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# Real hypersurfaces with parallel induced almost contact structures

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**Abstract.** Real affine hypersurfaces of the complex space $\mathbb{C}^{n+1}$ with a $J$-tangent transversal vector field and an induced almost contact structure $(\varphi, \xi, \eta)$ are studied. Some properties of hypersurfaces with $\varphi$ or $\eta$ parallel relative to an induced connection are proved. Also a local characterization of these hypersurfaces is given.

**1. Introduction.** We study real affine hypersurfaces of the complex space $\mathbb{C}^{n+1}$ with a $J$-tangent transversal vector field $C$ and an induced almost contact structure $(\varphi, \xi, \eta)$. The main purpose of this paper is to investigate some properties of hypersurfaces with $\nabla\varphi = 0$ or $\nabla\eta = 0$, where $\nabla$ is an affine connection induced by a transversal vector field $C$.

In Section 2 we briefly recall basic formulas of affine differential geometry, we introduce the notion of a $J$-tangent transversal vector field and give a lemma relating to differential equations required in the next sections.

In Section 3 we recall some results obtained in [SS] for an induced almost contact structure and show how induced almost contact structures are related to each other in case the $J$-tangent transversal vector field changes.

Section 4 contains the main results of this paper. In particular, we prove some properties of induced objects under the condition $\nabla\varphi = 0$ as well as $\nabla\eta = 0$. Moreover, we prove that the existence of a $J$-tangent transversal vector field $\varphi$ with $\nabla\varphi = 0$ is equivalent to the existence of a $J$-tangent transversal vector field $\eta$ with $\nabla\eta = 0$. At the end we give a local characterization of such hypersurfaces.

Throughout the paper we write $\alpha \equiv 0$ if $\alpha(x) = 0$ for all $x \in M$, and $\alpha \neq 0$ if $\alpha(x) \neq 0$ for every $x \in M$ (i.e. $\alpha$ is a nowhere vanishing function on $M$).

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**2. Preliminaries.** We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [NS]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable, connected differentiable $n$-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field $C$ we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X,Y)C, \tag{2.1}$$

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where $X, Y$ are vector fields tangent to $M$. For any transversal vector field, $\nabla$ is a torsion-free connection, $h$ is a symmetric bilinear form on $M$, called the *second fundamental form*, $S$ is a tensor of type $(1,1)$, called the *shape operator*, and $\tau$ is a 1-form, called the *transversal connection form*.

We shall now consider the change of a transversal vector field for a given immersion $f$.

Theorem 2.1 ([NS]). *Suppose we change a transversal vector field $C$ to*

$$\bar{C} = \Phi C + f_*(Z),$$

*where $Z$ is a tangent vector field on $M$ and $\Phi$ is a nowhere vanishing function on $M$. Then the affine fundamental form, the induced connection, the transversal connection form, and the affine shape operator change as follows:*

$$\begin{aligned}
\bar{h} &= \frac{1}{\Phi} h,\\
\bar{\nabla}_X Y &= \nabla_X Y - \frac{1}{\Phi} h(X,Y)Z,\\
\bar{\tau} &= \tau + \frac{1}{\Phi} h(Z,\cdot) + d\ln|\Phi|,\\
\bar{S} &= \Phi S - \nabla_{.} Z + \bar{\tau}(\cdot)Z.
\end{aligned}$$

If $h$ is non-degenerate, then we say that the hypersurface or the hypersurface immersion is *non-degenerate*. We have the following

Theorem 2.2 ([NS, §II.2, Theorem 2.1]). *For an arbitrary transversal vector field $C$ the induced connection $\nabla$, the second fundamental form $h$, the shape operator $S$, and the 1-form $\tau$ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.6}$$

Equations (2.3), (2.4), (2.5), and (2.6) are called, respectively, the equation of *Gauss*, *Codazzi for $h$*, *Codazzi for $S$ and Ricci*.

For a hypersurface immersion $f\colon M \to \mathbb{R}^{n+1}$ a transversal vector field $C$ is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$).

Let $\dim M = 2n+1$ and $f\colon (M,g) \to (\mathbb{R}^{2n+2}, \tilde{g})$ be a non-degenerate (relative to the second fundamental form) isometric immersion, where $\tilde{g}$ is the standard inner product on $\mathbb{R}^{2n+2}$. We assume that $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ is endowed with the standard complex structure $J$,

$$J(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = (-y_1, \dots, -y_{n+1}, x_1, \dots, x_{n+1}).$$

Let $C$ be a transversal vector field on $M$. We say that $C$ is *$J$-tangent* if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on $M$ as the biggest $J$-invariant distribution on $M$, that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap J(f_*(T_xM)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field $X$ is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields. We say that the distribution $\mathcal{D}$ is non-degenerate if $h$ is non-degenerate on $\mathcal{D}$. To simplify the writing, we will omit $f_*$ in front of vector fields in most cases.

We conclude this section with the following useful lemma relating to differential equations (we also give the proof for completeness):

LEMMA 2.3 ([S]). *Let $F\colon I \to \mathbb{R}^{2n}$ be a smooth function on the interval $I$ and let $\alpha, \beta \in C^\infty(I, \mathbb{R})$ be such that $\alpha^2 + \beta^2 \neq 0$ on $I$. If $F$ satisfies the differential equation*

$$F'(y) = -\alpha(y) JF(y) + \beta(y) F(y), \tag{2.7}$$

*then $F$ is of the form*

$$F(y) = Jve^{\hat{\beta}(y)} \cos(\hat{\alpha}(y)) + ve^{\hat{\beta}(y)} \sin(\hat{\alpha}(y)), \tag{2.8}$$

*where $v \in \mathbb{R}^{2n}$ and $\hat{\alpha}$, $\hat{\beta}$ are any integrals of $\alpha$ and $\beta$ on $I$, respectively.*

*Proof.* It is easily seen that functions of the form (2.8) satisfy the differential equation (2.7). On the other hand, since (2.7) is a first order ordinary differential equation, the Picard–Lindelöf theorem implies that any solution of (2.7) must be of the form (2.8). ∎

**3. Almost contact structures.** A $(2n+1)$-dimensional manifold $M$ is said to have an *almost contact structure* if there exist on $M$ a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1$$

for every $X \in TM$.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a hypersurface with a $J$-tangent transversal vector field $C$. Then we can define a vector field $\xi$, a 1-form $\eta$ and a tensor field $\varphi$ of type $(1,1)$ as follows:

$$\xi := JC, \quad \eta|_{\mathcal{D}} = 0 \quad \text{and} \quad \eta(\xi) = 1, \quad \varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \quad \text{and} \quad \varphi(\xi) = 0.$$

It is easy to see that $(\varphi, \xi, \eta)$ is an almost contact structure on $M$; it is said to be *induced* by $C$.

For an induced almost contact structure we have the following theorem:

THEOREM 3.1 ([SS]). *If $(\varphi, \xi, \eta)$ is an induced almost contact structure on $M$ then*

$$\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.1}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y)SX - h(X, Y)\xi, \tag{3.2}$$

$$\begin{aligned}\eta([X, Y]) = &-h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\ &+ \eta(Y)\tau(X) - \eta(X)\tau(Y),\end{aligned} \tag{3.3}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX, \tag{3.4}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.5}$$

$$\eta(SX) = h(X, \xi), \tag{3.6}$$

*for all $X, Y \in \mathcal{X}(M)$.*

LEMMA 3.2. *Let $C$ be a $J$-tangent transversal vector field. Then any other $J$-tangent transversal vector field $\bar{C}$ has the form*

$$\bar{C} = \phi C + f_* Z,$$

*where $\phi \neq 0$ and $Z \in \mathcal{D}$. Moreover, if $(\varphi, \xi, \eta)$ is the almost contact structure induced by $C$, then $\bar{C}$ induces the almost contact structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta})$, where*

$$\bar{\xi} = \phi\xi + \varphi Z, \qquad \bar{\eta} = \frac{1}{\phi}\eta, \qquad \bar{\varphi} = \varphi + \eta(\cdot)\frac{1}{\phi}Z.$$

*Proof.* Since $Z \in \mathcal{D}$ and $J = \varphi$ on $\mathcal{D}$, we have

$$\bar{\xi} = J\bar{C} = J(\phi C + f_* Z) = \phi JC + \varphi Z = \phi\xi + \varphi Z.$$

Directly from the definition of $\eta$ and $\bar{\eta}$ we get $\eta = \bar{\eta}$ on $\mathcal{D}$ and

$$\eta(\xi) = 1 = \bar{\eta}(\bar{\xi}) = \bar{\eta}(\phi\xi + \varphi Z) = \phi\bar{\eta}(\xi),$$

thus

$$\bar{\eta}(\xi) = \frac{1}{\phi}\eta(\xi),$$

and finally $\bar{\eta} = \frac{1}{\phi}\eta$. To prove the last equality of the statement, note that

$$0 = \varphi(\xi) = \bar{\varphi}(\bar{\xi}) = \bar{\varphi}(\phi\xi + \varphi Z) = \phi\bar{\varphi}(\xi) + \bar{\varphi}\varphi(Z).$$

From the definition of $\varphi$ and $\bar{\varphi}$ we have $\varphi = \bar{\varphi}$ on $\mathcal{D}$, which implies that

$$\bar{\varphi}(\xi) = \frac{1}{\phi}Z = \varphi(\xi) + \eta(\xi)\frac{1}{\phi}Z,$$

since $Z \in \mathcal{D}$. The last formula proves that

$$\bar{\varphi}(X) = \varphi(X) + \eta(X)\frac{1}{\phi}Z$$

is valid for $X = \xi$. Clearly, it is also valid for every $X \in \mathcal{D}$, and thus for every $X \in TM$. ■

**4. Parallel induced almost contact structures.** In this section we always assume that $(\varphi, \xi, \eta)$ is an almost contact structure induced by a $J$-tangent transversal vector field $C$. It is important to note that we do not assume that the second fundamental form $h$ is non-degenerate.

LEMMA 4.1. *Let $(\varphi, \xi, \eta)$ be an induced almost contact structure such that $\nabla\varphi = 0$. Then*

$$h|_{\mathcal{D}\times\mathcal{D}} = 0, \tag{4.1}$$

$$h(\xi, X) = h(X, \xi) = 0 \quad \textit{for all } X \in \mathcal{D}, \tag{4.2}$$

$$S|_{\mathcal{D}} = 0, \tag{4.3}$$

$$S\xi = h(\xi, \xi)\xi, \tag{4.4}$$

$$d\tau = 0. \tag{4.5}$$

*Proof.* From formula (3.2) we have

$$(\nabla_X\varphi)(Y) = -\eta(Y)SX + h(X, Y)\xi$$

for all $X, Y \in \mathcal{X}(M)$. Since $\nabla\varphi = 0$ we get $h(X, Y) = 0$ and $h(\xi, Y) = 0$ for all $X, Y \in \mathcal{D}$. Now, taking $X \in \mathcal{D}$ and $Y = \xi$ we have $SX = 0$. Taking $X = Y = \xi$ we easily get $S\xi = h(\xi, \xi)\xi$. The last equation follows immediately from the Ricci equation (2.6). ■

The above lemma implies that if $\nabla\varphi = 0$, then $C$ is a locally equiaffine transversal vector field, so locally we can find a nowhere vanishing function $\Phi$ such that $\bar{C} = \Phi C$ is an equiaffine $J$-tangent vector field. Now, using Theorem 2.1 and Lemma 3.2 we get the following corollary:

COROLLARY 4.2. *Let $C$ be a $J$-tangent transversal vector field such that $\nabla\varphi = 0$ and let $\Phi$ be a nowhere vanishing function on $M$. Denote by $\bar{C}$ the transversal vector field $\Phi C$. Then $\bar{\nabla}\bar{\varphi} = 0$. Thus, parallelism of $\varphi$ relative to $\nabla$ is the direction property. In particular, locally we can always choose $C$ equiaffine.*

We shall prove

LEMMA 4.3. *Let $(\varphi, \xi, \eta)$ be an induced almost contact structure such that $\nabla\eta = 0$. Then*

$$h|_{\mathcal{D}\times\mathcal{D}} = 0, \tag{4.6}$$

$$h(\xi, X) = h(X, \xi) = 0 \quad \textit{for every } X \in \mathcal{D}, \tag{4.7}$$

| | | |
|---|---|---|
| (4.8) | $\tau = 0,$ | |
| (4.9) | $\nabla_X Y \in \mathcal{D}$ | *for all* $X, Y \in \mathcal{D}$, |
| (4.10) | $\nabla_X \xi \in \mathcal{D}$ | *for every* $X \in \mathcal{X}(M)$, |
| (4.11) | $\nabla_\xi X \in \mathcal{D}$ | *for every* $X \in \mathcal{D}$, |
| (4.12) | $X(h(\xi, \xi)) = 0$ | *for every* $X \in \mathcal{D}$. |

*Proof.* Since $\nabla \eta = 0$ we have

$$\eta(\nabla_X Y) = X(\eta(Y)) \tag{4.13}$$

for all $X, Y \in \mathcal{X}(M)$. Now, using formula (3.1) we get

$$h(X, \varphi Y) = \eta(Y)\tau(X) \tag{4.14}$$

for all $X, Y \in \mathcal{X}(M)$. Hence, if $X, Y \in \mathcal{D}$, then $h(X, \varphi Y) = 0$, which proves (4.6). Taking $X = \xi$ and $Y \in \mathcal{D}$ in (4.14) we easily get (4.7). On the other hand, taking $Y = \xi$ we have $\tau(X) = 0$, that is, (4.8). Formulas (4.9)–(4.11) can be obtained directly from (4.13). To prove (4.12) note that from the Codazzi equation (2.4) for $h$ (and using (4.8)) we have

$$(\nabla_X h)(\xi, \xi) = (\nabla_\xi h)(X, \xi) = \xi(h(X, \xi)) - h(\nabla_\xi X, \xi) - h(X, \nabla_\xi \xi).$$

Now, if we take $X \in \mathcal{D}$ then because of (4.6)–(4.7) we get $h(X, \xi) = 0$ and $h(X, \nabla_\xi \xi) = 0$, whereas (4.11) implies that also $h(\nabla_\xi X, \xi) = 0$. Thus, we obtain

$$0 = (\nabla_X h)(\xi, \xi) = X(h(\xi, \xi)) - 2h(\nabla_X \xi, \xi)$$

for every $X \in \mathcal{D}$. Now, using (4.10) in the above formula leads to

$$X(h(\xi, \xi)) = 0$$

for every $X \in \mathcal{D}$. This finishes the proof of (4.12). ∎

Denote by $N$ the metric normal field for $f \colon M \to \mathbb{R}^{2n+2}$ (relative to the standard inner product on $\mathbb{R}^{2n+2}$). The metric normal field induces objects $\widehat{\nabla}$, $\widehat{h}$ and $\widehat{S}$ as the transversal vector field on $M$. Recall that the induced connection $\widehat{\nabla}$ is the Levi-Civita connection of the induced Riemannian metric $g$. It is clear that $N$ is $J$-tangent, thus induces an almost contact structure $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ on $M$.

THEOREM 4.4. *Let* $f \colon M \to \mathbb{R}^{2n+2}$ *be an affine immersion. Then the following conditions are equivalent:*

(1) *For every point on* $M$ *there exist a neighborhood* $U$ *and a* $J$*-tangent transversal vector field* $C$ *defined on* $U$ *such that* $\nabla \varphi = 0$.
(2) *For every point on* $M$ *there exist a neighborhood* $U$ *and a* $J$*-tangent transversal vector field* $C$ *defined on* $U$ *such that* $\nabla \eta = 0$.
(3) *An induced almost contact structure* $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ *is* $\widehat{\nabla}$*-parallel.*

*Proof.* Let $x$ be any point on $M$. Assume that in some neighborhood $U$ of $x$ there exists a $J$-tangent transversal vector field $C$ such that $\nabla\varphi = 0$. Then, by virtue of Corollary 4.2 we can assume (possibly shrinking $U$) that $C$ is equiaffine. Now, by Theorem 3.1 (formula (3.1)) we get

$$(\nabla_X \eta)(Y) = h(X, \varphi Y) - \eta(Y)\tau(X) = h(X, \varphi Y)$$

for all $X, Y \in \mathcal{X}(U)$. Using the first two formulas from Lemma 4.1 we get

$$\nabla\eta \equiv 0,$$

which proves the implication (1)⇒(2).

To prove (2)⇒(3) note that if $(\varphi, \xi, \eta)$ is an almost contact structure induced by a $J$-tangent transversal vector field $C$ defined on some neighborhood $U$ of $x$ and such that $\nabla\eta = 0$ then

$$\widehat{N}|_U = \varPhi\xi + \varphi Z,$$

where $Z \in \mathcal{D}$ and $\varPhi = \text{const}$. Also note that the condition $\nabla\eta = 0$ is invariant under scaling the field $C$ by a constant. Therefore, we can later assume that $C$ is chosen in such a way that

$$\widehat{N}|_U = \xi + \varphi Z.$$

By Theorem 2.1 and Lemma 3.2 we obtain $\widehat{h} = h$ and $\widehat{\eta} = \eta$. Since $N$ is the metric normal field we see that $g$, $\widehat{h} = h$ and $\widehat{S}$ are related by the formula

$$h(X, Y) = g(\widehat{S}X, Y)$$

for all $X, Y \in \mathcal{X}(U)$. The above equality and Lemma 4.3 imply

$$\widehat{S}X = h(\widehat{N}, X)\widehat{N}$$

for every $X \in \mathcal{X}(U)$. Now, using (3.2) and (3.5) for the structure $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ we easily get

$$\widehat{\varphi}(\widehat{\nabla}_X \widehat{N}) = \widehat{S}X - h(\widehat{N}, X)\widehat{N} = 0 \quad \text{and} \quad \widehat{\eta}(\widehat{\nabla}_X \widehat{N}) = 0$$

for every $X \in \mathcal{X}(U)$, that is, $\widehat{\nabla}_X \widehat{N} = 0$ for every $X \in \mathcal{X}(U)$. Lemma 4.3 implies that

$$\begin{aligned}(\widehat{\nabla}_X \widehat{\varphi})(Y) &= \widehat{h}(X, Y)\widehat{N} - \widehat{\eta}(Y)\widehat{S}X = h(X, Y)\widehat{N} - \eta(Y)h(\widehat{N}, X)\widehat{N} \\ &= (h(X, Y) - \eta(Y)h(\xi, X))\widehat{N} = 0\end{aligned}$$

for all $X, Y \in \mathcal{X}(U)$. Arbitrariness of $x \in M$ gives $\widehat{\nabla}\widehat{N} = 0$ and $\widehat{\nabla}\widehat{\varphi} = 0$ on the whole $M$. The condition $\widehat{\nabla}\widehat{\eta} = 0$ can easily be obtained from the equality $\widehat{\nabla}\widehat{\varphi} = 0$, the fact that $N$ is equiaffine and the proof of (1)⇒(2).

To prove (3)⇒(1) it is sufficient to take $C := N$. ∎

From the proof of Theorem 4.4 it follows that if there exists an equiaffine $J$-tangent transversal vector field $C$ with $\nabla\varphi = 0$, then we also have $\nabla\eta = 0$ for $C$. Moreover, condition (3) in the above theorem is equivalent to the global versions of conditions (1) and (2), that is,

(1′) There exists a $J$-tangent transversal vector field $C$ on $M$ such that $\nabla\varphi = 0$.
(2′) There exists a $J$-tangent transversal vector field $C$ on $M$ such that $\nabla\eta = 0$.

It follows from Lemmas 4.1 and 4.3 that rank $f \leq 1$. However, the converse is not true in general since we have the following

EXAMPLE 4.5. Let us consider an affine immersion defined as follows:

$$f\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \begin{bmatrix} x \\ y \\ z \\ e^z \end{bmatrix} \in \mathbb{R}^4.$$

Of course rank $f = 1$. Let $\{\partial_1, \partial_2, \partial_3\}$ be the canonical basis on $\mathbb{R}^3$ generated by the coordinate system $(x,y,z)$ on $\mathbb{R}^3$. It is not difficult to see that

$$N\colon \mathbb{R}^3 \ni (x,y,z) \mapsto \begin{bmatrix} 0 \\ 0 \\ -\frac{e^z}{\sqrt{e^{2z}+1}} \\ \frac{1}{\sqrt{e^{2z}+1}} \end{bmatrix} \in \mathbb{R}^4$$

is the metric normal field for $f$. Now,

$$\widehat{N} = \begin{bmatrix} \frac{e^z}{\sqrt{e^{2z}+1}} \\ -\frac{1}{\sqrt{e^{2z}+1}} \\ 0 \\ 0 \end{bmatrix}.$$

The above implies that

$$f_*(\partial_3) = f_z = \begin{bmatrix} 0 \\ 0 \\ 1 \\ e^z \end{bmatrix}$$

is orthogonal to $\widehat{N}$, thus it belongs to the distribution $\mathcal{D}$. We will show that $(\widehat{\nabla}_{\partial_3}\widehat{\varphi})(\partial_3) \neq 0$. By straightforward computations we get

$$\widehat{\nabla}_{\partial_3}\partial_3 = \frac{e^{2z}}{e^{2z}+1}\partial_3 \quad \text{and} \quad \widehat{\varphi}(\partial_3) = -\partial_1 - e^z\partial_2.$$

Now

$$\begin{aligned}
(\widehat{\nabla}_{\partial_3}\widehat{\varphi})(\partial_3) &= \widehat{\nabla}_{\partial_3}(\widehat{\varphi}(\partial_3)) - \widehat{\varphi}(\widehat{\nabla}_{\partial_3}\partial_3) \\
&= \widehat{\nabla}_{\partial_3}(-\partial_1 - e^z\partial_2) - \widehat{\varphi}\left(\frac{e^{2z}}{e^{2z}+1}\partial_3\right) \\
&= -\widehat{\nabla}_{\partial_3}\partial_1 - e^z\widehat{\nabla}_{\partial_3}\partial_2 - \partial_3(e^z)\partial_2 + \frac{e^{2z}}{e^{2z}+1}\partial_1 + \frac{e^{3z}}{e^{2z}+1}\partial_2 \\
&= \frac{e^{2z}}{e^{2z}+1}\partial_1 + \left(\frac{e^{3z}}{e^{2z}+1} - e^z\right)\partial_2 \neq 0,
\end{aligned}$$

since $\widehat{\nabla}_{\partial_3}\partial_1 = \widehat{\nabla}_{\partial_3}\partial_2 = 0$ and $\partial_1, \partial_2$ are linearly independent.

In later parts of this paper we will give a local characterization of affine hypersurfaces satisfying any (thus all) of the conditions from Theorem 4.4. We need the following lemma:

Lemma 4.6. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a hypersurface with a metric normal field $N$. Assume that an almost contact structure $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ induced by $N$ is $\widehat{\nabla}$-parallel. Then, for every point $x$ of $M$ and for any nowhere vanishing smooth function $\alpha$ defined in some neighborhood of $x$ and constant in the direction of $\mathcal{D}$ (i.e. $X(\alpha) = 0$ for every $X \in \mathcal{D}$), there exist a neighborhood of $x$ and a map $\psi(y, x_1, \ldots, x_{2n})$ defined on this neighborhood such that the vector fields $\partial/\partial y, \partial/\partial x_1, \ldots, \partial/\partial x_{2n}$ satisfy*

$$\frac{\partial}{\partial y} = \alpha\widehat{N} \quad \text{and} \quad \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}} \in \mathcal{D}.$$

*Proof.* Since $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ is $\widehat{\nabla}$-parallel, in particular (by (4.9)) the distribution $\mathcal{D}$ is involutive. Let $x$ be any point on $M$ and let $\alpha$ be a nowhere vanishing smooth function defined in some neighborhood of $x$ and constant in the direction of $\mathcal{D}$. The Frobenius theorem implies that for $x$ there exist an open neighborhood $U \subset M$ and linearly independent vector fields $X_1, \ldots, X_{2n}, X_{2n+1} = \alpha\widehat{N} \in \mathcal{X}(U)$ such that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, 2n+1$. For every $i = 1, \ldots, 2n$ we have

$$X_i = D_i + \alpha_i\widehat{N},$$

where $D_i \in \mathcal{D}$ and $\alpha_i \in C^\infty(U)$. Thus

$$0 = [X_i, X_{2n+1}] = [D_i, X_{2n+1}] - X_{2n+1}(\alpha_i)\widehat{N}. \tag{4.15}$$

From (4.10) and (4.11) it is clear that $[D_i, \widehat{N}] \in \mathcal{D}$. Since $D_i(\alpha) = 0$ we also have

$$[D_i, X_{2n+1}] = \alpha[D_i, \widehat{N}] + D_i(\alpha)\widehat{N} = \alpha[D_i, \widehat{N}] \in \mathcal{D}.$$

Now (4.15) implies that $[D_i, X_{2n+1}] = 0$ and $X_{2n+1}(\alpha_i) = 0$ for $i = 1, \ldots, 2n$. Moreover, for all $i, j = 1, \ldots, 2n$ we have

$$[D_i, D_j] = [X_i, X_j] - [\alpha_i\widehat{N}, X_j] - [X_i, \alpha_j\widehat{N}] + [\alpha_i\widehat{N}, \alpha_j\widehat{N}].$$

Since $[X_i, X_j] = 0$, $\mathcal{D}$ is involutive and the last three terms in the above equality are proportional to $\widehat{N}$, we obtain

$$[D_i, D_j] = 0$$

for all $i, j = 1, \ldots, 2n$. Of course the vector fields $D_1, \ldots, D_{2n}, X_{2n+1}$ are linearly independent over $C^\infty(U)$, so we can find a map $\psi(y, x_1, \ldots, x_{2n})$ on $U$ such that $\partial/\partial y = X_{2n+1}$ and $\partial/\partial x_i = D_i$ for $i = 1, \ldots, 2n$. ∎

In the next two theorems we give a local characterization of hypersurfaces for which there exists a $J$-tangent transversal vector field inducing an almost contact structure $(\varphi, \xi, \eta)$ such that $\nabla\varphi = 0$ or $\nabla\eta = 0$.

THEOREM 4.7. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a hypersurface such that the almost contact structure $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ is $\widehat{\nabla}$-parallel. Let $U$ be a non-empty open subset of $M$. If* $\operatorname{rank} f = 0$ *on $U$ then $f(U)$ is a piece of a hyperplane.*

*Proof.* Since $\operatorname{rank} \widehat{h} = 0$ and $\widehat{\nabla}\widehat{\varphi} = 0$ on $U$, Lemma 4.1 implies

$$D_X N = -\widehat{S}X = 0$$

for every $X \in \mathcal{X}(U)$. It follows that a metric normal field $N$ is constant on $U$, thus $f(U)$ is a hyperplane in $\mathbb{R}^{2n+2}$. ∎

THEOREM 4.8. *Let $f\colon M \to \mathbb{R}^{2n+2}$ be a hypersurface such that the almost contact structure $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$ is $\widehat{\nabla}$-parallel. Let $x$ be a point on $M$ such that* $\operatorname{rank} f = 1$ *at $x$. Then there exists an open neighborhood $U$ of $x$ such that $f$ can be expressed on $U$ in the form*

$$f(x_1, \ldots, x_{2n}, y) = x_1 b_1 + \cdots + x_{2n} b_{2n} - v\int \alpha(y)\cos y\, dy + Jv\int \alpha(y)\sin y\, dy, \tag{4.16}$$

*where $v \in \mathbb{R}^{2n+2}$, $\|v\| = 1$, $\alpha$ is some nowhere vanishing smooth function on $U$ and $b_1, \ldots, b_{2n} \in \mathbb{R}^{2n+2}$ are linearly independent vectors from $\{v, Jv\}^\perp$. Moreover, every hypersurface* (4.16) *has a $\widehat{\nabla}$-parallel almost contact structure $(\widehat{\varphi}, \widehat{N}, \widehat{\eta})$.*

*Proof.* First, note that since $\operatorname{rank} \widehat{h}_x = 1$, we have $\widehat{h}_x(\widehat{N}_x, \widehat{N}_x) \neq 0$. Since $\widehat{h}(\widehat{N}, \widehat{N})$ is smooth we can find a neighborhood $U$ of $x$ such that $\widehat{h}(\widehat{N}, \widehat{N}) \neq 0$ on $U$, thus $\operatorname{rank} \widehat{h} = 1$ on $U$. Moreover, by (4.12) the function $\widehat{h}(\widehat{N}, \widehat{N})$ is constant in a direction of the distribution $\mathcal{D}$.

Let us define a new function on $U$,

$$\alpha := \frac{1}{\widehat{h}(\widehat{N}, \widehat{N})}.$$

It is clear that $\alpha \neq 0$ and $\alpha$ is constant in a direction of $\mathcal{D}$. Using Lemma 4.6 and possibly shrinking $U$ we deduce that there exists a map $\psi$ on $U$ such that

$$\frac{\partial}{\partial y} = \alpha \widehat{N} \quad \text{and} \quad \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}} \in \mathcal{D}.$$

By the Weingarten formula (2.2) and formulas (4.3), (4.4) we get

$$D_{\partial/\partial x_i} N = -\widehat{S}\left(\frac{\partial}{\partial x_i}\right) = 0$$

for $i = 1, \ldots, 2n$ and

$$D_{\partial/\partial y} N = -\widehat{S}\left(\frac{\partial}{\partial y}\right) = -\alpha \widehat{S}(\widehat{N}) = -\alpha \widehat{h}(\widehat{N}, \widehat{N})\widehat{N} = -\widehat{N} = -JN,$$

thus $N_{x_i} = 0$ for $i = 1, \ldots, 2n$ and $N_y = -JN$. Now, Lemma 2.3 implies that

$$N = Jv \cos y + v \sin y,$$

where $v \in \mathbb{R}^{2n+2}$. Since $N$ is a metric normal field, we see that

$$1 = \|N\| = \|Jv \cos y + v \sin y\| = \|v\|.$$

Let $b_1, \ldots, b_{2n}$ be any linearly independent vectors from $\mathbb{R}^{2n+2}$ such that $b_i \in \{v, Jv\}^{\perp}$. We have

$$N \cdot b_i = 0 \quad \text{and} \quad \widehat{N} \cdot b_i = 0,$$

for every $i = 1, \ldots, 2n$, therefore the vectors $b_1, \ldots, b_{2n}$ span $f_*(\mathcal{D})$. Let $\partial_1, \ldots, \partial_{2n}$ be vector fields on $U$ such that $f_*(\partial_i) = b_i$ for $i = 1, \ldots, 2n$. Of course $\partial_1, \ldots, \partial_{2n}$ are linearly independent and span the distribution $\mathcal{D}$. For every $X \in TU$ and for every $i = 1, \ldots, 2n$ we have

$$D_X f_* \partial_i = D_X b_i = 0.$$

On the other hand by the Gauss formula (2.1) and due to the fact that $\widehat{h}|_{\mathcal{D}\times\mathcal{D}} = 0$ we obtain

$$D_X f_* \partial_i = f_*(\widehat{\nabla}_X \partial_i),$$

thus

$$\widehat{\nabla}_X \partial_i = 0$$

for every $X \in TU$. In particular, we have

$$\widehat{\nabla}_{\partial_i} \partial_j = 0$$

for all $i, j \in \{1, \ldots, 2n\}$ and

$$\widehat{\nabla}_{\partial/\partial y} \partial_i = 0$$

for $i = 1, \ldots, 2n$. Moreover

$$\widehat{\nabla}_{\partial_i} \frac{\partial}{\partial y} = \widehat{\nabla}_{\partial_i}(\alpha \widehat{N}) = \partial_i(\alpha)\widehat{N} + \alpha \widehat{\nabla}_{\partial_i} \widehat{N} = 0,$$

since $\alpha$ is constant in a direction of $\mathcal{D}$ and $\widehat{\nabla}\widehat{N} = 0$. To sum up, the vector fields

$$\partial_1, \ldots, \partial_{2n}, \frac{\partial}{\partial y}$$

are associated with some map $\tilde{\psi}$. Denoting again $\partial_1, \ldots, \partial_{2n}$ by $\partial/\partial x_1, \ldots$ $\ldots, \partial/\partial x_{2n}$ we see that the immersion $f$ satisfies the differential equations

$$f_{x_i} = b_i$$

for $i = 1, \ldots, 2n$ and

$$f_y = \alpha(y)\widehat{N} = \alpha(y)(-v \cos y + Jv \sin y).$$

Solving the above we get a local form of $f$ as follows:

$$f(x_1, \ldots, x_{2n}, y) = x_1 b_1 + \cdots + x_{2n} b_{2n} - v \int \alpha(y) \cos y \, dy + Jv \int \alpha(y) \sin y \, dy.$$

To prove the second part of the theorem note that the function described by (4.16) is an immersion, since $b_1, \ldots, b_{2n}$ and $-v\alpha(y)\cos y + Jv\alpha(y)\sin y$ are linearly independent. Now, it is enough to show that $\widehat{\nabla}\widehat{\eta} = 0$. It is not difficult to see that $N = Jv \cos y + v \sin y$, thus

$$\widehat{N} = -v \cos y + Jv \sin y;$$

moreover $\partial/\partial x_i \in \mathcal{D}$ and $\widehat{\nabla}_X(\partial/\partial x_i) = 0$ for $i = 1, \ldots, 2n$, which imply $(\widehat{\nabla}_X \widehat{\eta})(Y) = 0$ for all $X \in TM$ and $Y \in \mathcal{D}$. To complete the proof note that

$$(\widehat{\nabla}_X \widehat{\eta})(\widehat{N}) = X(\widehat{\eta}(\widehat{N})) - \widehat{\eta}(\widehat{\nabla}_X \widehat{N}) = -\widehat{\eta}(\widehat{\nabla}_X \widehat{N})$$

for every $X \in TM$. If $X \in \mathcal{D}$ then $\widehat{\nabla}_X \widehat{N} = 0$, because

$$D_X \widehat{N} = D_X(-v \cos y + Jv \sin y) = 0$$

for every $X \in \mathcal{D}$. If $X = \partial/\partial y$ then

$$D_{\partial/\partial y} \widehat{N} = Jv \cos y + v \sin y = N,$$

thus $\widehat{\nabla}_{\partial/\partial y} \widehat{N} = 0$. Summarizing, we have shown that $\widehat{\nabla}\widehat{\eta} = 0$, which completes the proof. ■

## References


[B] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, 2nd ed., Progr. Math. 203, Birkhäuser Boston, 2010.

[C] V. Cruceanu, *Real hypersurfaces in complex centro-affine spaces*, Results Math. 13 (1988), 224–234.

[NS] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge Univ. Press, 1994.

[SS] M. Szancer and Z. Szancer, *Real hypersurfaces with an induced almost contact structure*, Colloq. Math. 114 (2009), 41–51.

[S] Z. Szancer, *Real hypersurfaces with a special transversal vector field*, submitted.

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