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Analytic solutions of a second-order iterative functional differential equation near resonance

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Abstract. We study existence of analytic solutions of a second-order iterative functional differential equation

$$x''(z) = \sum_{j=0}^{k} \sum_{t=1}^{\infty} C_{t,j}(z)(x^{[j]}(z))^t + G(z)$$

in the complex field $\mathbb{C}$. By constructing an invertible analytic solution $y(z)$ of an auxiliary equation of the form

$$\alpha^2 y''(\alpha z)y'(z) = \alpha y'(\alpha z)y''(z) + [y'(z)]^3 \Big[\sum_{j=0}^{k} \sum_{t=1}^{\infty} C_{t,j}(y(z))(y(\alpha^j z))^t + G(y(z))\Big]$$

invertible analytic solutions of the form $y(\alpha y^{-1}(z))$ for the original equation are obtained. Besides the hyperbolic case $0 < |\alpha| < 1$, we focus on α on the unit circle S^1, i.e., $|\alpha| = 1$. We discuss not only those α at resonance, i.e. at a root of unity, but also near resonance under the Brjuno condition.

1. Introduction. Delay differential equations or more generally functional differential equations have been studied rather extensively in the past forty years and are used as models to describe many physical and biological systems. For example, delay differential equations of the form

$$x''(z) = f(z, x(z), x(z - \tau_1(z)), \ldots, x(z - \tau_k(z))) \tag{1.1}$$

have been extensively studied in [6], [1]. However, equations where the delay functions $\tau_j(z)$ $(j = 0, 1, \ldots, k)$ depend not only on the argument of the unknown function but also on the state, $\tau_j(z) = \tau_j(z, x(z))$, have been investigated not so much. In 1965, Petahov [9] studied the existence of solutions of the second-order equation

$$x''(z) = ax(x(z)).$$

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For the study of analytic solutions to this class of second-order equations, we refer to [10]–[14].

In this paper, we will discuss the existence of invertible analytic solutions to a functional differential equation of the form

$$x''(z) = \sum_{j=0}^{k} \sum_{t=1}^{\infty} C_{t,j}(z)(x^{[j]}(z))^t + G(z) \tag{1.2}$$

in the complex field, where $x^{[j]}(z)$ denotes the jth iterate of $x(z)$. The above equation is a special case of (1.1), with

$$f(z, y_1, \dots, y_k) = \sum_{t=1}^{\infty} C_{t,0}(z)z^t + \sum_{j=1}^{k} \sum_{t=1}^{\infty} C_{t,j}(z)y_j^t + G(z)$$

and $\tau_j(z) = z - x^{[j-1]}(z)$. A distinctive feature of (1.2) is to include the sum of infinitely many terms in contrast to the previously considered equations [10]–[14].

Throughout this paper, we will assume that

(**H**) the functions $C_{t,j}(z)$ ($t \in \mathbb{N}$, $j = 0, 1, \dots, k$) and $G(z)$ are all analytic in $|z| < \sigma$ ($\sigma > 0$), and for each $j = 0, 1, \dots, k$, the series $\sum_{t=1}^{\infty} C_{t,j}(z_1)z_2^t$ converges for every pair (z_1, z_2) of nonzero complex numbers with $|z_1| < \sigma$.

We need the convergence of the series in (**H**) so that (1.2) is meaningful.

As in our previous works [10]–[14], by means of $x(z) = y(\alpha y^{-1}(z))$, sometimes called the Schröder transformation, we reduce (1.2) to the auxiliary equation

$$\begin{aligned} &\alpha^2 y''(\alpha z)y'(z) \\ &\quad = \alpha y'(\alpha z)y''(z) + [y'(z)]^3 \Big[\sum_{j=0}^{k} \sum_{t=1}^{\infty} C_{t,j}(y(z))(y(\alpha^j z))^t + G(y(z)) \Big]. \end{aligned} \tag{1.3}$$

By constructing a convergent power series solution $y(z)$ of (1.3), invertible analytic solutions of the form $y(\alpha y^{-1}(z))$ for (1.2) are obtained. As we have discussed in [10]–[14], the existence of analytic solutions for such equations is closely related to the location of α in the complex plane. In this paper, we will replace the Diophantine condition by a weaker condition, the Brjuno condition, in the case where α is on the unit circle and is not a root of unity. More precisely, we distinguish three different cases for α:

(**C1**) $0 < |\alpha| < 1$;

(**C2**) $\alpha = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and θ is a Brjuno number ([2], [8]):
$$B(\theta) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty,$$
where $\{p_n/q_n\}$ denotes the sequence of partial fractions of the continued fraction expansion of θ;

(**C3**) $\alpha = e^{2\pi iq/p}$ for some integer $p \in \mathbb{N}$ with $p \geq 2$ and $q \in \mathbb{Z} \setminus \{0\}$, and $\alpha \neq e^{2\pi i\xi/v}$ for all $1 \leq v \leq p-1$ and $\xi \in \mathbb{Z} \setminus \{0\}$.

We observe that α is inside the unit circle S^1 in case (**C1**) but on S^1 in the remaining cases. More difficulties are encountered for α on S^1 because of the small divisor $\alpha^n - 1$ in (2.5). Under the Diophantine condition: "$\alpha = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and there exist constants $\zeta > 0$ and $\delta > 0$ such that $|\alpha^n - 1| \geq \zeta^{-1} n^{-\delta}$ for all $n \geq 1$," the number $\alpha \in S^1$ is "far" from all roots of unity and was considered in different settings [10]–[14]. Since then, we have been striving to give a result of analytic solutions for those α "near" a root of the unity, i.e., neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in (**C2**) provides such a chance for us. Moreover, we also discuss the so-called resonance case, i.e. (**C3**).

Remark 1.1. Let f be a germ of a holomorphic diffeomorphism of $(\mathbb{C}, O)$. One of the main questions in the study of local holomorphic dynamics is whether there exists a local holomorphic change of coordinates such that f is conjugate to its linear part. The answer depends on the eigenvalue of the linearized f at its fixed point O. The three cases mentioned above correspond to the hyperbolic, parabolic and elliptic cases of holomorphic dynamics. For more information on this and other aspects of local dynamics, see the monographs by Lennart Carleson and Theodore W. Gamelin [3] and S. Marmi [7]. In particular, S. Marmi [7] gives a discussion of the parabolic and the elliptic case which is very close to the one given here.

2. The auxiliary equation in cases (C1) and (C2). In this section, we discuss locally invertible analytic solutions of (1.3) with the initial condition

$$y(0) = 0, \quad y'(0) = \gamma \neq 0, \quad \gamma \in \mathbb{C}. \tag{2.1}$$

In order to study the existence of analytic solutions of (1.3) under the Brjuno condition, we first briefly recall the definition of Brjuno numbers and some basic facts. For a real number θ, we let $[\theta]$ denote its integer part and $\{\theta\} = \theta - [\theta]$ its fractional part. Every irrational θ has a unique expression as a Gauss continued fraction

$$\theta = d_0 + \theta_0 = d_0 + \frac{1}{d_1 + \theta_1} = \cdots,$$

denoted simply by $\theta = [d_0, d_1, \ldots, d_n, \ldots]$, where d_j's and θ_j's are calculated by the algorithm: (a) $d_0 = [\theta]$, $\theta_0 = \{\theta\}$, and (b) $d_n = \left[\frac{1}{\theta_{n-1}}\right]$, $\theta_n = \left\{\frac{1}{\theta_{n-1}}\right\}$ for all $n \geq 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} q_{-2} &= 1, \qquad q_{-1} = 0, \qquad q_n = d_n q_{n-1} + q_{n-2}, \\ p_{-2} &= 0, \qquad p_{-1} = 1, \qquad p_n = d_n p_{n-1} + p_{n-2}. \end{aligned}$$

It is easy to show that $p_n/q_n = [d_0, d_1, \ldots, d_n]$. For every $\theta \in \mathbb{R} \setminus \mathbb{Q}$ the series $\sum_{n\geq 0} \frac{\log q_{n+1}}{q_n}$ converges and defines an arithmetical function $B(\theta)$. We say that θ is a *Brjuno number* or that it satisfies the *Brjuno condition* if $B(\theta) < \infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $d_{n+1} \leq ce^{d_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [d_0, d_1, \ldots, d_n, \ldots]$ is a Brjuno number but is not a Diophantine number. So, case (**C2**) contains both a Diophantine condition and a condition which expresses that α is near resonance.

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $(q_n)_{n\in\mathbb{N}}$ be the sequence of partial denominators of the Gauss continued fraction for θ. As in [4], let

$$A_k = \{n \geq 0 \mid \|n\theta\| \leq 1/(8q_k)\}, \qquad E_k = \max(q_k, q_{k+1}/4), \qquad \eta_k = q_k/E_k.$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left((1+\eta_k)\frac{n}{q_k} - 2,\ (m_n\eta_k + n)\frac{1}{q_k} - 1\right),$$

where $m_n = \max\{j \mid 0 \leq j \leq n,\ j \in A_k^*\}$. We then define a function $h_k : \mathbb{N} \to \mathbb{R}_+$ as follows:

$$h_k(n) = \begin{cases} \frac{m_n + \eta_k n}{q_k} - 1 & \text{if } m_n + q_k \in A_k^*, \\ l_k(n) & \text{if } m_n + q_k \notin A_k^*. \end{cases}$$

Let $g_k(n) := \max(h_k(n), [n/q_k])$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is non-decreasing. The following result is known.

Lemma 2.1 (Davie's lemma [5]). *Let*

$$K(n) = n\log 2 + \sum_{k=0}^{k(n)} g_k(n)\log(2q_{k+1}).$$

Then

(a) *there is a universal constant* $\rho > 0$ (*independent of* n *and* θ) *such that*

$$K(n) \leq n\Big(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \rho\Big),$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all* n_1 *and* n_2,

(c) $-\log|\alpha^n - 1| \leq K(n) - K(n-1)$.

Now, we consider the existence of analytic solutions of equation (1.3) in case (**C1**) or (**C2**) holds.

THEOREM 2.1. *If* (**C1**) *or* (**C2**) *holds*, *then equation* (1.3) *has an analytic solution of the form*

$$y(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots, \qquad a_1 = \gamma, \tag{2.2}$$

in a neighborhood of the origin.

Proof. Let

$$G(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad C_{t,j}(z) = \sum_{n=0}^{\infty} c_{t,j,n} z^n,$$

for $t \in \mathbb{N}$ and $j = 0, 1, \ldots, k$. To find a power series solution of the form (2.2), we rewrite (1.3) as

$$\frac{\alpha^2 y''(\alpha z) y'(z) - \alpha y'(\alpha z) y''(z)}{(y'(z))^2} = y'(z)\Big[\sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(y(z))(y(\alpha^j z))^t + G(y(z))\Big]$$

or

$$\left(\frac{y'(\alpha z)}{y'(z)}\right)' = \frac{1}{\alpha}\, y'(z)\Big[\sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(y(z))(y(\alpha^j z))^t + G(y(z))\Big].$$

Since $y'(z) = \gamma \neq 0$, (1.3) reduces to the integro-differential equation

$$y'(\alpha z) = y'(z)\left[1 + \frac{1}{\alpha}\int_0^z y'(s)\Big(\sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(y(s))(y(\alpha^j s))^t + G(y(s))\Big)\, ds\right]. \tag{2.3}$$

By substituting the expansion of $G(z)$, $C_{t,j}(z)$ and (2.2) into (2.3), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1) a_{n+1} \alpha^{n+1} z^n \\
&\quad = \alpha \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=0}^{\infty}\Big[\sum_{u=0}^{n} (u+1) a_{u+1} a_{n-u+1} b_0\Big] z^{n+1} \\
&\quad + \sum_{n=0}^{\infty}\Big[\sum_{u=0}^{n} \frac{u+1}{n-u+2}\, a_{u+1} \sum_{i=0}^{n-u} (i+1) a_{i+1} \\
&\quad \times \sum_{j=0}^{k} \sum_{\substack{(l_m)\in A^t_{n-i-u+1} \\ 1\le t\le n-i-u+1}} c_{t,j,0} \prod_{m=1}^{t} \alpha^{j l_m} a_{l_m}\Big] z^{n+2}
\end{aligned}$$

$$+\sum_{n=0}^{\infty}\Big[\sum_{u=0}^{n}\frac{u+1}{n-u+2}\,a_{u+1}\sum_{i=0}^{n-u}(i+1)a_{i+1}\sum_{\substack{(l_m)\in A^s_{n-i-u+1}\\ 1\le s\le n-i-u+1}} b_s\prod_{m=1}^{s}a_{l_m}\Big]z^{n+2}$$

$$+\sum_{n=0}^{\infty}\Big[\sum_{u=0}^{n}\frac{u+1}{n-u+3}\,a_{u+1}\sum_{i=0}^{n-u}\sum_{h=0}^{n-u-i}(i+1)a_{i+1}$$

$$\times\sum_{j=0}^{k}\sum_{\substack{(l_m)\in(A)^t_{n-i-h-u+1}\\ 1\le t\le n-i-h-u+1}}\prod_{m=1}^{t}\alpha^{jl_m}a_{l_m}\sum_{\substack{(l_m)\in A^\tau_{h+1}\\ 1\le\tau\le h+1}} c_{t,j,\tau}\prod_{m=1}^{\tau}a_{l_m}\Big]z^{n+3}$$

where $\mathcal{A}^t_n := \{(n_1,\ldots,n_t)\in\mathbb{N}^t : n_1+\cdots+n_t=n\}$. Equating coefficients, we obtain

$$a_1\alpha = a_1\alpha,\quad a_2=\frac{a_1^2b_0}{2\alpha(\alpha-1)}, \tag{2.4}$$

$$\begin{aligned}
a_{n+1} &= \frac{1}{(n+1)(\alpha^{n+1}-\alpha)}\Big[\sum_{u=0}^{n-1}(u+1)a_{u+1}a_{n-u}b_0+\sum_{u=0}^{n-2}\frac{u+1}{n-u}a_{u+1}\\
&\quad\times\sum_{i=0}^{n-u-2}(i+1)a_{i+1}\sum_{j=0}^{k}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-u-1}\\ 1\le t\le n-i-u-1}} c_{t,j,0}\prod_{m=1}^{t}\alpha^{jl_m}a_{l_m}\\
&\quad+\sum_{u=0}^{n-2}\frac{u+1}{n-u}\,a_{u+1}\sum_{i=0}^{n-u-2}(i+1)a_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}} b_s\prod_{m=1}^{s}a_{l_m}\\
&\quad+\sum_{u=0}^{n-3}\frac{u+1}{n-u}\,a_{u+1}\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}(i+1)a_{i+1}\\
&\quad\times\sum_{j=0}^{k}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}\alpha^{jl_m}a_{l_m}\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}} c_{t,j,\tau}\prod_{m=1}^{\tau}a_{l_m}\Big],\qquad n\ge 2.
\end{aligned}\tag{2.5}$$

We can choose $a_1=\gamma\neq 0$, and the sequence $\{a_n\}_{n=2}^{\infty}$ is successively determined by (2.4) and (2.5) in a unique manner.

In what follows we prove the convergence of the series (2.2) in a neighborhood of the origin. By (**H**), for any given $r\in(0,\min\{|z_1|,|z_2|\})$, there exists a positive number M such that

$$|b_n|\le\frac{M}{r^n},\qquad |c_{t,j,n}|\le\frac{M}{(1+k)r^{n+t}},\qquad \forall t\in\mathbb{N},\ n\in\mathbb{N}\cup\{0\},\ j=0,1,\ldots,k.$$

By (2.5), we have

$$
(2.6)\qquad |a_{n+1}| \le \frac{M}{|\alpha^{n+1}-\alpha|}\Bigg[\sum_{u=0}^{n-1}|a_{u+1}|\,|a_{n-u}| + 2\sum_{u=0}^{n-2}|a_{u+1}|\sum_{i=0}^{n-u-2}|a_{i+1}|
\times \sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s}|a_{l_m}|
+\sum_{u=0}^{n-3}|a_{u+1}|\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}|a_{i+1}|\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}|a_{l_m}|
\times \sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le \tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau}|a_{l_m}|\Bigg],\qquad n\ge 2.
$$

First of all, in case (**C1**), we have

$$\lim_{n\to\infty}\left|\frac{1}{\alpha^{n+1}-\alpha}\right| = \frac{1}{|\alpha|}.$$

Thus, there exists a positive number L such that

$$\left|\frac{1}{\alpha^{n+1}-\alpha}\right| \le L.$$

In order to construct a governing series of (2.2), we consider the following implicit function equation for $H(z)$:

$$H(z) = |\gamma|z + \widetilde{L}M\left(\frac{H(z)}{1-H(z)/r}\right)^2$$

where $\widetilde{L}=L$ if (**C1**) holds and $\widetilde{L}=1$ as (**C2**) holds. Define

$$(2.7)\qquad \Theta(z,\omega;\gamma,\widetilde{L},M,r) = |\gamma|z-\omega+\widetilde{L}M\left(\frac{\omega}{1-\omega/r}\right)^2$$

for (z,ω) in a neighborhood of $(0,0)$. Then the function $H(z)$ satisfies

$$(2.8)\qquad \Theta(z,H(z);\gamma,\widetilde{L},M,r)=0.$$

Since $\Theta(0,0;\gamma,\widetilde{L},M,r)=0$ and $\Theta'_\omega(0,0;\gamma,\widetilde{L},M,r)=-1\ne 0$, by the implicit function theorem there exists a unique function $\Phi(z)$, analytic in a neighborhood of zero, such that

$$\Phi(0)=0,\qquad \Phi'(0) = -\frac{\Theta'_z(0,0;\gamma,\widetilde{L},M,r)}{\Theta'_\omega(0,0;\gamma,\widetilde{L},M,r)} = |\gamma|,$$

and $\Theta(z,\Phi(z);\gamma,\widetilde{L},M,r)=0$. According to (2.8), we have $H(z)=\Phi(z)$.

Let $H(z)=\sum_{n=1}^{\infty} C_n z^n$ be the power series expansion of $H(z)$. Substituting the series in (2.8) we have

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n+1} z^{n+1} = |\gamma| z + \widetilde{L} M \Bigg[& \sum_{n=0}^{\infty}\Big(\sum_{u=0}^{n} C_{u+1} C_{n-u+1}\Big) z^{n+2} \\ & + 2\sum_{n=0}^{\infty}\Bigg(\sum_{u=0}^{n} C_{u+1}\sum_{i=0}^{n-u} C_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u+1}\\ 1\le s\le n-i-u+1}}\frac{1}{r^s}\prod_{m=1}^{s} C_{l_m}\Bigg) z^{n+3} \\ & + \sum_{n=0}^{\infty}\Bigg(\sum_{u=0}^{n} C_{u+1}\sum_{i=0}^{n-u}\sum_{h=0}^{n-u-i} C_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u+1}\\ 1\le t\le n-i-h-u+1}}\prod_{m=1}^{t} C_{l_m} \\ & \times \sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau} C_{l_m}\Bigg) z^{n+4}\Bigg].\end{aligned}$$

Equating coefficients, we obtain $C_1=|\gamma|$ and

$$\begin{aligned}(2.9)\quad & C_{n+1} \\ & = \widetilde{L} M \Bigg[\sum_{u=0}^{n-1} C_{u+1} C_{n-u} + 2\sum_{u=0}^{n-2} C_{u+1}\sum_{i=0}^{n-u-2} C_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s} C_{l_m} \\ & + \sum_{u=0}^{n-3} C_{u+1}\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3} C_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t} C_{l_m} \\ & \times \sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau} C_{l_m}\Bigg], \qquad n\ge 1.\end{aligned}$$

In the case of (**C1**), from (2.6) we have, for $n\ge 2$,

$$\begin{aligned}|a_{n+1}| \le \widetilde{L} M \Bigg[& \sum_{u=0}^{n-1} |a_{u+1}|\,|a_{n-u}| \\ & + 2\sum_{u=0}^{n-2} |a_{u+1}| \sum_{i=0}^{n-u-2} |a_{i+1}| \sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s} |a_{l_m}|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{u=0}^{n-3}|a_{u+1}|\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}|a_{i+1}|\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}|a_{l_m}|\\
&\times\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau}|a_{l_m}|\Bigg].
\end{aligned}
$$

Then by immediate induction we obtain $|a_n|\le C_n$ for all n. This implies that (2.2) converges in a neighborhood of the origin.

In the case of (**C2**), we will deduce that $|a_n|\le C_n e^{K(n-1)}$ for $n\ge 1$, where $K:\mathbb{N}\to\mathbb{R}$ is defined in Lemma 2.1.

In fact, $|a_1|=|\gamma|=C_1$. For a proof by induction we assume that $|a_{q_1}|\le C_{q_1}e^{K(q_1-1)}$, $q_1\le n$ and from Lemma 2.1 we know

$$
\begin{aligned}
|a_{n+1}|\le{}&\frac{M}{|\alpha^n-1|}\Bigg[\sum_{u=0}^{n-1}|a_{u+1}|\,|a_{n-u}|\\
&+2\sum_{u=0}^{n-2}|a_{u+1}|\sum_{i=0}^{n-u-2}|a_{i+1}|\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s}|a_{l_m}|\\
&+\sum_{u=0}^{n-3}|a_{u+1}|\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}|a_{i+1}|\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}|a_{l_m}|\\
&\times\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau}|a_{l_m}|\Bigg]\\
\le{}&\frac{M}{|\alpha^n-1|}\Bigg[\sum_{u=0}^{n-1}C_{u+1}e^{K(u)}C_{n-u}e^{K(n-u-1)}\\
&+2\sum_{u=0}^{n-2}C_{u+1}e^{K(u)}\sum_{i=0}^{n-u-2}C_{i+1}e^{K(i)}\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s}C_{l_m}e^{K(l_m-1)}\\
&+\sum_{u=0}^{n-3}C_{u+1}e^{K(u)}\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}C_{i+1}e^{K(i)}\\
&\times\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}C_{l_m}e^{K(l_m-1)}\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau}C_{l_m}e^{K(l_m-1)}\Bigg].
\end{aligned}
$$

Note that

$$\begin{aligned}
K(u) + K(n-u-1) &\le K(n-1), \\
K(i) + K(u) + [K(l_1-1) + \cdots + K(l_s-1)] & \\
&\le K(i) + K(u) + K(n-i-u-s-1) \\
&\le K(n-s-1) \le K(n-1)
\end{aligned}$$

and

$$\begin{aligned}
K(i)+K(u)+&[K(l_1-1)+\cdots+K(l_t-1)]+[K(l_1-1)+\cdots+K(l_\tau-1)] \\
&\le K(i)+K(u)+K(n-i-h-u-t-2)+K(h+1-\tau) \\
&\le K(n-t-\tau-1) \le K(n-1).
\end{aligned}$$

Therefore

$$\begin{aligned}
|a_{n+1}| \le \frac{M}{|\alpha^n-1|}\, e^{K(n-1)} \Bigg[& \sum_{u=0}^{n-1} C_{u+1} C_{n-u} \\
&+ 2\sum_{u=0}^{n-2} C_{u+1} \sum_{i=0}^{n-u-2} C_{i+1} \sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1} \\ 1\le s\le n-i-u-1}} \frac{1}{r^s} \prod_{m=1}^{s} C_{l_m} \\
&+ \sum_{u=0}^{n-3} C_{u+1} \sum_{i=0}^{n-u-3} \sum_{h=0}^{n-u-i-3} C_{i+1} \\
&\times \sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2} \\ 1\le t\le n-i-h-u-2}} \prod_{m=1}^{t} C_{l_m} \sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1} \\ 1\le\tau\le h+1}} \frac{1}{r^{t+\tau}} \prod_{m=1}^{\tau} C_{l_m} \Bigg] \\
&\le C_{n+1}|\alpha^n-1|^{-1} e^{K(n)+\log|\alpha^n-1|} \le C_{n+1}e^{K(n)}
\end{aligned}$$

as required. Since $\sum_{n=1}^{\infty} C_n z^n$ is convergent in a neighborhood of the origin, there exists a constant $\Lambda > 0$ such that

$$C_n < \Lambda^n, \qquad n \ge 1.$$

Moreover, from Lemma 2.1, we know that $K(n) \le n(B(\theta)+\rho)$ for some universal constant $\rho > 0$. Thus

$$|a_n| \le C_n e^{K(n-1)} \le \Lambda^n e^{(n-1)(B(\theta)+\rho)},$$

that is,

$$\limsup_{n\to\infty} |a_n|^{1/n} \le \limsup_{n\to\infty} (\Lambda^n e^{(n-1)(B(\theta)+\rho)})^{1/n} = \Lambda e^{B(\theta)+\rho}.$$

This implies that the convergence radius of (2.2) is at least $(\Lambda e^{B(\theta)+\rho})^{-1}$. ■

3. The auxiliary equation in case (C3). This section is devoted to case (**C3**). In this case neither the Diophantine condition nor the Brjuno condition are satisfied.

We need to define a sequence $\{\widetilde{a}_n\}_{n=1}^{\infty}$ by $\widetilde{a}_1 = |\gamma|$ and

$$\begin{aligned}(3.1)\quad \widetilde{a}_{n+1} = \Gamma M\Bigg[&\sum_{u=0}^{n-1}\widetilde{a}_{u+1}\widetilde{a}_{n-u} + 2\sum_{u=0}^{n-2}\widetilde{a}_{u+1}\sum_{i=0}^{n-u-2}\widetilde{a}_{i+1}\\ &\times \sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s}\widetilde{a}_{l_m}\\ &+\sum_{u=0}^{n-3}\widetilde{a}_{u+1}\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}\widetilde{a}_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}\widetilde{a}_{l_m}\\ &\times\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau}\widetilde{a}_{l_m}\Bigg],\qquad n\ge 1,\end{aligned}$$

where

$$(3.2)\qquad \Gamma := \max\left\{1, \frac{1}{|\alpha-1|}, \frac{1}{|\alpha^2-1|}, \dots, \frac{1}{|\alpha^{p-1}-1|}\right\},$$

p is given by (**C3**), and M is defined in Theorem 2.1.

THEOREM 3.1. *Assume that* (**C3**) *holds. Let* $\{a_n\}_{n=1}^{\infty}$ *be determined by* $a_1 = \gamma$, $a_2 = \gamma^2 b_0/(2(\alpha^2-\alpha))$ *and*

$$(3.3)\qquad (\alpha^{n+1}-\alpha)(n+1)a_{n+1} = \Psi(n,\alpha), \quad n = 2,3,\dots,$$

where

$$\begin{aligned}\Psi(n,\alpha) = &\sum_{u=0}^{n-1}(u+1)a_{u+1}a_{n-u}b_0\\ &+\sum_{u=0}^{n-2}\frac{u+1}{n-u}a_{u+1}\sum_{i=0}^{n-u-2}(i+1)a_{i+1}\sum_{j=0}^{k}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-u-1}\\ 1\le t\le n-i-u-1}}c_{t,j,0}\prod_{m=1}^{t}\alpha^{jl_m}a_{l_m}\\ &+\sum_{u=0}^{n-2}\frac{u+1}{n-u}a_{u+1}\sum_{i=0}^{n-u-2}(i+1)a_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}b_s\prod_{m=1}^{s}a_{l_m}\end{aligned}$$

$$+\sum_{u=0}^{n-3}\frac{u+1}{n-u}\,a_{u+1}\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}(i+1)a_{i+1}$$
$$\times\sum_{j=0}^{k}\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}\alpha^{jl_m}a_{l_m}\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}c_{t,j,\tau}\prod_{m=1}^{\tau}a_{l_m},\qquad n\ge 2.$$

If $\Psi(lp,\alpha)=0$ *for all* $l\in\mathbb{N}=\{1,2,\dots\}$, *then* (1.3) *has an analytic solution of the form*

$$y(z)=\gamma z+\frac{\gamma^2 b_0}{2(\alpha^2-\alpha)}\,z^2+\sum_{n=lp+1,\,l\in\mathbb{N}}\mu_{lp+1}z^n+\sum_{n\ne lp+1,\,l\in\mathbb{N}}a_n z^n$$

in a neighborhood of the origin, where all μ_{lp+1}*'s are arbitrary constants satisfying the inequality* $|\mu_{lp+1}|\le\widetilde{a}_{lp+1}$ *and the sequence* $\{\widetilde{a}_n\}_{n=1}^{\infty}$ *is defined in* (3.1). *Otherwise, if* $\Psi(lp,\alpha)\ne 0$ *for some* $l=1,2,\dots$, *then* (1.3) *has no analytic solutions in any neighborhood of the origin.*

Proof. Analogously to the proof of Theorem 2.1, let (2.2) be the expansion of a formal solution $y(z)$ of (1.3). We also have (2.5) or (3.3). If $\Psi(lp,\alpha)\ne 0$ for some natural number l, then (3.3) does not hold for $n=lp$ since $\alpha^{lp+1}-\alpha=0$. In that case, (1.3) has no formal solutions.

If $\Psi(lp,\alpha)=0$ for all natural numbers l, then there are infinitely many choices of corresponding a_{lp+1} in (2.5), and the formal solutions (2.2) form a family of functions of infinitely many parameters. We can arbitrarily choose $a_{lp+1}=\mu_{lp+1}$ such that $|\mu_{lp+1}|\le\widetilde{a}_{lp+1}$, $l=1,2,\dots$. In what follows we prove that the formal solution (2.2) converges in a neighborhood of the origin. First of all, note that

$$|\alpha^n-1|^{-1}\le\Gamma.$$

It follows from (2.5) that

$$\tag{3.4}|a_{n+1}|\le\Gamma M\Bigg[\sum_{u=0}^{n-1}|a_{u+1}|\,|a_{n-u}|+2\sum_{u=0}^{n-2}|a_{u+1}|\sum_{i=0}^{n-u-2}|a_{i+1}|\times\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{r^s}\prod_{m=1}^{s}|a_{l_m}|$$
$$+\sum_{u=0}^{n-3}|a_{u+1}|\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}|a_{i+1}|\sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}|a_{l_m}|\times\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}}\frac{1}{r^{t+\tau}}\prod_{m=1}^{\tau}|a_{l_m}|\Bigg]$$

for all $n \neq lp$, $l = 1, 2, \ldots$. Further, we can prove that

$$|a_n| \le \widetilde{a}_n, \quad n = 1, 2, \ldots. \tag{3.5}$$

In fact, for a proof by induction we assume that $|a_{i_2}| \le \widetilde{a}_{i_2}$ for all $1 \le i_2 \le n$. When $n = lp$, we have $|a_{n+1}| = |\mu_{n+1}| \le \widetilde{a}_{n+1}$. On the other hand, when $n \neq lp$, from (3.4) we get

$$\begin{aligned}|a_{n+1}| \le \Gamma M \Bigg[&\sum_{u=0}^{n-1} \widetilde{a}_{u+1}\widetilde{a}_{n-u} + 2\sum_{u=0}^{n-2} \widetilde{a}_{u+1} \sum_{i=0}^{n-u-2} \widetilde{a}_{i+1} \sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}} \frac{1}{r^s} \prod_{m=1}^{s} \widetilde{a}_{l_m} \\ &+ \sum_{u=0}^{n-3} \widetilde{a}_{u+1} \sum_{i=0}^{n-u-3} \sum_{h=0}^{n-u-i-3} \widetilde{a}_{i+1} \sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}} \prod_{m=1}^{t} \widetilde{a}_{l_m} \\ &\times \sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le\tau\le h+1}} \frac{1}{r^{t+\tau}} \prod_{m=1}^{\tau} \widetilde{a}_{l_m} \Bigg] = \widetilde{a}_{n+1},\end{aligned}$$

implying (3.5). Set

$$F(z) = \sum_{n=1}^{\infty} \tilde{a}_n z^n. \tag{3.6}$$

It is easy to check that (3.6) satisfies

$$\Theta(z, F(z); \gamma, \Gamma, M, r) = 0, \tag{3.7}$$

where the function Θ is defined in (2.7). Moreover, similarly to the proof of Theorem 2.1, we can prove that (3.7) has a unique analytic solution $F(z)$ in a neighborhood of the origin such that $F(0) = 0$ and $F'(0) = |\gamma| \neq 0$. Thus (3.6) converges in a neighborhood of the origin. By the convergence of (3.6) and inequality (3.5), the series (2.2) converges in a neighborhood of the origin. This completes the proof. ∎

4. Analytic solutions of equation (1.2). In this section, we give a theorem on existence of analytic solutions for equation (1.2).

THEOREM 4.1. *Suppose that the conditions of either Theorem* 2.1 *or Theorem* 3.1 *are satisfied. Then equation* (1.2) *has an invertible analytic solution of the form*

$$x(z) = y(\alpha y^{-1}(z))$$

in a neighborhood of the origin, where $y(z)$ is an analytic solution of (1.3) *satisfying the initial conditions* (2.1).

Proof. In view of Theorems 2.1 and 3.1, we can find an analytic solution $y(z)$ of the auxiliary equation (1.3) in the form (2.2) such that $y(0) = 0$

and $y'(0) = \gamma \neq 0$. Clearly the inverse $y^{-1}(z)$ exists and is analytic in a neighborhood of $y(0) = 0$. Define

$$x(z) := y(\alpha y^{-1}(z)). \tag{4.1}$$

Then $x(z)$ is analytic and invertible in a neighborhood of the origin. From (4.1) and (1.3), it is easy to see

$$\begin{aligned} &x(0) = y(\alpha y^{-1}(0)) = y(0) = 0, \\ &x'(0) = \alpha y'(\alpha y^{-1}(0))(y^{-1})'(0) = \frac{\alpha y'(\alpha y^{-1}(0))}{y'(y^{-1}(0))} = \frac{\alpha y'(0)}{y'(0)} = \alpha \neq 0, \end{aligned}$$

and

$$\begin{aligned} x''(z) &= \frac{\alpha^2 y''(\alpha y^{-1}(z))y'(y^{-1}(z)) - \alpha y'(\alpha y^{-1}(z))y''(y^{-1}(z))}{(y'(y^{-1}(z))^3} \\ &= \sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(y(y^{-1}(z)))(y(\alpha^j y^{-1}(z)))^t + G(y(y^{-1}(z))) \\ &= \sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(z)(x^{[j]}(z))^t + G(z), \end{aligned}$$

that is, the function $x(z)$ defined in (4.1) satisfies equation (1.2). ∎

The following two examples show how to construct an analytic solution of (1.2) for a concrete equation.

Example 4.1. Consider the equation

$$x''(z) = \sum_{j=0}^{2}\sum_{t=1}^{\infty} C_{t,j}(z)(x^{[j]}(z))^t + G(z), \quad \forall z \in \mathbb{C}, \tag{4.2}$$

where

$$C_{t,j}(z) = \frac{j}{3^{t-1}(3-z)} = \sum_{n=0}^{\infty} \frac{j}{3^{t+n}}\, z^n, \qquad t \in \mathbb{N},\ j = 0, 1, 2,$$

and

$$G(z) = \frac{1}{2}\, e^z = \sum_{n=0}^{\infty} \frac{z^n}{2n!}.$$

Clearly the functions $C_{t,j}(z)$ $(t \in \mathbb{N},\ j = 0, 1, 2)$ and $G(z)$ are analytic for $|z| < 3$, and for each $j = 0, 1, 2$, the series

$$\sum_{t=1}^{\infty} C_{t,j}(z_1) z_2^t = \sum_{t=1}^{\infty} \frac{j}{3^{t-1}(3-z_1)}\, z_2^t = \frac{3j}{3-z_1} \cdot \frac{z_2}{3-z_2}$$

converges for a pair of nonzero complex z_1, z_2 with $|z_1| < 3$ and $|z_2| < 3$. Take $\alpha = 1/2$. By Theorem 2.1, the auxiliary equation

$$\tag{4.3} y'\left(\frac{1}{2}z\right) = y'(z)\left[1 + 2\int_0^z y'(s)\left(\sum_{j=0}^{2}\sum_{t=1}^{\infty}\frac{j}{3^{t-1}(3-y(s))}\,(y(2^{-j}s))^t + \frac{1}{2}\,e^{y(s)}\right)ds\right]$$

has a solution of the form (2.2) where $a_1 = \gamma \neq 0$ is given arbitrarily and $a_2, a_3, \ldots$ are determined by (2.4) and (2.5) recursively, i.e. $a_2 = -\gamma^2$, and

$$\tag{4.4} \begin{aligned} a_{n+1} = {} & \left(\frac{1}{2}(n+1)\left(\frac{1}{2^n}-1\right)\right)^{-1}\left[\frac{1}{2}\sum_{u=0}^{n-1}(u+1)a_{u+1}a_{n-u}\right. \\ & + \sum_{u=0}^{n-2}\frac{u+1}{n-u}\,a_{u+1}\sum_{i=0}^{n-u-2}(i+1)a_{i+1}\sum_{j=0}^{2}\frac{1}{2^{j(n-i-u-1)}} \\ & \times \sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-u-1}\\ 1\le t\le n-i-u-1}}\frac{j}{3^t}\prod_{m=1}^{t}a_{l_m} \\ & + \sum_{u=0}^{n-2}\frac{u+1}{n-u}\,a_{u+1}\sum_{i=0}^{n-u-2}(i+1)a_{i+1}\sum_{\substack{(l_m)\in\mathcal{A}^s_{n-i-u-1}\\ 1\le s\le n-i-u-1}}\frac{1}{2s!}\prod_{m=1}^{s}a_{l_m} \\ & + \sum_{u=0}^{n-3}\frac{u+1}{n-u}\,a_{u+1}\sum_{i=0}^{n-u-3}\sum_{h=0}^{n-u-i-3}(i+1)a_{i+1}\sum_{j=0}^{2}\frac{1}{2^{j(n-i-h-u-2)}} \\ & \left.\times \sum_{\substack{(l_m)\in\mathcal{A}^t_{n-i-h-u-2}\\ 1\le t\le n-i-h-u-2}}\prod_{m=1}^{t}a_{l_m}\sum_{\substack{(l_m)\in\mathcal{A}^\tau_{h+1}\\ 1\le \tau\le h+1}}\frac{j}{3^{t+\tau}}\prod_{m=1}^{\tau}a_{l_m}\right], \qquad n \ge 2. \end{aligned}$$

In particular, from (4.4) we have

$$a_3 = \frac{y'''(0)}{3!} = \frac{26}{27}\,\gamma^3, \quad \ldots.$$

Since $y(0) = 0, y'(0) = \gamma \neq 0$, and the inverse $y^{-1}(z)$ is analytic near the origin, we can calculate

$$(y^{-1})'(0) = \frac{1}{y'(y^{-1}(0))} = \frac{1}{y'(0)} = \frac{1}{\gamma},$$

$$(y^{-1})''(0) = -\frac{y''(y^{-1}(0))(y^{-1})'(0)}{(y'(y^{-1}(0)))^2} = -\frac{y''(0)(y^{-1})'(0)}{(y'(0))^2} = \frac{2}{\gamma},$$

$$\begin{aligned}(y^{-1})'''(0) &= -\frac{[y'''(y^{-1}(0))((y^{-1})'(0))^2 + y''(y^{-1}(0))(y^{-1})''(0)](y'(y^{-1}(0)))^2}{(y'(y^{-1}(0)))^4} \\ &\quad + \frac{y''(y^{-1}(0))((y^{-1})'(0)) \cdot 2y'(y^{-1}(0))y''(y^{-1}(0))(y^{-1})'(0)}{(y'(y^{-1}(0)))^4} \\ &= -\frac{[y'''(0)\gamma^{-2} + y''(0) \cdot (\frac{2}{\gamma})](y'(0))^2 - y''(0)\gamma^{-1} \cdot 2y'(0)y''(0)\gamma^{-1}}{(y'(0)))^4} \\ &= \frac{56}{9\gamma}, \quad \ldots .\end{aligned}$$

Furthermore, we get

$$\begin{aligned} x(0) &= y\left(\frac{1}{2}\, y^{-1}(0)\right) = y(0) = 0, \\ x'(0) &= y'\left(\frac{1}{2} y^{-1}(0)\right) \cdot \frac{1}{2}(y^{-1})'(0) = \frac{1}{2}\, y'(0)(y^{-1})'(0) = \frac{1}{2}, \\ x''(0) &= \frac{1}{4}\, y''\left(\frac{1}{2}\, y^{-1}(0)\right)[(y^{-1})'(0)]^2 + \frac{1}{2}\, y'\left(\frac{1}{2}\, y^{-1}(0)\right)(y^{-1})''(0) = \frac{1}{2}, \\ x'''(0) &= \frac{1}{8}\, y'''\left(\frac{1}{2}\, y^{-1}(0)\right)[(y^{-1})'(0)]^3 + \frac{1}{2}\, y''\left(\frac{1}{2}\, y^{-1}(0)\right)(y^{-1})'(0)(y^{-1})''(0) \\ &\quad + \frac{1}{4}\, y''\left(\frac{1}{2}\, y^{-1}(0)\right)(y^{-1})'(0)(y^{-1})''(0) + \frac{1}{2}\, y'\left(\frac{1}{2}\, y^{-1}(0)\right)(y^{-1})'''(0) \\ &= \frac{1}{8}\, y'''(0)[(y^{-1})'(0)]^3 + \frac{3}{4}\, y''(0)(y^{-1})'(0)(y^{-1})''(0) + \frac{1}{2}\, y'(0)(y^{-1})'''(0) \\ &= \frac{5}{6}, \quad \ldots . \end{aligned}$$

Thus near 0 equation (4.2) has an analytic solution

$$x(z) = \frac{1}{2}\, z + \frac{1}{4}\, z^2 + \frac{5}{36}\, z^3 + \cdots .$$

EXAMPLE 4.2. Consider the equation

$$x''(z) = zx^{[k]}(z) - \alpha z^2, \qquad \forall z \in \mathbb{C}, \tag{4.5}$$

where $k > 2$ is an integer and α is a primitive root of unity of order $k - 1$.

We can get the auxiliary equation

$$y'(\alpha z) = y'(z)\left[1 + \frac{1}{\alpha}\int_0^z y'(s)(y(s)y(\alpha^k s) - \alpha y^2(s))ds\right]. \tag{4.6}$$

If we substitute $y(z) = \sum_{n=1}^{\infty} a_n z^n$ in (4.6), where $a_1 = \alpha$, we obtain $a_2 = a_3 = 0$ and

$$(\alpha^{n+1} - \alpha)(n + 1)a_{n+1} = \Psi(n, \alpha), \qquad n = 3, 4, \ldots, \tag{4.7}$$

where

$$\Psi(n,\alpha)$$
$$=\sum_{u=0}^{n-3}\sum_{i=0}^{n-u-3}\sum_{h=1}^{n-i-u-2}\frac{u+1}{n-u}(i+1)(\alpha^h-\alpha)a_h a_{i+1}a_{u+1}a_{n-u-i-h-1}, \qquad n\geq 3.$$

It is easy to find

$$\Psi(n,\alpha)=0, \qquad n=3,4,\dots .$$

Obviously, for all $l=1,2,\dots$, we have $\Psi(l(k-1),\alpha)=0$ and $a_2=a_3=\dots=0$. This implies that (4.6) has an analytic solution

$$y(z)=\alpha z.$$

Thus, (4.5) has an analytic solution

$$x(z)=y(\alpha y^{-1}(z))=y\left(\alpha\left(\frac{1}{\alpha}z\right)\right)=y(z)=\alpha z.$$

Notice in the first example that if the functions $G(z)=\sum_{n=0}^{\infty}b_n z^n$ and $C_{t,j}(z)=\sum_{n=0}^{\infty}c_{t,j,n}z^n$ for $t\in\mathbb{N}$, $j=0,1,\dots,k$, are all given near 0 by convergent series with real coefficients then by Theorem 4.1 equation (1.2) has an invertible analytic real solution. Clearly by (2.4) and (2.5) we can define a real sequence $\{a_n\}_{n=1}^{\infty}$ and obtain a solution $y(z)$ of the form (2.2) with real coefficients. The restriction of the function $y(z)$ to the reals is real-valued. Hence the function $x(z)=y(\alpha y^{-1}(z))$ is also a real function, and Theorem 4.1 implies that it is analytic and invertible.

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