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ω-pluripolar sets and subextension of ω-plurisubharmonic functions on compact Kähler manifolds

by Le Mau Hai, Nguyen Van Khue and Pham Hoang Hiep (Hanoi)

Abstract. We establish some results on ω-pluripolarity and complete ω-pluripolarity for sets in a compact Kähler manifold X with fundamental form ω. Moreover, we study subextension of ω-psh functions on a hyperconvex domain in X and prove a comparison principle for the class $\mathcal{E}(X,\omega)$ recently introduced and investigated by Guedj–Zeriahi.

1. Introduction. Plurisubharmonic (psh) and holomorphic functions are very important objects of complex analysis. In order to study singularities of psh functions Demailly, Lempert and Shiffman in [DLS] introduced the notion of quasi-psh functions, which are locally a sum of a psh function and a smooth function. Regarding this notion recently Kołodziej [Ko] and Guedj–Zeriahi [GZ1], [GZ2] introduced and investigated ω-psh functions on a compact Kähler manifold with fundamental form ω. They studied some problems of pluripotential theory in a local setting (for bounded hyperconvex domains in $\mathbb{C}^n$) for ω-psh functions, in particular, the Dirichlet problem.

The aim of this paper is to study some other problems of pluripotential theory of ω-psh functions. Namely in Section 3 we study ω-pluripolar and complete ω-pluripolar sets in a compact Kähler manifold. In particular, we prove that a subset S of a compact Kähler manifold X with fundamental form ω is locally pluripolar if and only if there exists a $\varphi \in \mathcal{E}^\infty(X,\omega)$ (see Definition 2.3) such that $\varphi = -\infty$ on S. This result in a weaker form was proved by Guedj–Zeriahi in [GZ2]. Section 4 is devoted to investigating complete ω-pluripolar sets in the projective space $\mathbb{CP}^n$. We prove that a subset $S \subset \mathbb{CP}^n$ is complete ω-pluripolar in $\mathbb{CP}^n$ if and only if $S \cap U_j$ is complete pluripolar in the coordinate neighbourhood $U_j = \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_j \neq 0\}$ for $0 \leq j \leq n$. It is shown that a subset $S \subset \mathbb{CP}^n$ is complete ω-pluripolar in $\mathbb{CP}^n$ iff $\widetilde{S} = \pi^{-1}(S) \cup \{0\}$ is complete pluripolar in $\mathbb{C}^{n+1}$ where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ denotes the canonical projection. Next in Sec-

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tion 5 we study the problem of subextension for ω-psh functions. We show that every psh function in the class $\mathcal{F}$ (see Definition 2.2) on a hyperconvex domain in a compact Kähler manifold X can be subextended to an ω-psh function on X. Finally, in Section 6 we establish a comparison principle for the class $\mathcal{E}(X,\omega)$ introduced and investigated recently by Guedj–Zeriahi (see [GZ2]).

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2. Preliminaries. In this section we recall some elements of pluripotential theory in the local setting that can be found in Bedford–Taylor [BT], Klimek [Kl], and Cegrell [Ce1], [Ce2].

2.1. Let Ω be a bounded domain in $\mathbb{C}^n$. The C_n*-capacity* in the sense of Bedford and Taylor on Ω is the set function given by

$$C_n(E) = C_n(E,\Omega) = \sup\Big\{ \int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0\Big\}$$

for every Borel set E in Ω. It is known [BT] that

$$C_n(E) = \int_\Omega (dd^c h^*_{E,\Omega})^n,$$

where $h^*_{E,\Omega}$ is the relative extremal psh function for E (relative to Ω) defined as the smallest upper semicontinuous majorant of

$$h_{E,\Omega}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ u \le 0,\ u \le -1 \text{ on } E\}.$$

2.2. Let $p \ge 1$. In [Ce1] and [Ce2] Cegrell introduced the following classes of psh functions on a bounded hyperconvex domain Ω in $\mathbb{C}^n$:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j\ge1} \int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \forall z_0 \in \Omega,\ \text{there exists a neighbourhood } U \ni z_0 \text{ and } \mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } U \text{ with } \sup_{j\ge1} \int_\Omega (dd^c\varphi_j)^n < \infty\Big\}.$$

Recently Błocki [Bl] has proved that (belonging to) $\mathcal{E}$ is a local property. This result motivates the introduction of the following space:

$$\mathcal{D} = \mathcal{D}(\Omega) = \{\varphi \in \mathrm{PSH}(\Omega) : \forall z_0 \in \Omega,\ \text{there is a neighbourhood } U \ni z_0 \text{ such that } \varphi|_U \in \mathcal{E}(U) + \mathbb{R}\}.$$

2.3. Let X be a compact Kähler manifold with fundamental form ω. For example, X is a projective space with the Fubini–Study Kähler form $\omega = \omega_{\mathrm{FS}}$. An upper semicontinuous function $\varphi : X \to [-\infty, \infty)$ is said to be *ω-psh* if $\varphi \in L^1(X)$ and

$$\omega + dd^c\varphi \geq 0.$$

We denote by $\mathrm{PSH}(X,\omega)$ the set of all ω-psh functions on X. Along the lines of [Ce1], the following classes of ω-psh functions were considered by Guedj and Zeriahi in [GZ2]:

$$\mathcal{E}(X,\omega) = \{\varphi \in \mathrm{PSH}(X,\omega) : \forall z_0 \in X, \text{ there is a neighbourhood } U \text{ of } z_0 \text{ and a potential } \theta \text{ of } \omega \text{ on } U \text{ such that } \varphi + \theta|_U \in \mathcal{D}(U)\},$$

$$\mathcal{E}^p(X,\omega) = \Big\{\varphi \in \mathrm{PSH}(X,\omega) : \exists\ \mathrm{PSH}(X,\omega) \cap L^\infty(X) \ni \varphi_j \searrow \varphi,\ \sup_{j\geq 1} \int_X |\varphi_j|^p \omega_{\varphi_j}^n < \infty\Big\}$$

and

$$\mathcal{E}^\infty(X,\omega) = \bigcap_{p\geq 1} \mathcal{E}^p(X,\omega).$$

2.4. Following Bedford and Taylor [BT], Kołodziej [Ko] considered the Cap_ω-capacity on X given by

$$\mathrm{Cap}_\omega(E) = \sup\Big\{\int_E \omega_\varphi^n : \varphi \in \mathrm{PSH}(X,\omega),\ 0 \leq \varphi \leq 1\Big\}$$

for all Borel sets $E \subset X$. In [Ko] (see also [GZ2]), it is proved that if $\{U_\alpha\}$ is a finite cover of X by strictly pseudoconvex open subsets $U_\alpha = \{z \in X : \varphi_\alpha(z) < 0\}$ where φ_α is a strictly psh smooth function on a neighbourhood of U_α then for every $\delta > 0$ there exists $C > 0$ such that

$$\frac{1}{C}\,\mathrm{Cap}_\omega(\cdot) \leq \mathrm{Cap}_{\mathrm{BT}}(\cdot) \leq C\,\mathrm{Cap}_\omega(\cdot),$$

where

$$\mathrm{Cap}_{\mathrm{BT}}(E) = \sum_\alpha C_n(E \cap U_\alpha, U_\alpha^\delta), \qquad U_\alpha^\delta = \{z \in U_\alpha : \varphi_\alpha(z) < -\delta\}.$$

The following equality was proved by Guedj and Zeriahi in [GZ1]:

$$\mathrm{Cap}_\omega(E) = \int_X (-h_{E,\omega}^*)\omega_{h_{E,\omega}^*}^n$$

for all Borel sets $E \subset X$, where

$$h_{E,\omega}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}(X,\omega),\ \varphi \leq 0 \text{ and } \varphi \leq -1 \text{ on } E\}.$$

2.5. Let $S \subset X$. We say that S is *ω-pluripolar* if there exists $\varphi \in \mathrm{PSH}(X,\omega)$ such that $S \subset \varphi^{-1}(-\infty)$ and $\varphi \not\equiv -\infty$. If φ can be chosen such that $S = \varphi^{-1}(-\infty)$ then S is said to be a *complete ω-pluripolar* set.

In [GZ1] the authors have shown that S is ω-pluripolar if and only if S is locally pluripolar.

2.6. Given a domain Ω in X and an ω-psh function φ on Ω, an ω-psh function $\widetilde{\varphi}$ on X is said to be a *subextension* of φ if $\widetilde{\varphi} \le \varphi$ on Ω.

2.7. In this paper we use Proposition 6.5 and Theorem 5.1 of [GZ2]. The latter is claimed to hold for $n > 2$ (see Theorem 7.5 in [GZ2]). However, it is not mentioned that Proposition 6.5 also holds for $n > 2$. We now prove that, using the notation of [GZ2]. Namely we establish the following. Let μ be a probability measure on a compact connected Kähler manifold, $\dim_{\mathbb{C}} X = n$, equipped with the Kähler form ω. Assume that there exist $\alpha > p/(p+1)$ and $A > 0$ such that

$$\mu(E) \le A\,\mathrm{Cap}_\omega(E)^\alpha$$

for all Borel sets $E \subset X$. Then $\mathcal{E}^p(X,\omega) \subset L^p(X)$.

First we recall that integration by parts on a compact manifold always holds since there is no boundary. Now the above claim follows from the following three results.

1) Let $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Then

$$\int_X (-\varphi)^p \omega^n \le \int_X (-\varphi)^p \omega_\varphi \wedge \omega^{n-1} \le \cdots \le \int_X (-\varphi)^p \omega_\varphi^n.$$

Indeed, let T be a closed positive current. Then

$$\begin{aligned}\int_X (-\varphi)^p \omega_\varphi \wedge T &= \int_X (-\varphi)^p \omega \wedge T + \int_X (-\varphi)^p dd^c\varphi \wedge T\\ &= \int_X (-\varphi)^p \omega \wedge T + p\int_X (-\varphi)^{p-1} d\varphi \wedge d^c\varphi \wedge T\\ &\ge \int_X (-\varphi)^p \omega \wedge T,\end{aligned}$$

and 1) follows.

2) Let $\varphi \in \mathcal{E}^p(X,\omega)$. Then

$$\mathrm{Cap}_\omega(\varphi < -2t) \le C(\varphi)/t^{p+1}.$$

Indeed,

$$\begin{aligned}\mathrm{Cap}_\omega(\varphi < -2t) &= \sup\Big\{\int_{\{\varphi<-2t\}} \omega_u^n : u \in \mathrm{PSH}(X,\omega),\ -1 \le u \le 0\Big\}\\ &\le \sup\Big\{\int_{\{\varphi/t<u-1\}} \omega_u^n : u \in \mathrm{PSH}(X,\omega),\ -1 \le u \le 0\Big\}\\ &\le \sup\Big\{\int_{\{\varphi/t<u-1\}} \omega_{\varphi/t}^n : u \in \mathrm{PSH}(X,\omega),\ -1 \le u \le 0\Big\}\end{aligned}$$

$$\le \int_{\{\varphi/t<-1\}} (\omega+\omega_\varphi/t)^n \le \int_{\{\varphi<-t\}} \sum_{j=0}^{n} C_n^j \frac{\omega_\varphi^j}{t^j} \wedge \omega^{n-j}$$

$$= \sum_{j=0}^{n} \frac{C_n^j}{t^j} \int_{\{\varphi<-t\}} \omega_\varphi^j \wedge \omega^{n-j} = \int_{\{\varphi<-t\}} \omega^n + \sum_{j=1}^{n} \frac{C_n^j}{t^j} \int_{\{\varphi<-t\}} \omega_\varphi^j \wedge \omega^{n-j}$$

$$\le \int_{\{\varphi<-t\}} \omega^n + \sum_{j=1}^{n} \frac{C_n^j}{t^{j+p}} \int_X (-\varphi)^p \omega_\varphi^j \wedge \omega^{n-j}$$

$$\le \int_{\{\varphi<-t\}} \omega^n + \frac{\sum_{j=1}^n C_n^j \int_X (-\varphi)^p \omega_\varphi^n}{t^{j+p}}$$

$$\le \frac{1}{t^{1+p}} \Big[\int_X (-\varphi)^p \omega^n + \sum_{j=1}^{n} C_n^j \int_X (-\varphi)^p \omega_\varphi^n \Big] \le \frac{C(\varphi)}{t^{p+1}}.$$

Proof of Proposition 6.5 of [GZ2] *for* $n > 2$. Let $\varphi \in \mathcal{E}^p(X,\omega)$ with $\sup_X \varphi = -1$. By the Fubini theorem we have

$$\int_X (-\varphi)^p \, d\mu = p \int_1^\infty t^{p-1} \mu(\varphi < -t)\, dt + \mu(X)$$
$$\le pA \int_1^\infty t^{p-1} (\mathrm{Cap}_\omega(\varphi < -t))^\alpha \, dt + \mu(X).$$

From 2) it follows that

$$\int_X (-\varphi)^p \, d\mu \le 1 + C(\varphi) \int_1^\infty \frac{1}{t^{\alpha(p+1)+1-p}}\, dt < \infty$$

because from the hypothesis we have $\alpha(p+1)+1-p > 1$.

3. ω-pluripolar and complete ω-pluripolar sets. In this section we investigate ω-pluripolar and complete ω-pluripolar sets on a compact Kähler manifold with fundamental form ω. Before stating the first result we would like to explain its origin. In Theorem 6.2 of [GZ1] the authors proved that every locally pluripolar set is an ω-pluripolar set. Here we give another proof of this fact by applying a recent result on solution of the Monge–Ampère equation presented in [GZ2].

3.1. THEOREM. *Let S be a locally ω-pluripolar set in X. Then there exists $\varphi \in \mathcal{E}^\infty(X,\omega)$ such that $\varphi \equiv -\infty$ on S and $\varphi \not\equiv -\infty$.*

In order to prove the theorem we need the following lemma.

3.2. Lemma. *Let Ω be a domain in X which is biholomorphic to a ball in $\mathbb{C}^n$ and $D \Subset \Omega$. Let $\varphi \in \mathcal{F}_\infty(\Omega)$. Then there exists $u \in \mathcal{E}^\infty(X, a\omega)$ for some $a > 0$ such that $u \le \varphi$ on D. Here*

$$\mathcal{F}_\infty = \mathcal{F}_\infty(\Omega) = \bigcap_{p\ge 1} \mathcal{F}_p(\Omega)$$

with

$$\mathcal{F}_p = \mathcal{F}_p(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j\ge 1} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < \infty\Big\}.$$

Proof. By hypothesis, $\varphi \le 0$ on Ω. We can assume that $\varphi \not\equiv 0$. Put

$$h_{D,\varphi}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ u \le 0 \text{ and } u|_D \le \varphi\}.$$

Since $\varphi \le h^*_{D,\varphi}$ and $\varphi \in \mathcal{F}_\infty(\Omega)$ it follows that $h^*_{D,\varphi} \in \mathcal{F}_\infty(\Omega)$ ([Ce1]). Moreover, $h_{D,\varphi} = \varphi$ on D and $\operatorname{supp}(dd^c h^*_{D,\varphi})^n \subset \overline{D}$. It is easy to see that $(dd^c h^*_{D,\varphi})^n \ne 0$. Indeed, otherwise Lemma 3.3 in [Ah] implies that $h^*_{D,\varphi} \equiv 0$ on Ω, hence $\varphi \equiv 0$, which is a contradiction. Consider the probability measure $\mu = \alpha^{-1}(dd^c h^*_{D,\varphi})^n$ with $\alpha = \int_\Omega (dd^c h^*_{D,\varphi})^n \ne 0$. It follows from the Hölder inequality (see [Ko]) that for each $p \ge 1$ there exist $A_p, B_p > 0$ such that

$$\begin{aligned}
\mu(E) = \mu(E \cap \overline{D}) &\le \frac{1}{\alpha} \int_\Omega (-h^*_{E\cap\overline{D},\Omega})^p (dd^c h^*_{D,\varphi})^n \\
&\le \frac{A_p}{\alpha}\Big(\int_\Omega (-h^*_{D,\varphi})^p (dd^c h^*_{D,\varphi})^n\Big)^{n/(p+n)} \\
&\quad \times \Big(\int_\Omega (-h^*_{E\cap\overline{D},\Omega})^p (dd^c h^*_{E\cap\overline{D},\Omega})^n\Big)^{p/(p+n)} \\
&\le \frac{A_p}{\alpha}\Big(\int_\Omega (-h^*_{D,\varphi})^p (dd^c h^*_{D,\varphi})^n\Big)^{n/(p+n)} C_n(E\cap\overline{D}, \Omega)^{p/(p+n)} \\
&\le B_p \operatorname{Cap}_\omega(E \cap \overline{D}, X)^{p/(p+n)}
\end{aligned}$$

for all Borel sets $E \subset X$. Proposition 6.5 and Theorem 5.1 in [GZ2] imply that there exists $v \in \mathcal{E}^\infty(X, \omega)$ with $\omega_v^n = \mu$. Let θ be a negative potential of ω on Ω, $\omega = dd^c\theta$. Since

$$(dd^c(v+\theta))^n = (dd^c v + \omega)^n = \frac{1}{\alpha}\,(dd^c h^*_{D,\varphi})^n$$

on Ω, by the comparison principle we have

$$v + \theta \le \frac{1}{\alpha^{1/n}}\, h^*_{D,\varphi}$$

on Ω. Notice that $\mathcal{E}^\infty(X, A\omega) = A\mathcal{E}^\infty(X,\omega)$ for all $A > 0$, and hence for $u = \alpha^{1/n}(v+c)$ where $c = \inf_{\overline{D}} \theta$ it follows that $u \in \mathcal{E}^\infty(X, \alpha^{1/n}\omega)$. Moreover, $u \le h^*_{D,\varphi}$ on D, and therefore $u \le \varphi$ almost everywhere on D. Thus $u \le \varphi$ on D.

Proof of Theorem 3.1. Let S be a locally ω-pluripolar set in X. Then by [H] we can find hyperconvex subsets $V_s \Subset U_s$ and $\varphi_s \in \mathcal{F}_\infty(U_s)$ such that $\varphi_s = -\infty$ on $S \cap U_s$ and $X = \bigcup_{s=1}^k V_s$. We may assume that every U_s is biholomorphic to a ball in $\mathbb{C}^n$. For each $s = 1, \ldots, k$ applying Lemma 3.2 we can find $u_s \in \mathcal{E}^\infty(X, a_s\omega)$ with $a_s > 0$ such that $u_s \le \varphi_s$ on V_s. Hence $\{\varphi_s = -\infty\} \cap V_s = \{u_s = -\infty\} \cap V_s$ for $s = 1, \ldots, k$. Put

$$u = \frac{1}{k}\sum_{s=1}^{k}\frac{u_s}{a_s}.$$

From the convexity of $\mathcal{E}^\infty(X,\omega)$, we infer that $u \in \mathcal{E}^\infty(X,\omega)$ and $u = -\infty$ on S. This completes the proof of Theorem 3.1.

Remark. Theorem 3.1 also follows from [GZ2]. Indeed, by Example 6.3 in [GZ2], we can find $\varphi \in \mathcal{E}^1(X,\omega)$ such that $\varphi \equiv -\infty$ on S. It is enough to consider the function $u := -\log(-\varphi)$.

Next we investigate the completeness of ω-pluripolar sets. Given a pluripolar set $S \subset X$, as in the local setting put

$$S^* = \{z \in X : \varphi(z) = -\infty,\ \forall \varphi \in \mathrm{PSH}(X,\omega),\ \varphi|_S = -\infty\}.$$

In the local setting (for pseudoconvex domains in $\mathbb{C}^n$) Zeriahi [Ze] proved that if S is an F_σ and G_δ pluripolar set such that $S = S^*$ then S is complete pluripolar. By a similar argument using the approximation theorem of Demailly [De] for ω-psh functions we also obtain

3.3. Proposition. *Let S be an F_σ and G_δ ω-pluripolar set such that $S = S^*$. Then S is complete ω-pluripolar.*

Proof. Since S is F_σ and G_δ, we can write S and $X \setminus S$ as increasing unions of compact subsets

$$S = \bigcup_{j=1}^\infty K_j, \qquad X \setminus S = \bigcup_{j=1}^\infty L_j.$$

Let $a \in L_j$. Then $a \notin S^*$. Hence, there exists $u_a^{(j)} \in \mathrm{PSH}(X,\omega)$ such that $u_a^{(j)}|_S \equiv -\infty$ and $u_a^{(j)}(a) > -\infty$. Since $\varepsilon u_a^{(j)} \in \mathrm{PSH}(X,\omega)$ for all $0 < \varepsilon < 1$, we can assume that

$$u_a^{(j)}|_S \equiv -\infty, \qquad u_a^{(j)}(a) > -1, \qquad u_a^{(j)} \le 0.$$

By [De] there exists a sequence $\{u_k^{(j)}\} \subset \mathrm{PSH} \cap C^\infty(X,\omega)$ that decreases pointwise to $u_a^{(j)}$ on X. Applying Dini's theorem we find k_a such that

$$u_{k_a}^{(j)}|_{K_j} \le -2^j, \qquad u_{k_a}^{(j)}(a) > -1, \qquad u_{k_a}^{(j)} \le 0.$$

Let U_a be a neighbourhood of a such that $u_{k_a}^{(j)} > -1$ on U_a. Now a standard argument using the compactness of L_j implies that there exists a continuous function $v_j \in \mathrm{PSH}(X,\omega)$ such that

(i) $v_j|_{K_j} \le -2^j$.
(ii) $v_j|_{L_j} > -1$.
(iii) $v_j \le 0$.

Set

$$v = \sum_{j=1}^{\infty} 2^{-j} v_j.$$

Then $v \in \mathrm{PSH}(X,\omega)$ and $S = \{v = -\infty\}$, and the proposition follows.

3.4. Proposition. *Let S be a closed complete locally ω-pluripolar set in X. Then S is complete ω-pluripolar.*

Proof. From the proof of Theorem 1 in [Co] it follows that there exist finite open covers $D_i'' \Subset D_i' \Subset D_i$, $1 \le i \le m$, of X and negative psh functions φ_i on D_i such that

(i) $S \cap D_i = \{\varphi_i = -\infty\}$, $X = \bigcup_{i=1}^m D_i''$.
(ii) $\varphi_i - \varphi_j$ is bounded on $D_i \cap D_j \setminus S$.
(iii) $\omega = dd^c\theta_i$ on D_i where θ_i is a strictly psh function on D_i and $\theta_i < 0$.

As in the proof of Theorem 1 in [Co] we can choose $p_i \in C_0^\infty(X)$ with $p_i \ge 0$ and $\operatorname{supp} p_i \subset D_i'$ such that

$$\varphi_i + p_i < \varphi_j + p_j \quad \text{on } (\partial D_i' \cap D_j'') \setminus S. \tag{1}$$

Set

$$\varphi(z) = \frac{1}{M} \sup_{1\le i\le m} \{\varphi_i(z) + p_i(z) : z \in D_i'\}$$

where $M > 0$ is chosen such that $p_i/M + \theta_i$ is psh on D_i' for $1 \le i \le m$. From (1) we see that φ is upper semicontinuous on X. Moreover, (iii) implies that $\varphi \in \mathrm{PSH}(X,\omega)$. It is easy to check that $S = \varphi^{-1}(-\infty)$.

Remark. Proposition 3.4 was in fact proved in [DLS] by Demailly–Lempert–Shiffman.

Now we investigate complete pluripolarity in the case $\dim X = 1$. We have the following result.

3.5. Proposition. *Let $\dim X = 1$ and S an ω-pluripolar set in X. Then*

(i) $S = S^*$.
(ii) *S is complete ω-pluripolar if and only if S is a G_δ.*

Proof. (i) Take an ω-psh function u on X such that $u \not\equiv -\infty$ and $S \subset \{u = -\infty\}$. Let $z \notin S$. Since $\dim X = 1$, by [Lan] there exists a decreasing neighbourhood basis U_j of z such that $\inf_{\partial U_j} u > -\infty$. Take $\varepsilon_j > 0$ such that $\inf_{\partial U_j} \varepsilon_j u > -1$. Define

$$v_j(z) = \begin{cases} \max\{\varepsilon_j u(z), -1\} & \text{on } U_j, \\ \varepsilon_j u(z) & \text{on } X \setminus U_j. \end{cases}$$

It follows that v_j is ω-psh on X with $v_j \geq -1$ on U_j and $v_j = -\infty$ on $S \setminus U_j$. Let

$$v = \sum_{j=1}^{\infty} \frac{v_j}{2^j}.$$

From the convexity of $\mathrm{PSH}(X, \omega)$ it follows that v is ω-psh on X with $v(z) > -1$ and $v = -\infty$ on S. Hence $z \notin S^*$ and the desired conclusion follows.

(ii) The necessity is obvious. It remains to prove the sufficiency. Assume that S is a G_δ ω-pluripolar set. Fix $z \in X$. Take a coordinate neighbourhood U_z of z in X and a smooth subharmonic function θ_z on a neighbourhood of $\overline{U}_z$ such that $\omega = dd^c\theta_z$. Since $S \cap U_z$ is a G_δ polar set, Deny's theorem (see [Lan]) implies that there exists a subharmonic function u_z on U_z such that $U_z \cap S = \{u_z = -\infty\}$. Let φ be an ω-psh function on X such that $S \subset \{\varphi = -\infty\}$ and $\varphi \not\equiv -\infty$. As in the proof of (i) we can find an ω-psh function φ_z on X such that $\varphi_z \geq -1$ on $\overline{U'_z}$ and $U'_z \cap S \subset \{\varphi_z = -\infty\}$ where U'_z is some neighbourhood of z with $U'_z \Subset U_z$. Define

$$\psi_z = \begin{cases} \max\{u_z - \sup_{\overline{U'_z}} u_z - 1 - \theta_z + \inf_{\overline{U'_z}} \theta_z, \varphi_z\} & \text{on } U'_z, \\ \varphi_z & \text{on } X \setminus U'_z. \end{cases}$$

It follows that ψ_z is ω-psh on X with $U'_z \cap S = \{\psi_z = -\infty\}$. By the compactness of X we can find a finite open cover U'_{z_j}, $j = 1, \ldots, m$, of X. Put

$$\psi = \frac{1}{m} \sum_{j=1}^{m} \psi_{z_j}.$$

Then ψ is ω-psh with $S = \{\psi = -\infty\}$.

4. Complete ω-pluripolar sets in $\mathbb{CP}^n$. This section is devoted to studying the complete ω-pluripolarity of subsets in $\mathbb{CP}^n$ equipped with the Fubini–Study Kähler form $\omega = \omega_{\mathrm{FS}}$. First we prove the following

4.1. Proposition. *Let $S \subset \mathbb{CP}^n$. Then S is complete ω-pluripolar if and only if $S \cap U_j$ is complete pluripolar in U_j for $0 \leq j \leq n$ where*

$$U_j = \{z = [z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_j \neq 0\}$$

Proof. Necessity. Let S be a complete ω-pluripolar subset in $\mathbb{CP}^n$. Then there exists an ω-psh function φ on $\mathbb{CP}^n$ such that $\varphi \not\equiv -\infty$ and $S = \{\varphi = -\infty\}$. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection. Then $\pi|_{V_j} : V_j \to U_j$ is biholomorphic where

$$V_j = \{(z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n)\} \subset \mathbb{C}^{n+1} \setminus \{0\}.$$

The function $\psi(z) = \varphi(\pi(z)) + \frac{1}{2}\log(\sum_{k=0}^{n} |z_k|^2)$, $z \in V_j$, is plurisubharmonic on V_j and $(\pi|_{V_j})^{-1}(S) \cap V_j = \{\psi = -\infty\}$. Hence $(\pi|_{V_j})^{-1}(S) \cap V_j$ is complete pluripolar in V_j. From $S \cap U_j = \pi|_{V_j}((\pi|_{V_j})^{-1}(S) \cap V_j)$ it follows that $S \cap U_j$ is complete pluripolar in U_j for $0 \le j \le n$.

Sufficiency. Assume that $S \cap U_j$ is complete pluripolar for $0 \le j \le n$. Since $\mathbb{CP}^n \setminus U_j$ is complete ω-pluripolar, we can find an ω-psh function u_j on $\mathbb{CP}^n$ such that

$$\{u_j = -\infty\} = \mathbb{CP}^n \setminus U_j.$$

By [Si] there exists $v_j \in \mathcal{L}(U_j)$ such that $\{v_j = -\infty\} = S \cap U_j$. The example 1.2 in [GZ1] shows that the function

$$\widetilde{v}_j(z) = \begin{cases} v_j(z) - \frac{1}{2}\log(1 + \|z\|^2) & \text{for } z \in U_j, \\ \overline{\lim\limits_{U_j \ni w \to z}} \big(v_j(w) - \frac{1}{2}\log(1 + \|w\|^2)\big) & \text{for } z \in \mathbb{CP}^n \setminus U_j, \end{cases}$$

belongs to $\mathrm{PSH}(\mathbb{CP}^n, \omega)$. Moreover $\{\widetilde{v}_j = -\infty\} \cap U_j = S \cap U_j$. Let

$$\varphi_j = \frac{u_j + \widetilde{v}_j}{2}.$$

Then

$$\begin{aligned} &\varphi_j \in \mathrm{PSH}(\mathbb{CP}^n, \omega), \quad \{\varphi_j = -\infty\} \cap U_j = S \cap U_j, \\ &\varphi_j = -\infty \quad \text{on } \mathbb{CP}^n \setminus U_j. \end{aligned} \tag{2}$$

By (2) if $\varphi = \max\{\varphi_j : 0 \le j \le n\}$ then φ is ω-psh on $\mathbb{CP}^n$ and $\{\varphi = -\infty\} = S$. The proof of Proposition 4.1 is complete.

Next we establish a result on complete ω-pluripolarity of a subset in $\mathbb{CP}^n$.

4.2. PROPOSITION. *Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection and $S \subset \mathbb{CP}^n$. Then S is complete ω-pluripolar if and only if $\widetilde{S} = \pi^{-1}(S) \cup \{0\}$ is complete pluripolar in $\mathbb{C}^{n+1}$.*

Proof. Assume that S is complete ω-pluripolar. Take an ω-psh function φ on $\mathbb{CP}^n$ with $\varphi \not\equiv -\infty$ and $S = \{\varphi = -\infty\}$. Consider $\widetilde{\varphi}(z) = \varphi(\pi(z)) + \log\|z\|$ for $z \in \mathbb{C}^{n+1} \setminus \{0\}$. Since φ is an ω-psh function on $\mathbb{CP}^n$ it follows that $\widetilde{\varphi}$ is plurisubharmonic on $\mathbb{C}^{n+1} \setminus \{0\}$, and hence on $\mathbb{C}^{n+1}$. Because $\widetilde{\varphi}(0) = \overline{\lim}_{z \to 0}(\varphi(\pi(z)) + \log\|z\|) = -\infty$ we infer that $\widetilde{S} = \{\widetilde{\varphi} = -\infty\}$. Hence $\widetilde{S}$ is complete pluripolar in $\mathbb{C}^{n+1}$.

Conversely, assume that $\widetilde{S}$ is complete pluripolar in $\mathbb{C}^{n+1}$. For each $0 \le j \le n$, let $V_j = \{(z_0, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n)\} \subset \mathbb{C}^{n+1} \setminus \{0\}$ and $U_j = \{z = [z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_j \ne 0\}$. Then $\pi|_{V_j} : V_j \to U_j$ is biholomorphic and $\widetilde{S} \cap V_j = \pi^{-1}(S) \cap V_j$ is complete pluripolar in V_j. This implies that $S \cap U_j$ is complete pluripolar in U_j. Proposition 4.1 implies that S is complete ω-pluripolar in $\mathbb{CP}^n$.

4.3. PROPOSITION. *Let S be an ω-pluripolar set in $\mathbb{CP}^n$. Then*

$$[S]^*_{\mathbb{CP}^n} \cap U_j = [S \cap U_j]^*_{U_j} \quad \textit{for } j = 0, \ldots, n,$$

and hence

$$[S]^*_{\mathbb{CP}^n} = \bigcup_{j=0}^{n} [S \cap U_j]^*_{U_j}.$$

Proof. It is easy to see that $[S \cap U_j]^*_{U_j} \subset [S]^*_{\mathbb{CP}^n} \cap U_j$. Hence, it remains to show that $[S]^*_{\mathbb{CP}^n} \cap U_j \subset [S \cap U_j]^*_{U_j}$ for $0 \le j \le n$. Let $z_0 \in [S]^*_{\mathbb{CP}^n} \cap U_j$ and $u \in \mathrm{PSH}(U_j)$ with $u = -\infty$ on $S \cap U_j$. By [Si] we may assume that $u \in \mathcal{L}(U_j)$. As in the proof of Proposition 4.1 the function

$$\widetilde{u}(z) = \begin{cases} u(z) - \frac{1}{2}\log(1 + \|z\|^2) & \text{for } z \in U_j, \\ \overline{\lim}_{U_j \ni w \to z} \big(u(w) - \frac{1}{2}\log(1 + \|w\|^2)\big) & \text{for } z \in \mathbb{CP}^n \setminus U_j, \end{cases}$$

is ω-psh on $\mathbb{CP}^n$ and $\widetilde{u} = -\infty$ on $S \cap U_j$. Let v be an ω-psh function on $\mathbb{CP}^n$ such that $\{v = -\infty\} = \mathbb{CP}^n \setminus U_j$, and set $\varphi = (\widetilde{u} + v)/2$. Then φ is psh on $\mathbb{CP}^n$ and $\varphi = -\infty$ on S. Hence $\varphi(z_0) = -\infty$. Thus $\widetilde{u}$, and therefore u, is equal to $-\infty$ at z_0. This shows that $z_0 \in [S \cap U_j]^*_{U_j}$.

4.4. PROPOSITION. *Let S be a G_δ set which is a countable union of compact complete pluripolar sets in $\mathbb{C}^n$. Then S is complete ω-pluripolar in $\mathbb{CP}^n = \mathbb{C}^n \cup H_\infty$.*

Proof. We write $S = \bigcup_{j=1}^{\infty} S_j$, where S_j are compact complete pluripolar sets in $\mathbb{C}^n$. Proposition 3.4 implies that S_j is complete ω-pluripolar in $\mathbb{CP}^n$. On the other hand,

$$[S]^*_{\mathbb{CP}^n} = \bigcup_{j=1}^{\infty} [S_j]^*_{\mathbb{CP}^n} = \bigcup_{j=1}^{\infty} S_j = S.$$

Now the desired conclusion follows from Proposition 3.3.

4.5. EXAMPLES. (a) Let f be an entire function on $\mathbb{C}$ and $E = \{(z, f(z)) : z \in \mathbb{C}\} = \{[1 : z : f(z)] : z \in \mathbb{C}\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$. We have

$$[E]^*_{\mathbb{CP}^2} \cap U_0 = [E \cap U_0]^*_{U_0} = E,$$
$$\begin{aligned} [E]^*_{\mathbb{CP}^2} \cap U_1 &= [E \cap U_1]^*_{U_1} r = [\{(1/z, f(z)/z) : z \in \mathbb{C}^*\}]^*_{\mathbb{C}^2} \\ &= [\{(z, zf(1/z)) : z \in \mathbb{C}^*\}]^*_{\mathbb{C}^2} = E \cap U_1 \quad \text{(by [Wie])}. \end{aligned}$$

Let now $f(z) = e^z$. We have

$$\begin{aligned} [E]^*_{\mathbb{CP}^2} \cap U_2 &= [E \cap U_2]^*_{U_2} = [\{(e^{-z}, ze^{-z}) : z \in \mathbb{C}\}]^*_{\mathbb{C}^2} \\ &= [\{(e^{z}, -ze^{z}) : z \in \mathbb{C}\}]^*_{\mathbb{C}^2} \quad \text{(by Corollary 2.6 in [Edi])} \\ &= \{(e^{z}, -ze^{z}) : z \in \mathbb{C}\} = \{(e^{-z}, ze^{-z}) : z \in \mathbb{C}\} = E \cap U_2. \end{aligned}$$

Thus $\{(z, e^z) : z \in \mathbb{C}\}$ is complete ω-pluripolar in $\mathbb{CP}^2$.

Now we give an example in which the pluripolar hull of a graph for the class of ω-psh functions may not coincide with the graph. Let $P(t) = c_d t^d + \cdots + c_0$ be a polynomial of degree $d > 1$. Consider the graph

$$E = \{(\lambda, P(\lambda)) : \lambda \in \mathbb{C}\} = \{[1 : \lambda : P(\lambda)] : \lambda \in \mathbb{C}\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$$

We show that

$$[E]^*_{\mathbb{CP}^2} = E \cup \{[0 : 0 : 1]\}$$

where $[E]^*_{\mathbb{CP}^2}$ denotes the pluripolar envelope of E for the class $\mathrm{PSH}(\mathbb{CP}^2, \omega)$. It is easy to see that $E \subset [E]^*_{\mathbb{CP}^2}$. We show that $\{[0 : 0 : 1]\} \in [E]^*_{\mathbb{CP}^2}$. Let $u \in \mathrm{PSH}(\mathbb{CP}^2, \omega)$ be such that $u([1 : \lambda : P(\lambda)]) = -\infty$ for $\lambda \in \mathbb{C}$. Define

$$\varphi(\xi, \eta) = u([\xi : \eta : 1]) + \frac{1}{2}\log(1 + |\xi|^2 + |\eta|^2)$$

for $(\xi, \eta) \in \mathbb{C}^2$. From the ω-plurisubharmonicity of u it follows that φ is psh on $\mathbb{C}^2$ and

$$\begin{aligned}\varphi\left(\frac{1}{P(\lambda)}, \frac{\lambda}{P(\lambda)}\right) &= u\left(\left[\frac{1}{P(\lambda)} : \frac{\lambda}{P(\lambda)} : 1\right]\right) + \frac{1}{2}\log\left(1 + \frac{1 + |\lambda|^2}{|P(\lambda)|^2}\right) \\ &= u([1 : \lambda : P(\lambda)]) + \frac{1}{2}\log\left(1 + \frac{1 + |\lambda|^2}{|P(\lambda)|^2}\right) = -\infty\end{aligned}$$

for $\lambda \in \mathbb{C} \setminus P^{-1}(0)$. Take $R > 0$ sufficiently large such that $P(\lambda) = 0$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < R$. Thus

$$\varphi\left(\frac{1}{P(1/\lambda)}, \frac{1}{\lambda P(1/\lambda)}\right) = -\infty \quad \text{on } \mathbb{C} \setminus \{|\lambda| < R\},$$

and hence

$$\varphi\left(\frac{1}{P(1/\lambda)}, \frac{1}{\lambda P(1/\lambda)}\right) = -\infty \quad \text{for } 0 < |\lambda| < 1/R.$$

Consider the function

$$\psi(\lambda) = \varphi\left(\frac{\lambda^d}{c_d + \cdots + c_0\lambda^d}, \frac{\lambda^{d-1}}{c_d + \cdots + c_0\lambda^d}\right)$$

for $|\lambda| < 1/R$. Then ψ is subharmonic on $\{|\lambda| < 1/R\}$ and $\psi(\lambda) = -\infty$ for $0 < |\lambda| < 1/R$. Therefore $\psi(0) = -\infty$, and consequently

$$u([0 : 0 : 1]) = \varphi(0, 0) = \psi(0) = -\infty.$$

Thus $\{[0 : 0 : 1]\} \in [E]^*_{\mathbb{CP}^2}$.

Conversely, we show that $[E]^*_{\mathbb{CP}^2} \subset E \cup \{[0 : 0 : 1]\}$. Assume that $[x_0 : y_0 : z_0] \in \mathbb{CP}^2 \setminus E \cup \{[0 : 0 : 1]\}$. Consider the function

$$u([x,y,z]) = \begin{cases} \frac{1}{d}\log\left|\frac{z}{x} - P\left(\frac{y}{x}\right)\right| - \frac{1}{2}\log\left(1 + \frac{|y|^2+|z|^2}{|x|^2}\right) & \text{for } x \neq 0, \\ \varlimsup_{[x',y',z']\to[0,y,z]} \left\{\frac{1}{d}\log\left|\frac{z'}{x'} - P\left(\frac{y'}{x'}\right)\right| - \frac{1}{2}\log\left(1+\frac{|y'|^2+|z'|^2}{|x'|^2}\right)\right\} & \text{for } x = 0. \end{cases}$$

Example 1.2 in [GZ1] implies that $u \in \mathrm{PSH}(\mathbb{CP}^2,\omega)$. We have $u|_E = -\infty$. Now we check that $u([x_0 : y_0 : z_0]) > -\infty$. First we assume that $x_0 \neq 0$ and $u([x_0 : y_0 : z_0]) = -\infty$. Then $d^{-1}\log|z_0/x_0 - P(y_0/x_0)| = -\infty$, and consequently $z_0/x_0 = P(y_0/x_0)$. Hence $[x_0 : y_0 : z_0] \in E$, which is impossible. In the case $x_0 = 0$ and $y_0 \neq 0$ we have

$$\begin{aligned} u([0 : y_0 : z_0]) &\geq \varlimsup_{x\to 0}\left\{\frac{1}{d}\log\left|\frac{z_0}{x} - P\left(\frac{y_0}{x}\right)\right| - \frac{1}{2}\log(|x|^2+|y_0|^2+|z_0|^2) + \log|x|\right\} \\ &\geq \frac{1}{d}\log|c_d| + \frac{1}{2}\log\frac{|y_0|^2}{|y_0|^2+|z_0|^2} > -\infty. \end{aligned}$$

Finally, if $y_0 = 0$ then $z_0 \neq 0$ and $[0 : 0 : z_0] = [0 : 0 : 1]$, which is also impossible. Thus $[E]^*_{\mathbb{CP}^2} = E \cup \{[0 : 0 : 1]\}$.

(b) Let $f(z) = e^{1/z}$, $z \neq 0$, and $E = \{(z, e^{1/z}) : z \neq 0\} = \{[1 : z : e^{1/z}] : z \neq 0\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$. We have

$$\begin{aligned} E^*_{\mathbb{CP}^2} \cap U_0 &= [E \cap U_0]^*_{U_0} = [\{(z, e^{1/z}) : z \neq 0\}]^*_{\mathbb{C}^2} = E \cap U_0 \quad \text{(by [Wie])}, \\ E^*_{\mathbb{CP}^2} \cap U_1 &= [E \cap U_1]^*_{U_1} = [\{(1/z, e^{1/z}/z) : z \neq 0\}]^*_{\mathbb{C}^2} \\ &= [\{(z, ze^z) : z \neq 0\}]^*_{\mathbb{C}^2} = (E \cap U_1) \cup \{[0 : 1 : 0]\}, \\ [E]^*_{\mathbb{CP}^2} \cap U_2 &= [E \cap U_2]^*_{U_2} = [\{(e^{-1/z}, ze^{-1/z}) : z \neq 0\}]^*_{\mathbb{C}^2} \\ &= [\{(e^z, -e^z/z) : z \neq 0\}]^*_{\mathbb{C}^2} = [\pi(P)]^*_{\mathbb{C}^2}, \end{aligned}$$

where $\pi : \mathbb{C}^2 \to \mathbb{C}^2$, $\pi(z,w) = (e^z, w)$ and $P = \{(z, -e^z/z) : z \neq 0\}$. Since P is locally closed in $\mathbb{C}^2$ and π is an A-covering map (see the precise definition in [Edi]) and by Theorem 2.5 in [Edi] we have

$$[\pi(P)]^*_{\mathbb{C}^2} = \pi(P^*) = \pi(P) = E \cap U_2.$$

Thus $[E]^*_{\mathbb{CP}^2} \cap U_2 = E \cap U_2$. Therefore

$$[E]^*_{\mathbb{CP}^2} = E \cup \{[0 : 1 : 0]\}.$$

5. Subextension of ω-psh functions. Let X be a compact Kähler manifold with fundamental form ω, and Ω a hyperconvex domain in X. Assume that $\varphi \in \mathrm{PSH}(\Omega)$. In this section we investigate the existence of an ω-psh function $\widetilde{\varphi}$ on X such that $\widetilde{\varphi} \leq \varphi$ on Ω. Such an ω-psh function is said to be a *subextension* of φ. Now we have

5.1. THEOREM. *Let Ω be a hyperconvex domain in X such that ω has a negative potential θ on Ω. Assume that $\varphi \in \mathcal{F}(\Omega)$. Then there exist $a > 0$ and $\widetilde{\varphi} \in \mathrm{PSH}(X, a\omega)$ such that $\widetilde{\varphi} \not\equiv -\infty$ and $\widetilde{\varphi} \le \varphi$ on Ω.*

Proof. Let $\mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi$ be such that $\alpha = \int_\Omega (dd^c\varphi)^n < \infty$. Take an increasing exhaustion sequence $\{\Omega_j\}$ of Ω by relatively compact subdomains $\Omega_j \Subset \Omega$. For each $j \ge 1$, put

$$h_j = h_{\Omega_j,\varphi_j} = \sup\{v \in \mathrm{PSH}(\Omega) : v \le 0 \text{ and } v|_{\Omega_j} \le \varphi_j\}.$$

Then $\mathcal{E}_0(\Omega) \ni h_j \searrow \varphi$ and

$$\alpha_j = \int_\Omega (dd^c h_j)^n \le \int_\Omega (dd^c h_{j+1})^n = \alpha_{j+1} \to \alpha$$

(see Proposition 5.1 in [Ce2]). Consider the probability measure $\mu_j = (1/\alpha_j)(dd^c h_j)^n$ on X. Notice that $\mathrm{supp}(dd^c h_j)^n \subset \overline{\Omega}_j$. Theorem 5.1 in [Ko] and Proposition 2.10 in [GZ1] imply that for each $j, p \ge 1$ there exist $A_p, B_p^j > 0$ such that

$$\begin{aligned}
\mu_j(E) = \mu_j(E \cap \overline{\Omega}_j) &\le \frac{1}{\alpha_j} \int_\Omega (-h^*_{E\cap\overline{\Omega}_j})^p (dd^c h_j)^n \\
&\le \frac{A_p}{\alpha_j} \Big(\int_\Omega (-h_j)^p (dd^c h_j)^n \Big)^{n/(p+n)} \Big(\int_\Omega (-h^*_{E\cap\overline{\Omega}_j})^p (dd^c h^*_{E\cap\Omega_j})^n \Big)^{p/(p+n)} \\
&\le \frac{A_p}{\alpha_j} \Big(\int_\Omega (-h_j)^p (dd^c h_j)^n \Big)^{n/(p+n)} C_n(E \cap \overline{\Omega}_j, \Omega)^{p/(p+n)} \\
&\le B_p^j \,\mathrm{Cap}_\omega(E \cap \overline{\Omega}_j, X)^{p/(p+n)}
\end{aligned}$$

for all Borel sets $E \subset X$. Proposition 6.5 and Theorem 5.1 in [GZ2] imply that there exists $v_j \in \mathcal{E}^p(X,\omega)$ such that

$$\omega^n_{v_j} = \mu_j \quad \text{and} \quad \sup_X v_j = -1.$$

Since

$$(dd^c(v_j + \theta))^n = \omega^n_{v_j} = \mu_j = \Big(dd^c \Big(\frac{1}{\alpha_j^{1/n}} h_j \Big) \Big)^n$$

on Ω, by the comparison principle in [BT] it follows that

$$v_j + \theta \le \frac{1}{\alpha_j^{1/n}} h_j$$

on Ω. Thus for $u_j = \alpha_j^{1/n}(v_j + c)$ with $c = \inf_\Omega \theta < 0$ we have $u_j \in \mathrm{PSH}(X, \alpha_j^{1/n}\omega) \cap L^\infty(X) \subset \mathrm{PSH}(X, \alpha^{1/n}\omega) \cap L^\infty(X)$ and $\sup_X u_j = \alpha_j^{1/n}(c-1)$, $u_j \le h_j \le \varphi_j$ on Ω_j. Define $\widetilde{\varphi} = (\overline{\lim}_{j\to\infty} u_j)^*$. Since $\sup_X u_j = \alpha_j^{1/n}(c-1) \to \alpha^{1/n}(c-1)$ as $j \to \infty$, we have $\widetilde{\varphi} \not\equiv -\infty$ and it is easy to see that $\widetilde{\varphi} \in \mathrm{PSH}(X, a\omega)$ with $a = \alpha^{1/n}$ and $\widetilde{\varphi} \le \varphi$ on Ω. Theorem 5.1 is completely proved.

5.2. COROLLARY. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $\varphi \in \mathcal{F}(\Omega)$. Then there exists $\widetilde{\varphi} \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ such that $\widetilde{\varphi} \le \varphi$ on Ω. Here*

$$\varepsilon = \Big[\int_\Omega (dd^c\varphi)^n\Big]^{1/n},$$

$$\mathcal{L}_\varepsilon = \{u \in \mathrm{PSH}(\mathbb{C}^n) : u(z) \le \varepsilon \log^+ \|z\| + O(1)\}.$$

Proof. Consider Ω as a domain in $\mathbb{CP}^n = \mathbb{C}^n \cup H_\infty$. By Theorem 5.1 there exists $\psi \in \mathrm{PSH}(\mathbb{CP}^n, \varepsilon\omega)$ such that $\psi \le \varphi$ on Ω and $\psi \not\equiv -\infty$. Define

$$\widetilde{\varphi}(z) = \psi(z) + \frac{\varepsilon}{2}\log(1 + \|z\|^2) - c$$

with

$$c = \sup_\Omega \frac{\varepsilon}{2}\log(1 + \|z\|^2).$$

It follows that $\widetilde{\varphi} \in \mathcal{L}_\varepsilon(\mathbb{C}^n)$ and $\widetilde{\varphi} \le \varphi$ on Ω.

REMARK. Corollary 5.2 was proved as Theorem 5.1 of [CKZ].

6. Appendix: The comparison principle in the class $\mathcal{E}(X,\omega)$. In [Ko] Kołodziej proved the comparison principle for bounded ω-psh functions by using the approximation theorem of Demailly [De]. The aim of this section is to establish this principle in the class $\mathcal{E}(X,\omega)$. Notice that here we give a direct proof without using Demailly's theorem.

6.1. THEOREM. *Let $\varphi, \psi, \varphi_1, \ldots, \varphi_{n-1} \in \mathcal{E}(X,\omega)$ and $T = \omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_{n-1}}$. Then*

$$\int_{\{\varphi<\psi\}} \omega_\psi \wedge T \le \int_{\{\varphi<\psi\}} \omega_\varphi \wedge T + \int_{\{\varphi=\psi=-\infty\}} \omega_\varphi \wedge T.$$

Proof. We split the proof into the following two steps.

STEP 1. First we prove that

$$\int_{\{\varphi<\psi\}} \omega_\psi \wedge T \le \int_{\{\varphi\le\psi\}} \omega_\varphi \wedge T. \tag{3}$$

For this, we establish the equality

$$\int_X (dd^c\varphi + \omega) \wedge T = \int_X \omega \wedge T. \tag{4}$$

Assume for the moment that (4) is true. Put $\varphi_\varepsilon = \max(\varphi + \varepsilon, \psi)$, $\varepsilon > 0$. From (4) it follows that

$$\int_X (dd^c\varphi_\varepsilon + \omega) \wedge T = \int_X \omega^n = \int_X (dd^c\varphi + \omega) \wedge T.$$

This equality together with the equality

$$(dd^c\varphi_\varepsilon + \omega) \wedge T|_{\{\varphi+\varepsilon>\psi\}} = (dd^c\varphi + \omega) \wedge T|_{\{\varphi+\varepsilon>\psi\}} \quad \text{(see [KH])}$$

implies that

$$\int_{\{\varphi+\varepsilon\le\psi\}} (dd^c\varphi_\varepsilon + \omega) \wedge T \le \int_{\{\varphi\le\psi\}} (dd^c\varphi + \omega) \wedge T.$$

On the other hand,

$$(dd^c\varphi_\varepsilon + \omega) \wedge T|_{\{\varphi+\varepsilon<\psi\}} = (dd^c\psi + \omega) \wedge T|_{\{\varphi+\varepsilon<\psi\}}$$

so we obtain

$$\int_{\{\varphi+\varepsilon<\psi\}} (dd^c\psi + \omega) \wedge T = \int_{\{\varphi+\varepsilon<\psi\}} (dd^c\varphi_\varepsilon + \omega) \wedge T \le \int_{\{\varphi\le\psi\}} (dd^c\varphi + \omega) \wedge T.$$

Letting ε tend to 0 we obtain

$$\int_{\{\varphi<\psi\}} \omega_\psi \wedge T \le \int_{\{\varphi\le\psi\}} \omega_\varphi \wedge T,$$

because $\{\varphi + \varepsilon < \psi\} \nearrow \{\varphi < \psi\}$ as $\varepsilon \to 0$. Thus (3) follows.

To prove (4), we first observe that by Stokes' formula, if φ is bounded then

$$\int_X dd^c\varphi \wedge T = 0. \tag{5}$$

Next consider the case $\varphi \in \mathcal{E}(X,\omega)$. Set $\varphi_j = \max(\varphi, -j)$. Notice that $\varphi_j \in \mathcal{E}(X,\omega) \cap L^\infty(X)$ and $\varphi_j \searrow \varphi$. Therefore $dd^c\varphi_j \wedge T$ weakly converges to $dd^c\varphi \wedge T$. Using the above result we have $\int_X dd^c\varphi_j \wedge T = 0$ for all j, and hence $\int_X dd^c\varphi \wedge T = 0$.

Step 2. Applying Step 1 to $\varphi + \varepsilon$ and ψ we get

$$\int_{\{\varphi+\varepsilon<\psi\}} \omega_\psi \wedge T \le \int_{\{\varphi+\varepsilon\le\psi\}} \omega_\varphi \wedge T.$$

Letting ε tend to 0 we have

$$\int_{\{\varphi<\psi\}} \omega_\psi \wedge T \le \int_{\{\varphi<\psi\}} \omega_\varphi \wedge T + \int_{\{\varphi=\psi=-\infty\}} \omega_\varphi \wedge T,$$

because $\{\varphi + \varepsilon \le \psi\} \nearrow \{\varphi < \psi\} \cup \{\varphi = \psi = -\infty\}$ as $\varepsilon \to 0$.

6.2. Corollary. *Let $\varphi, \psi \in \mathcal{E}(X,\omega)$. Then*

$$\int_{\{\varphi<\psi\}} \omega_\psi^n \le \int_{\{\varphi<\psi\}} \omega_\varphi^n + \int_{\{\varphi=\psi=-\infty\}} \omega_\varphi^n.$$

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Department of Mathematics
Hanoi University of Education (Dai hoc Su Pham Hanoi)
Tuliem, Ha Noi, Vietnam
E-mail: mauhai@fpt.vn

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