k+l+2n-3}e^{-sr^2}\,dr \int_{\mathbb{X}} p_k(\xi)(z\bullet\bar{\xi})^l\,d\mu(\xi)
\end{aligned}$$

where

$$a_{k,l}(s) = \frac{2(2l+n-1)s^{l+n-1}}{l!(n-2)!(n-1)}.$$

On the other hand, by (2.5) in [MY] we see that

$$\int_{\mathbb{X}} p_k(\xi)(z\bullet\bar{\xi})^l\,d\mu(\xi) = \begin{cases} \dfrac{k!(n-1)!}{(k+n-2)!(2k+n-1)} p_k(z) & \text{if } l = k, \\ 0 & \text{else.} \end{cases}$$

Finally, an easy computation shows that

$$\langle f, K_s(\cdot,z)\rangle_{\mathbb{H}} = \sum_{k=0}^{\infty} \frac{2s^{k+n-1}p_k(z)}{(k+n-2)!} \int_0^{\infty} r^{2(k+n-2)} e^{-sr^2} r\,dr$$
$$= \sum_{k=0}^{\infty} \frac{p_k(z)}{(k+n-2)!} \int_0^{\infty} u^{k+n-2} e^{-u}\,du = \sum_{k=0}^{\infty} p_k(z) = f(z). \quad \blacksquare$$

LEMMA 2.3. *Suppose that* $\beta \geq 0$. *Then*

$$\int_{\mathbb{H}} |z_{n+1}|^{2\beta} e^{-\gamma|z|^2} \alpha(z)\wedge\bar{\alpha}(z) = \frac{m_n \pi^{(n+1)/2}\Gamma(\beta+n-1)\Gamma(\beta+1)}{2^{\beta}\gamma^{\beta+n-1}\Gamma(\beta+(n+1)/2)}$$

for all $\gamma > 0$.

Proof. We observe by Lemma 2.1 that

$$\int_{\mathbb{H}} |z_{n+1}|^{2\beta} e^{-\gamma|z|^2} \alpha(z)\wedge\bar{\alpha}(z) = m_n \int_0^{\infty} r^{2\beta+2n-3} e^{-\gamma r^2}\,dr \int_{\mathbb{X}} |\xi_{n+1}|^{2\beta}\,d\mu(\xi)$$
$$= m_n \frac{\Gamma(\beta+n-1)}{2\gamma^{\beta+n-1}} \int_{\mathbb{X}} |\xi_{n+1}|^{2\beta}\,d\mu(\xi).$$

Let $\sigma = \sigma_n$ denote the rotation invariant measure on $\mathbb{S}^n$. For each $\eta \in \mathbb{S}^n$, there exists a rotation $U \in O(n+1,\mathbb{R})$ such that $U(\eta) = (0,\ldots,0,1)$. Therefore,

$$\xi_{n+1} = (0,\ldots,0,1)\bullet\xi = \xi\bullet U(\eta),$$

so that by the $O(n+1)$-invariance on $\mathbb{X}$, we obtain

$$\int_{\mathbb{X}} |\xi_{n+1}|^{2\beta}\,d\mu(\xi) = \int_{\mathbb{X}} |\xi\bullet U(\eta)|^{2\beta}\,d\mu(\xi) = \int_{\mathbb{X}} |U^{-1}(\xi)\bullet\eta|^{2\beta}\,d\mu(\xi)$$
$$= \int_{\mathbb{X}} |\xi\bullet\eta|^{2\beta}\,d\mu(\xi).$$

Integrating over $\mathbb{S}^n$ with respect to the variable η and the measure σ yields

$$\int_{\mathbb{X}} |\xi_{n+1}|^{2\beta}\,d\mu(\xi) = \int_{\mathbb{S}^n}\int_{\mathbb{X}} |\xi\bullet\eta|^{2\beta}\,d\mu(\xi)\,d\sigma(\eta) = \int_{\mathbb{X}}\int_{\mathbb{S}^n} |\xi\bullet\eta|^{2\beta}\,d\sigma(\eta)\,d\mu(\xi)$$
$$= 2^{-\beta}\int_{\mathbb{S}^n} (\eta_{n+1}^2+\eta_n^2)^{\beta}\,d\sigma(\eta),$$

where the last equality holds due to the rotation invariance of σ because each $\xi \in \mathbb{X}$ has a unique decomposition $\xi = x+iy$ with $x, y \in \mathbb{R}^{n+1}$, $x\bullet x = y\bullet y = 1/2$ and $x\bullet y = 0$. It is also clear that if β is a nonnegative integer, then

$$\int_{\mathbb{S}^n} (\eta_{n+1}^2+\eta_n^2)^{\beta}\,d\sigma(\eta) = \frac{2\pi^{(n+1)/2}\Gamma(\beta+1)}{\Gamma(\beta+(n+1)/2)}.$$

The latter formula holds for all $\beta \geq 0$ due to the uniqueness theorem for bounded analytic functions on the half-plane $\operatorname{Re}\beta > 0$. Therefore,

$$\int_{\mathbb{X}} |\xi_{n+1}|^{2\beta}\, d\mu(\xi) = 2^{-\beta} \int_{\mathbb{S}^n} (\eta_{n+1}^2 + \eta_n^2)^{\beta}\, d\sigma(\eta) = \frac{\pi^{(n+1)/2}\Gamma(\beta+1)}{2^{\beta-1}\Gamma(\beta+(n+1)/2)}.$$

Finally

$$\int_{\mathbb{H}} |z_{n+1}|^{2\beta} e^{-\gamma|z|^2} \alpha(z)\wedge\bar{\alpha}(z) = \frac{m_n \pi^{(n+1)/2}\Gamma(\beta+n-1)\Gamma(\beta+1)}{2^{\beta}\gamma^{\beta+n-1}\Gamma(\beta+(n+1)/2)}$$

for all real numbers $\beta \geq 0$. ∎

Now, we study necessary conditions for the boundedness of P_t and T_t on $L^p(\mathbb{H},\omega_s)$. We first observe that P_t is the integral operator

$$P_t f(z) = \int_{\mathbb{H}} H_{t,s}(z,\xi) f(\xi)\omega_s(\xi)$$

where $H_{t,s}(z,\xi)$ is the hermitian kernel given by

$$H_{t,s}(z,\xi) = (t/s)^{n-1} e^{(s-t)|z|^2} K_t(z,\xi).$$

The operator T_t is also an integral operator with kernel $|H_{s,t}(z,\xi)|$.

LEMMA 2.4. *Assume that $p \geq 1$. If P_t is bounded on $L^p(\mathbb{H},\omega_s)$, then $pt \leq 2s$.*

Proof. Let $a > 0$ be a real number and k be a positive integer. Consider the function

$$f_{k,a}(z) = z_{n+1}^{2k+1} e^{-a|z|^2}, \qquad z \in \mathbb{H}.$$

Then Lemma 2.3 implies that

$$\int_{\mathbb{H}} |f_{k,a}(z)|^p \omega_s(z) = C_{k,p}\, \frac{s^{n-1}}{(ap+s)^{kp+p/2+n-1}}$$

where

$$C_{k,p} := \frac{2\pi^{(n+1)/2}\Gamma(kp+p/2+n-1)\Gamma(\beta+1)}{2^{kp+p/2}(n-2)!\,\Gamma(\beta+(n+1)/2)}. \tag{2.2}$$

Hence $f_{k,a} \in L^p(\mathbb{H},\omega_s)$. By the reproducing formula we see that

$$\begin{aligned} P_t f_{k,a}(z) &= \frac{t^{n-1}}{(t+a)^{n-1}} \int_{\mathbb{H}} K_{t+a}\left(\frac{t}{t+a}z, w\right) \xi_{n+1}^{2k+1}\omega_{t+a}(\xi) \\ &= \left(\frac{t}{t+a}\right)^{2k+n} z_{n+1}^{2k+1}. \end{aligned}$$

It follows again from Lemma 2.3 that

$$\int_{\mathbb{H}} |P_t f_{k,a}(z)|^p \omega_s(z) = C_{k,p} \frac{1}{s^{kp+p/2}} \left(\frac{t}{t+a}\right)^{2kp+pn}.$$

Now, the assumption that P_t is bounded on $L^p(\mathbb{H}, \omega_s)$ implies that there exists a positive constant C, not depending on a or k, such that

$$\int_{\mathbb{H}} |P_t(f_{k,a})(z)|^p \omega_s(z) \le C \int_{\mathbb{H}} |f_{k,a}(z)|^p \omega_s(z).$$

This leads to

$$\left(\frac{t}{t+a}\right)^{p(2k+n)} \le C\left(\frac{s}{s+ap}\right)^{kp+p/2+n-1},$$

from which it follows that

$$\left(\frac{t}{t+a}\right)^{2p+np/k} \le C^{1/k}\left(\frac{s}{s+ap}\right)^{p+p/2k+(n-1)/k}.$$

Taking the limit as $k \to \infty$ we see that

$$\left(\frac{t}{t+a}\right)^{2} \le \frac{s}{s+ap},$$

which in turn implies that $pt^2 \le 2st + sa$. Letting $a \to 0$ yields $pt \le 2s$. ∎

In the following we need explicit formulas for the adjoint operators of P_t and T_t with respect to the integral pairing

$$\langle f, g\rangle_s = \int_{\mathbb{H}} f(z)\overline{g(z)}\omega_s(z).$$

Throughout the rest of this section, if $1 \le p \le \infty$ we let $q = p/(p-1)$ with the understanding that $q = \infty$ when $p = 1$ and $q = 1$ when $p = \infty$. Indeed, we have the following.

LEMMA 2.5. *Suppose T_t is bounded on $L^p(\mathbb{H}, \omega_s)$. Then the adjoint operators of P_t and T_t with respect to the pairing $\langle \cdot, \cdot\rangle_s$ are given by*

$$P_t^* f(z) = \left(\frac{t}{s}\right)^{n-1} e^{(s-t)|z|^2} \int_{\mathbb{H}} K_t(z,w) f(w)\omega_s(w),$$

$$T_t^* f(z) = \left(\frac{t}{s}\right)^{n-1} e^{(s-t)|z|^2} \int_{\mathbb{H}} |K_t(z,w)| f(w)\omega_s(w).$$

Furthermore, both T_t^* and P_t^* are bounded on $L^q(\mathbb{H}, \omega_s)$, where $1/p + 1/q = 1$.

The proof of the above lemma follows from classical functional analysis arguments (see [HS]).

LEMMA 2.6. *Suppose $1 < p < \infty$ and P_t is bounded on $L^p(\mathbb{H}, \omega_s)$. Then $pt > s$.*

Proof. Suppose that $p > 1$ and P_t is bounded on $L^p(\mathbb{H}, \omega_s)$. Then P_t^* is bounded on $L^q(\mathbb{H}, \omega_s)$ where $q = p/(p-1)$. Note that the constant function $f = z_{n+1}$ belongs to $L^q(\mathbb{H}, \omega_s)$, and

$$P_t^* f(z) = (t/s)^n e^{(s-t)|z|^2} z_{n+1}$$

is in $L^q(\mathbb{H}, \omega_s)$. By Lemma 2.1 it easily follows that

$$q(s-t) < s.$$

Thus $pt > s$. ∎

LEMMA 2.7. *Suppose that $1 < p < 2$ and P_t is bounded on $L^p(\mathbb{H}, \omega_s)$. Then $pt = 2s$.*

Proof. Once again, consider the function

$$f_{k,a}(z) = z_{n+1}^{2k+1} e^{-a|z|^2}, \qquad z \in \mathbb{H},$$

where $a > 0$ and k is a positive integer. Then from Lemma 2.5 and the reproducing formula it follows that

$$P_t^* f_{k,a}(z) = \left(\frac{t}{s+a}\right)^{2k+n} e^{(s-t)|z|^2} z_{n+1}^{2k+1}.$$

On other hand, by Lemma 2.4 and (2.2), we have seen that

$$\int_{\mathbb{H}} |f_{k,a}(z)|^q \omega_s(z) = C_{k,q} \frac{s^{n-1}}{(aq+s)^{kq+q/2+n-1}}$$

and

$$\int_{\mathbb{H}} |P_t^* f_{k,a}(z)|^q \omega_s(z) = C_{k,q} \frac{s^{n-1}}{(s-q(s-t))^{kq+q/2+n-1}} \left(\frac{t}{s+a}\right)^{q(2k+n)}.$$

If P_t is bounded on $L^p(\mathbb{H}, \omega_s)$, then P_t^* is bounded on $L^q(\mathbb{H}, \omega_s)$. So, there exists a positive constant C, not depending on a and k, such that

$$\int_{\mathbb{H}} |P_t^* f_{k,a}(z)|^q \omega_s(z) \le C \int_{\mathbb{H}} |f_{k,a}(z)|^q \omega_s(z).$$

It follows that

$$\left(\frac{t}{s+a}\right)^{q(2k+n)} \le C \left(\frac{s-q(s-t)}{s+aq}\right)^{kq+q/2+n-1}.$$

Now arguing as in the proof of Lemma 2.4, and in the proof of Lemma 9 in [DZ], we see that Lemma 2.7 follows. ∎

LEMMA 2.8. *Suppose that $2 < p < \infty$ and P_t is bounded on $L^p(\mathbb{H}, \omega_s)$. Then $pt = 2s$.*

Proof. From Lemma 2.6 we have $s-q(s-t)>0$. For each $f\in L^q(\mathbb{H},\omega_s)$, let

$$f(z)=g(z)e^{(s-t)|z|^2}$$

with $g\in L^q(\mathbb{H},\omega_{s-q(s-t)})$. By assumption, there exists a positive constant C such that

$$\int_{\mathbb{H}}|P_t^*g(z)|^q\omega_{s-q(s-t)}(z)\le C\int_{\mathbb{H}}|g(z)|^q\omega_{s-q(s-t)}(z)$$

for all $g\in L^q(\mathbb{H},\omega_{s-q(s-t)})$. Since $1<q<2$, Lemma 2.7 yields

$$qt=2(s-q(s-t)),$$

showing that $pt=2s$. ∎

LEMMA 2.9. *Suppose $s>0$. Then there exist three positive constants C, C' and C'' such that*

$$C\frac{e^{s|z|^2/4}}{1+s|z|^2}-C'\le\int_{\mathbb{H}}|K_s(z,w)|\omega_s(w)\le C''e^{s|z|^2/4}$$

for all $z\in\mathbb{H}$.

Proof. Let

$$I_s(z):=\int_{\mathbb{H}}\left|\left(1+\frac{2s}{n-1}z\bullet\bar{w}\right)^2e^{sz\bullet\bar{w}}\right|\omega_s(w),$$
$$J_s(z):=\int_{\mathbb{H}}|e^{sz\bullet\bar{w}}|\omega_s(w).$$

A little computing shows that

$$\begin{aligned}J_s(z)&=\int_{\mathbb{H}}|e^{\frac{s}{2}z\bullet\bar{w}}|^2\omega_s(w)=\int_{\mathbb{H}}\left|\sum_{k=0}^{\infty}\frac{s^k}{2^kk!}(z\bullet\bar{w})^k\right|^2\omega_s(w)\\&=\sum_{k=0}^{\infty}\frac{s^{2k+n-1}}{(2^kk!)^2}\int_0^{\infty}r^{2k+2n-3}e^{-sr^2}\,dr\int_{\mathbb{X}}|z\bullet\bar{\xi}|^{2k}\,d\mu(\xi)\\&=\frac{1}{2}\sum_{k=0}^{\infty}\frac{s^k}{(2^kk!)^2}(k+n-2)!\int_{\mathbb{X}}|z\bullet\bar{\xi}|^{2k}\,d\mu(\xi)\\&=\frac{(n-1)!}{2}\sum_{k=0}^{\infty}\frac{1}{k!(2k+n-1)}\left(\frac{s|z|^2}{4}\right)^k\simeq\frac{e^{s|z|^2/4}}{1+s|z|^2}.\end{aligned}$$

More precisely, by easy estimates we have

$$\frac{(n-1)!}{2(n+1)}\,\frac{e^{s|z|^2/4}}{1+s|z|^2}-\frac{1}{2}(n-2)!e^{n-1}\le J_s(z)\le 2e(n-1)!\frac{e^{s|z|^2/4}}{1+s|z|^2}.$$

On the other hand,

$$
\begin{aligned}
I_s(z) &= \int\limits_{\mathbb{H}} \left| \left(1 + \frac{2s}{n-1} z \bullet \bar{w}\right) e^{\frac{s}{2} z \bullet \bar{w}} \right|^2 \omega_s(w) \\
&= \frac{1}{(n-1)^2} \int\limits_{\mathbb{H}} \left| \sum_{k=0}^{\infty} \frac{n-1+2k}{k!} \left(\frac{s}{2} z \bullet \bar{w}\right)^k \right|^2 \omega_s(w) \\
&= \frac{1}{(n-1)^2} \sum_{k=0}^{\infty} \frac{(2k+n-1)^2 s^{2k+n-1}}{(2^k k!)^2} \\
&\quad \times \int\limits_0^{\infty} r^{2k+2n-3} e^{-sr^2}\, dr \int\limits_{\mathbb{X}} |z \bullet \bar{\xi}|^{2k}\, d\mu(\xi) \\
&= \frac{1}{2(n-1)^2} \sum_{k=0}^{\infty} \frac{(2k+n-1)^2 s^k}{(2^k k!)^2} (k+n-2)! \int\limits_{\mathbb{X}} |z \bullet \bar{\xi}|^{2k}\, d\mu(\xi) \\
&= \frac{1}{2(n-1)^2} \sum_{k=0}^{\infty} \frac{2k+n-1}{k!} \left(\frac{s|z|^2}{4}\right)^k \simeq (1 + s|z|^2) e^{s|z|^2/4}.
\end{aligned}
$$

Also, it is clear that

$$\frac{1}{4}(1+s|z|^2)e^{s|z|^2/4} \le I_s(z) \le (n-1)(1+s|z|^2)e^{s|z|^2/4}.$$

Now by Hölder's inequality and the above estimates we see that

$$\int\limits_{\mathbb{H}} |K_s(z,w)|\omega_s(w) \le \sqrt{I_s(z) J_s(z)} \le \sqrt{2e(n-1)!(n-1)}\, e^{s|z|^2/4}.$$

Let $\mathbb{E} := \{w \in \mathbb{H} : |1 + \frac{2s}{n-1} z \bullet \bar{w}| \ge 1\}$. It is clear that

$$\int\limits_{\mathbb{H}\setminus\mathbb{E}} |K_s(z,w)|\omega_s(w) \le \frac{1}{2}(n-2)!e^{n-1}.$$

Thus

$$
\begin{aligned}
\int\limits_{\mathbb{H}} |K_s(z,w)|\omega_s(w) \ge \int\limits_{\mathbb{E}} |K_s(z,w)|\omega_s(w) &= J_s(z) - \int\limits_{\mathbb{H}\setminus\mathbb{E}} |K_s(z,w)|\omega_s(w) \\
&\ge J_s(z) - \frac{1}{2}(n-2)!e^{n-1} \\
&\ge \frac{(n-1)!}{2(n+1)} \frac{e^{s|z|^2/4}}{1+s|z|^2} - \frac{1}{2}(n-2)!e^{n-1}. \quad \blacksquare
\end{aligned}
$$

We set

$$F_s(z) := \int\limits_{\mathbb{H}} |K_s(z,w)|\omega_s(w).$$

Lemma 2.10. *If P_t is bounded on $L^1(\mathbb{H}, \omega_s)$, then $t = 2s$.*

Proof. The hypothesis of the lemma implies that P_t^* is bounded on $L^\infty(\mathbb{H},\omega_s)$. Fix $w_0\in\mathbb{H}$ and consider the function

$$f_{w_0}(z)=\frac{K_t(z,w_0)}{|K_t(z,w_0)|},\qquad z\in\mathbb{H}.$$

Then

$$P_t^*f_{w_0}(w_0)=\left(\frac{t}{s}\right)^{n-1}e^{(s-t)|w_0|^2}F_s\left(\frac{t}{s}w_0\right)\quad\text{and}\quad\|f_{w_0}\|_\infty=1.$$

By Lemma 2.9 and the boundedness of P_t^* on $L^\infty(\mathbb{H},\omega_s)$, there exists a positive constant C such that

$$(t/s)^{n-1}e^{(s-t)|w_0|^2}e^{\frac{s}{4}|\frac{t}{s}w_0|^2}\le C$$

for all $w_0\in\mathbb{H}$. The above inequality is possible only if

$$s-t+\frac{t^2}{4s}\le 0,$$

which is equivalent to $(2s-t)^2\le 0$ and hence $t=2s$. ∎

To study the boundedness of the operator T_t on $L^p(\mathbb{H},\omega_s)$, $1<p<\infty$, we need the following well-known Schur lemma.

LEMMA 2.11. *Suppose $H(z,w)$ is a positive kernel and*

$$Tf(z)=\int_\Omega H(z,w)f(w)\,d\nu(w)$$

is the associated integral operator. Let $1<p<\infty$ with $1/p+1/q=1$. If there exists a positive function $h(z)$ and positive constants C_1 and C_2 such that

$$\int_\Omega H(z,w)(h(w))^q\,d\nu(w)\le C_1(h(z))^q,\qquad z\in\Omega,$$

$$\int_\Omega H(z,w)(h(z))^p\,d\nu(z)\le C_2(h(w))^p,\qquad w\in\Omega,$$

then the operator T is bounded on $L^p(\Omega,d\nu)$. Moreover, the norm of T on $L^p(\Omega,d\nu)$ does not exceed $C_1^{1/q}C_2^{1/p}$.

Proof. See [R], for example. ∎

LEMMA 2.12. *Suppose $1<p<\infty$. If $pt=2s$, then T_t is bounded on $L^p(\mathbb{H},\omega_s)$.*

Proof. Consider the positive function

$$h(z)=e^{\lambda|z|^2},\qquad z\in\mathbb{H},$$

where λ is a constant to be specified later. To evaluate

$$\int_{\mathbb{H}}|K_t(z,w)|h^q(w)\omega_s(w),$$

write

$$T_t f(z) = \int_{\mathbb{H}} H(z,w) f(w) \omega_s(w)$$

where

$$H(z,w) = (t/s)^{n-1} |K_t(z,w) e^{(s-t)|w|^2}|.$$

If

$$t - q\lambda > 0, \tag{2.3}$$

then

$$\int_{\mathbb{H}} H(z,w) h^q(w) \omega_s(w) = \left(\frac{t}{t-q\lambda}\right)^{n-1} F_{t-q\lambda}\left(\frac{t}{t-q\lambda} z\right).$$

So, it follows from Lemma 2.9 that

$$\int_{\mathbb{H}} H(z,w) h^q(w) \omega_s(w) \le C \left(\frac{t}{t-q\lambda}\right)^{n-1} e^{\frac{t^2}{4(t-q\lambda)}|z|^2}. \tag{2.4}$$

If we choose λ such that

$$\frac{t^2}{4(t-q\lambda)} = q\lambda, \tag{2.5}$$

then

$$\int_{\mathbb{H}} H(z,w) h^q(w) \omega_s(w) \le C \left(\frac{t}{t-q\lambda}\right)^{n-1} h^q(z). \tag{2.6}$$

On the other hand, if

$$s - p\lambda > 0, \tag{2.7}$$

write

$$\int_{\mathbb{H}} H(z,w) h^p(z) \omega_s(z) = \left(\frac{t}{s-p\lambda}\right)^{n-1} e^{(s-t)|w|^2} F_{s-p\lambda}\left(\frac{t}{s-p\lambda} w\right).$$

Then from Lemma 2.9 we have

$$\int_{\mathbb{H}} H(z,w) h^p(z) \omega_s(z) \le C \left(\frac{t}{s-p\lambda}\right)^{n-1} e^{[s-t+\frac{t^2}{4(s-p\lambda)}]|w|^2}.$$

Once again, if we choose λ so that

$$s - t + \frac{t^2}{4(s-p\lambda)} = p\lambda, \tag{2.8}$$

then

$$\int_{\mathbb{H}} H(z,w) h^p(z) \omega_s(z) \le C \left(\frac{t}{s-p\lambda}\right)^{n-1} h^p(w). \tag{2.9}$$

The conclusion now follows from Lemma 2.11. ■

3. Sharpness of the norm and proof of the results. In this section we compute the operator norm and prove Theorems A and B. We consider the operator U_0 defined in the introduction and let $U := CU_0$ be the operator defined on functions $\widetilde{f}$ on $\mathbb{C}^n$ by

$$U\widetilde{f}(z) := Cz_{n+1}\widetilde{f}(z_1, \dots, z_n)$$

for $z = (z_1, \dots, z_n, z_{n+1}) \in \mathbb{H}$, where

$$C = \left(\frac{(n-2)!m_n}{4(n+1)^2 s^{n-1}}\right)^{1/p}.$$

The operator U will play a key role in our proof. Indeed, we need the following:

Lemma 3.1. *For each $p \geq 1$ and $s > 0$, the linear operator U is a unitary isometry from $\widetilde{\mathcal{F}}_{p,s}(\mathbb{C}^n)$ onto $\mathcal{F}_{p,s}(\mathbb{H})$. Moreover, $U\widetilde{P}_s = P_s U$ on $\mathcal{F}_{p,s}(\mathbb{H})$.*

Proof. From Lemma 4.1 in [MY], we only need to prove that U is onto. To this end, it suffices to observe that

$$U(\widetilde{f}_{k,a}) = C(-1)^k f_{k,a} \quad \text{for all } k \text{ and } a. \ \blacksquare$$

As a consequence of the above result we have the following.

Lemma 3.2. *Suppose that $t, s > 0$ and $p \geq 1$. Then P_t is bounded on $\mathcal{F}_{p,s}(\mathbb{H})$ if only if $\widetilde{P}_t$ is bounded on $\widetilde{\mathcal{F}}_{p,s}(\mathbb{C}^n)$.*

Lemma 3.3. *Suppose $1 \leq p < \infty$ and $pt = 2s$. Then the linear operators $P_t : L^p(\mathbb{H}, \omega_s) \to \mathcal{A}^p_s(\mathbb{H})$ and $\widetilde{P}_t : L^p_s(\mathbb{C}^n) \to \mathcal{A}^p_s(\mathbb{C}^n)$ have the same norm*

$$\|\widetilde{P}_t\|_p = \|P_t\|_p = 2^{n-1}\sqrt{2e(n-1)!(n-1)}.$$

Proof. From the proof of Lemma 2.9 combined with an appropriate choice of λ, the constants in (2.6) and (2.9) both reduce to $2^{n-1}\sqrt{2e(n-1)!(n-1)}$. Therefore, by Lemma 2.11,

$$\|P_t\|_p \leq 2^{n-1}\sqrt{2e(n-1)!(n-1)}$$

as long as $1 < p < \infty$. The case $p = 1$ follows from Fubini's theorem and Lemma 2.9. Conversely, from the inequality

$$\frac{\|P_t f_{0,xt}\|_{p,s}}{\|f_{0,xt}\|_{p,s}} \leq \|P_t\|_p$$

where x is a large enough positive constant, we deduce by Lemma 2.3 that

$$\|P_t\|_p \geq 2^{n-1}\sqrt{2e(n-1)!(n-1)}.$$

By the estimates

$$\frac{\|P_t f_{0,xt}\|_{p,s}}{\|f_{0,xt}\|_{p,s}} \leq \|\widetilde{P}_t\|_p \leq \|P_t\|_p,$$

arising from the isometry U, we also have

$$\|\widetilde{P}_t\|_p = 2^{n-1}\sqrt{2e(n-1)!(n-1)}. \ \blacksquare$$

Proof of Theorem A. Suppose $p = 1$. That (a) implies (b) and (b) implies (c) is obvious. That (c) implies (d) follows from Lemma 2.4, and that (d) implies (a) can be seen from Fubini's theorem and Lemma 2.9. Now consider $1 < p < \infty$. That (a) implies (b) and (b) implies (c) is still obvious. That (c) implies (d) follows from Lemma 2.4, and that (d) implies (a) follows from Lemma 2.12. To complete the proof we appeal to Lemma 3.3. ■

Proof of Theorem B. This follows from Theorem A and Lemmas 3.2 and 3.3.

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