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Pullback attractors for non-autonomous 2D MHD equations on some unbounded domains

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Abstract. We study the 2D magnetohydrodynamic (MHD) equations for a viscous incompressible resistive fluid, a system with the Navier–Stokes equations for the velocity field coupled with a convection-diffusion equation for the magnetic fields, in an arbitrary (bounded or unbounded) domain satisfying the Poincaré inequality with a large class of non-autonomous external forces. The existence of a weak solution to the problem is proved by using the Galerkin method. We then show the existence of a unique minimal pullback D_σ-attractor for the process associated to the problem. An upper bound on the fractal dimension of the pullback attractor is also given.

1. Introduction. Let Ω be an arbitrary (bounded or unbounded) domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. In this paper we consider the following non-autonomous 2D magnetohydrodynamic (MHD) equations:

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \dfrac{1}{R_e}\Delta u + \nabla\left(p + \dfrac{S}{2}|B|^2\right) - S(B\cdot\nabla)B = f, \\ \dfrac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u + \dfrac{1}{R_m}\tilde{\operatorname{curl}}(\operatorname{curl} B) = 0, \\ \operatorname{div} u = 0, \\ \operatorname{div} B = 0, \end{cases}
$$

with the initial conditions

$$
(1.2)\qquad u(x,\tau) = u_0(x), \quad B(x,\tau) = B_0(x), \quad \forall x \in \Omega,
$$

and the boundary conditions

$$
(1.3)\qquad \begin{cases} u = 0 \quad \text{on } \partial\Omega, \\ B\cdot n = 0 \quad \text{and} \quad \operatorname{curl} B = 0 \quad \text{on } \partial\Omega, \end{cases}
$$

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where $u = (u_1(x,t), u_2(x,t))$ is the velocity of the particulate of fluid which is at point x at time t, $B = (B_1(x,t), B_2(x,t))$ is the magnetic field at point x at time t, $p = p(x,t)$ is the pressure of the fluid, the term $|B|^2/2$ denotes the magnetic pressure, $f = f(x,t)$ represents a volume density force, n is the unit outward normal on $\partial\Omega$, R_e is the Reynolds number, R_m is the magnetic Reynolds number, $S = M^2/(R_e R_m)$, where M is the Hartman number, and

$$\operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \qquad \text{for every vector function } u = (u_1, u_2),$$

$$\tilde{\operatorname{curl}}\, \phi = \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right) \qquad \text{for every scalar function } \phi,$$

$$\tilde{\operatorname{curl}}(\operatorname{curl} u) = \operatorname{grad} \operatorname{div} u - \Delta u.$$

The MHD equations govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasma (see [C]). Because of their significant role in physics, mathematical questions related to MHD equations have attracted interest of many mathematicians in the past years. The existence and uniqueness of both weak and strong solutions to the MHD equations in bounded domains was proved in [DL, ST], while the well-posedness of the initial-value problem in some other function spaces was proved in [CMZ, MY]. The regularity of solutions has been studied extensively in recent years (see e.g. [CW, G, JZ, K]). The large-time behavior of solutions, including the decay properties of solutions and the stability of stationary solutions, has been studied in [AgS, SSS, ZL].

As is well known, a useful way for studying the long-time behavior of solutions is to use the theory of attractors. The classical global attractor for autonomous dynamical systems is an invariant compact set which attracts all bounded sets and contains some important information about the long-time behavior of the solutions. For the autonomous 2D MHD equations in bounded domains, the existence of a finite-dimensional global attractor for the semigroup generated by the equations was proved in [ST, T97]. The aim of this paper is to extend this result to the non-autonomous case, i.e. the external force depends on the time variable, and in some domains not necessarily bounded.

More precisely, the domain Ω can be an arbitrary bounded or unbounded open set in $\mathbb{R}^2$ without any regularity assumption on its boundary $\partial\Omega$ and with the assumption that the Poincaré inequality holds on it, i.e., there exists $\lambda_1 > 0$ such that

$$\int_\Omega |\phi|^2\,dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2\,dx \quad \text{for all } \phi \in H_0^1(\Omega). \tag{1.4}$$

We also require that the domain Ω satisfies the cone condition so that Lemma 2.2 in Section 2 is valid on Ω (see [AF, Chapter 5] for details).

Let us now explain the method used in this paper. Because the external force is time-dependent, we will use the theory of pullback attractors to study the long-time behavior of solutions to problem (1.1)–(1.3). This theory is a natural generalization of the theory of global attractors for autonomous dynamical systems and allows considering a number of different problems of non-autonomous dynamical systems for a large class of non-autonomous forcing terms (see e.g. the recent monograph [CLR]). To prove the existence of a pullback attractor, the usual approach is to obtain a family of bounded pullback absorbing sets in a more regular space and then use the compactness of the Sobolev embeddings. Here, because the domain considered may be unbounded, the Sobolev embedding is no longer compact, and therefore the method used in [ST, T97] for 2D MHD equations in bounded domains no longer works. To overcome this difficulty, we exploit the energy equation method introduced by Ball [B] with suitable modifications so as to handle non-autonomous equations. As a result, we obtain the pullback asymptotic compactness of the process; combined with the existence of a family of pullback absorbing sets, this will lead to the existence of a pullback attractor. Such an approach has been used to prove the existence of pullback attractors for the 2D Navier–Stokes equations [CŁR], the Bénard or Bousinessq system [AS], and the Navier–Stokes–Voigt equations [AT] in the non-autonomous case. Finally, following the general lines of the approach in [LŁR], we show that the pullback attractor has a finite fractal dimension under some additional conditions on the external force and on the domain. The results obtained in this paper extend some existing ones for the 2D Navier–Stokes equations [CŁR, LŁR, Ros], and in particular, when f is time-independent, we recover and extend the results for 2D MHD equations in [ST, T97] to some unbounded domains.

The paper is organized as follows. In Section 2, for the convenience of the reader, we recall some auxiliary results on function spaces and operators related to the problem, and abstract results on the existence and fractal dimension of pullback attractors. In Section 3, we prove the existence and uniqueness of a weak solution to problem (1.1)–(1.3) by using the Galerkin method. In Section 4, using the energy equation method, we prove the existence of a pullback D_σ-attractor for the process associated to the problem. An upper bound on the fractal dimension of the pullback D_σ-attractor is given in the last section.

2. Preliminaries

2.1. Function spaces and operators. In this subsection we recall several function spaces necessary to write the MHD equations in their variational formulation.

We denote

$$\mathbb{L}^2(\Omega) = (L^2(\Omega))^2, \quad \mathbb{H}^m(\Omega) = (H^m(\Omega))^2, \quad \mathbb{H}_0^m(\Omega) = (H_0^m(\Omega))^2.$$

The spaces used in the theory of the MHD equations are a combination of spaces used for the Navier–Stokes equations and spaces used in the theory of Maxwell equations. They are

$$\begin{aligned}
\mathcal{V}_1 &= \{v \in (\mathcal{C}_0^\infty(\Omega))^2 : \nabla \cdot v = 0\},\\
V_1 &= \text{closure of } \mathcal{V}_1 \text{ in the } \mathbb{H}_0^1(\Omega) \text{ norm} = \{v \in \mathbb{H}_0^1(\Omega) : \nabla \cdot v = 0\},\\
H_1 &= \text{closure of } \mathcal{V}_1 \text{ in the } \mathbb{L}^2(\Omega) \text{ norm}\\
&= \{v \in \mathbb{L}^2(\Omega) : \nabla \cdot v = 0 \text{ and } v \cdot n|_{\partial\Omega} = 0\},\\
\mathcal{V}_2 &= \{C \in (\mathcal{C}^\infty(\bar{\Omega}))^2 : \nabla \cdot C = 0 \text{ and } C \cdot n|_{\partial\Omega} = 0\},\\
V_2 &= \text{closure of } \mathcal{V}_2 \text{ in the } \mathbb{H}^1(\Omega) \text{ norm}\\
&= \{C \in \mathbb{H}^1(\Omega) : \nabla \cdot C = 0 \text{ and } C \cdot n|_{\partial\Omega} = 0\},\\
H_2 &= \text{closure of } \mathcal{V}_2 \text{ in the } \mathbb{L}^2(\Omega) \text{ norm} = H_1,\\
V &= V_1 \times V_2, \qquad H = H_1 \times H_2.
\end{aligned}$$

The inner product and norm in V_1 are given by

$$\begin{aligned}
((u, \tilde{u}))_1 &= \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla \tilde{u}_i \, dx, \qquad \forall u, \tilde{u} \in V_1,\\
\|u\|_1 &= ((u, u))_1^{1/2}, \qquad \forall u \in V_1.
\end{aligned}$$

Due to (1.4), this norm is equivalent to the usual one in $\mathbb{H}_0^1(\Omega)$. The inner product and norm in V_2 are given by

$$\begin{aligned}
((B, \tilde{B}))_2 &= \int_\Omega \operatorname{curl} B \cdot \operatorname{curl} \tilde{B} \, dx, \qquad \forall B, \tilde{B} \in V_2,\\
\|B\|_2 &= ((B, B))_2^{1/2}, \qquad \forall B \in V_2.
\end{aligned}$$

Since the domain Ω is simply connected, the above bilinear form is actually a scalar product on V_2; it defines a norm which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on V_2 (see [DL]). Repeating the notations again for simplicity, we define the inner product and norm in V by

$$\begin{aligned}
((z, \tilde{z})) &= ((u, \tilde{u}))_1 + S((B, \tilde{B}))_2, \qquad \forall z = (u, B),\ \tilde{z} = (\tilde{u}, \tilde{B}) \in V,\\
\|z\| &= ((z, z))^{1/2}, \qquad \forall z \in V.
\end{aligned}$$

The inner products and norms in H_1 and H_2 are the usual ones inherited from $\mathbb{L}^2(\Omega)$. We define the inner product and norm in H by

$$\begin{aligned}
(z, \tilde{z}) &= (u, \tilde{u}) + S(B, \tilde{B}), \qquad \forall z = (u, B),\ \tilde{z} = (\tilde{u}, \tilde{B}) \in H,\\
|z| &= (z, z)^{1/2}, \qquad \forall z \in H.
\end{aligned}$$

Since S is positive, the inner products and norms defined above for H and V are equivalent to the usual ones defined on these product spaces.

It follows from (1.4) and the equivalence of norms in $\mathbb{H}^1$ and V_2 that there exists a positive constant c_0 such that

$$\lambda_1|u|^2 \le \|u\|_1^2, \qquad c_0|B|^2 \le \|B\|_2^2 \tag{2.1}$$

for all $u \in V_1$ and $B \in V_2$. Applying the Riesz representation theorem, we can identify the dual space H' with H and obtain the following relation: $V \subset H = H' \subset V'$, where the injections are continuous and each space is dense in the following ones.

Define the bilinear forms $a_i : V_i \times V_i \to \mathbb{R}$, for $i = 1, 2$, $a : V \times V \to \mathbb{R}$, and the corresponding linear operator $A : V \to V'$ by

$$\begin{aligned}
a_1(u, \tilde{u}) &= ((u, \tilde{u}))_1 = \int_\Omega \sum_{i=1}^2 \nabla u_i \cdot \nabla \tilde{u}_i \, dx, \\
a_2(B, \tilde{B}) &= ((B, \tilde{B}))_2 = \int_\Omega \operatorname{curl} B \cdot \operatorname{curl} \tilde{B} \, dx, \\
a(z, \tilde{z}) &= \langle Az, \tilde{z}\rangle_{V',V} = \frac{1}{R_e} a_1(u, \tilde{u}) + \frac{S}{R_m} a_2(B, \tilde{B}).
\end{aligned}$$

The operator A is clearly a homomorphism from V into V', and the bilinear form a is coercive since

$$\min\left(\frac{1}{R_e}, \frac{1}{R_m}\right)\|z\|^2 \le a(z, z) = \langle Az, z\rangle_{V',V} \le \max\left(\frac{1}{R_e}, \frac{1}{R_m}\right)\|z\|^2. \tag{2.2}$$

We now define the trilinear form b by

$$b(u, v, w) = \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

whenever the integrals make sense, and $\mathbb{B}_0 : V \times V \times V \to \mathbb{R}$ and $\mathbb{B}(z) = \mathbb{B}(z, z)$, its associated bilinear operator $\mathbb{B} : V \times V \to V'$, by

$$\begin{aligned}
\mathbb{B}_0(z_1, z_2, z_3) = b(u_1, u_2, u_3) &- Sb(B_1, B_2, u_3) + Sb(u_1, B_2, B_3) \\
&- Sb(B_1, u_2, B_3), \qquad \forall z_i = (u_i, B_i) \in V,\ i = 1, 2, 3.
\end{aligned}$$

It is easy to check that if $u, v, w \in V_i$, $i = 1, 2$, then

$$b(u, v, w) = -b(u, w, v). \tag{2.3}$$

Hence

$$b(u, v, v) = 0. \tag{2.4}$$

The following result is well known.

Lemma 2.1 (Ladyzhenskaya's inequality). *For any open set $\Omega \subset \mathbb{R}^2$ we have*

$$\|u\|_{L^4(\Omega)} \leq \frac{1}{2^{1/4}} \|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)}^{1/2}, \qquad \forall u \in H_0^1(\Omega). \tag{2.5}$$

Using Lemma 2.1 and the Poincaré inequality (1.4) we obtain

$$\|u\|_{L^4(\Omega)} \leq \left(\frac{1}{2\lambda_1}\right)^{1/4} \|\nabla u\|_{L^2(\Omega)}, \qquad \forall u \in H_0^1(\Omega).$$

Lemma 2.2 ([AF, Theorem 5.8]). *Let Ω be a domain in $\mathbb{R}^2$ satisfying the cone condition. Then there exists a constant K depending on the dimension of the cone (provided that the cone condition holds for Ω) such that for all $\phi \in H^1(\Omega)$,*

$$\|\phi\|_{L^4(\Omega)} \leq K \|\phi\|_{L^2(\Omega)}^{1/2} \|\phi\|_{H^1(\Omega)}^{1/2}.$$

The following result plays an important role later.

Lemma 2.3. *For any open set $\Omega \subset \mathbb{R}^2$ satisfying the cone condition and for $z, \tilde{z} \in V$, we have*

$$|\mathbb{B}_0(z, z, \tilde{z})| \leq C|z| \, \|z\| \, \|\tilde{z}\|.$$

Proof. We observe that

$$\frac{1}{\sqrt{2}}(\|\tilde{u}\|_1 + \sqrt{S}\, \|\tilde{B}\|_2) \leq \sqrt{\|\tilde{u}\|_1^2 + S\|\tilde{B}\|_2^2} = \|\tilde{z}\|, \tag{2.6}$$

$$|b(u, \hat{u}, \tilde{u})| \leq \|u\|_{\mathbb{L}^4} \|\nabla \hat{u}\|_{\mathbb{L}^2} \|\tilde{u}\|_{\mathbb{L}^4}. \tag{2.7}$$

From (2.7) and (2.5) we have

$$|b(u, u, \tilde{u})| \leq \frac{1}{\sqrt{2}} |u| \, \|u\|_1 \|\tilde{u}\|_1.$$

Applying Lemma 2.2 shows

$$\|B\|_{\mathbb{L}^4(\Omega)} \leq K|B|^{1/2} \|B\|_{\mathbb{H}^1(\Omega)}^{1/2}, \qquad \forall B \in \mathbb{H}^1(\Omega).$$

This inequality and (2.7) tell us that

$$|b(B, B, \tilde{u})| \leq \frac{1}{\sqrt{2}} |B| \, \|B\|_{\mathbb{H}^1} \|\tilde{u}\|_1 \leq \frac{C}{\sqrt{2}} |B| \, \|B\|_2 \|\tilde{u}\|_1.$$

Using the Cauchy inequality, we obtain

$$|b(u, u, \tilde{u}) - Sb(B, B, \tilde{u})| \leq \frac{C}{\sqrt{2}} |z| \, \|z\| \, \|\tilde{u}\|_1.$$

In a similar way we get

$$\begin{aligned}|Sb(u,B,\tilde{B})-Sb(B,u,\tilde{B})| &\le \frac{CS}{\sqrt{2}}(|u|\,\|u\|_1|B|\,\|B\|_{\mathbb{H}^1})^{1/2}\|\tilde{B}\|_{\mathbb{H}^1}\\ &\le \frac{CS}{\sqrt{2}}(|u|\,\|u\|_1|B|\,\|B\|_2)^{1/2}\|\tilde{B}\|_2\\ &\le \frac{C\sqrt{S}}{\sqrt{2}}|z|\,\|z\|\,\|\tilde{B}\|_2.\end{aligned}$$

Hence

$$|\mathbb{B}_0(z,z,\tilde{z})| \le \frac{C}{\sqrt{2}}|z|\,\|z\|\big(\|\tilde{u}\|_1+\sqrt{S}\|\tilde{B}\|_2\big).$$

Using (2.6) we complete the proof. ∎

From the relation

$$\mathbb{B}_0(z_1,z_2,z_2)=b(u_1,u_2,u_2)+Sb(u_1,B_2,B_2)-S[b(B_1,B_2,u_2)+b(B_1,u_2,B_2)],$$

and from (2.3)–(2.4), we get

$$\begin{cases}\mathbb{B}_0(z_1,z_2,z_2)=0, & \forall z_1,z_2\in V,\\ \mathbb{B}_0(z_1,z_2,z_3)=-\mathbb{B}_0(z_1,z_3,z_2), & \forall z_1,z_2,z_3\in V.\end{cases} \tag{2.8}$$

Using (2.8) and Lemma 2.3, we see that

$$\langle \mathbb{B}(z,z),z\rangle_{V',V}=0, \tag{2.9}$$

$$\|\mathbb{B}(z)\|_{V'}\le C|z|\,\|z\|, \quad \forall z\in V. \tag{2.10}$$

2.2. Pullback attractors. Let (X,d) be a metric space. For $A,B\subset X$, we define the Hausdorff semidistance between A and B by

$$\operatorname{dist}(A,B):=\sup_{a\in A}\inf_{b\in B} d(a,b).$$

A *process* on X is a two-parameter family of mappings $\{Z(t,\tau)\}$ in X having the following properties:

$$\begin{aligned}Z(t,r)Z(r,\tau)&=Z(t,\tau) &&\text{for all } t\ge r\ge\tau,\\ Z(\tau,\tau)&=\mathrm{Id} &&\text{for all } \tau\in\mathbb{R}.\end{aligned}$$

Suppose $\mathcal{B}(X)$ is the family of all non-empty bounded subsets of X, and $\mathcal{D}$ is a non-empty class of parameterized sets $\hat{\mathcal{D}}=\{D(t):t\in\mathbb{R}\}\subset\mathcal{B}(X)$.

DEFINITION 2.1. The process $\{Z(t,\tau)\}$ is said to be *pullback $\mathcal{D}$-asymptotically compact* if for any $t\in\mathbb{R}$, any $\hat{\mathcal{D}}\in\mathcal{D}$, any sequence $\tau_n\to-\infty$, and any sequence $x_n\in D(\tau_n)$, the sequence $\{Z(t,\tau_n)x_n\}_n$ is relatively compact in X.

DEFINITION 2.2. The family of bounded sets $\hat{\mathcal{B}}=\{B(t):t\in\mathbb{R}\}\in\mathcal{D}$ is called *pullback $\mathcal{D}$-absorbing* for the process $Z(t,\tau)$ if for any $t\in\mathbb{R}$ and any

$\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \le t$ such that

$$\bigcup_{\tau \le \tau_0} Z(t,\tau)D(\tau) \subset B(t).$$

Definition 2.3. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a *pullback $\mathcal{D}$-attractor* for the process $\{Z(t,\tau)\}$ if

(i) $A(t)$ is compact for all $t \in \mathbb{R}$;
(ii) $\hat{\mathcal{A}}$ is invariant, i.e., $Z(t,\tau)A(\tau) = A(t)$ for all $t \ge \tau$;
(iii) $\hat{\mathcal{A}}$ is *pullback $\mathcal{D}$-attracting*, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}(Z(t,\tau)D(\tau), A(t)) = 0 \quad \text{for all } \hat{\mathcal{D}} \in \mathcal{D} \text{ and all } t \in \mathbb{R};$$

(iv) if $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Theorem 2.1 ([CŁR]). *Let $\{Z(t,\tau)\}$ be a continuous process such that $\{Z(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{Z(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$, and*

$$A(t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} Z(s,\tau)B(\tau)}.$$

We now recall from [LŁR] some abstract results on the fractal dimension of pullback attractors.

Let H be a separable real Hilbert space. Given a compact set $K \subset H$ and $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimum number of open balls in H with radius ε that are necessary to cover K.

Definition 2.4. For any non-empty compact set $K \subset H$, the *fractal dimension* of K is the number

$$d_{\mathrm{F}}(K) = \limsup_{\varepsilon \downarrow 0} \frac{\log(N_\varepsilon(K))}{\log(1/\varepsilon)}.$$

Consider a separable real Hilbert space $V \subset H$ such that the injection of V in H is continuous, and V is dense in H. We identify H with its topological dual H', and we consider V as a subspace of H', identifying $\eta \in V$ with the element $f_\eta \in H'$ defined by

$$f_\eta(h) = (\eta, h), \qquad h \in H.$$

Let $F : V \times \mathbb{R} \to V'$ be a given family of non-linear operators such that, for all $\tau \in \mathbb{R}$ and any $z_0 \in H$, there exists a unique function $z(t) = z(t;\tau, z_0)$

satisfying

(2.11)
$$\begin{cases} z \in L^2(\tau, T; V) \cap C([\tau, T]; H), \quad F(z(t), t) \in L^1(\tau, T; V') \quad \text{for all } T > \tau, \\ \dfrac{dz}{dt} = F(z(t), t), \quad t > \tau, \\ z(\tau) = z_0. \end{cases}$$

Let us define

$$Z(t, \tau)z_0 = z(t; \tau, z_0), \qquad \tau \le t, \ z_0 \in H.$$

Fix $T^* \in \mathbb{R}$. We assume that there exists a family $\{A(t) : t \le T^*\}$ of non-empty compact subsets of H with the invariance property

$$Z(t, \tau)A(\tau) = A(t) \quad \text{for all } \tau \le t \le T^*,$$

and such that, for all $\tau \le t \le T^*$ and any $z_0 \in A(\tau)$, there exists a continuous linear operator $L(t; \tau, z_0) \in \mathcal{L}(H)$ such that

$$|Z(t,\tau)\bar{z}_0 - Z(t,\tau)z_0 - L(t;\tau,z_0)(\bar{z}_0 - z_0)| \le \chi(t-\tau, |\bar{z}_0 - z_0|)|\bar{z}_0 - z_0| \tag{2.12}$$

for all $\bar{z}_0 \in A(\tau)$, where $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\chi(s, \cdot)$ is non-decreasing for all $s \ge 0$, and

$$\lim_{r \to 0} \chi(s, r) = 0 \quad \text{for any } s \ge 0. \tag{2.13}$$

We assume that, for all $t \le T^*$, the mapping $F(\cdot, t)$ is Gateaux differentiable in V, i.e., for any $z \in V$ there exists a continuous linear operator $F'(z, t) \in \mathcal{L}(V; V')$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(z + \epsilon\eta, t) - F(z, t) - \epsilon F'(z, t)\eta] = 0 \quad \text{in } V'.$$

Moreover, we suppose that the mapping

$$F' : V \times (-\infty, T^*] \ni (z, t) \mapsto F'(z, t) \in \mathcal{L}(V; V')$$

is continuous (thus, in particular, for each $t \le T^*$, the mapping $F(\cdot, t)$ is continuously Fréchet differentiable in V).

Then, for all $\tau \le T^*$ and $z_0, \eta_0 \in H$, there exists a unique $\eta(t) = \eta(t; \tau, z_0, \eta_0)$ which is a solution of

$$\begin{cases} \eta \in L^2(\tau, T; V) \cap C([\tau, T]; H) \quad \text{for all } \tau < T \le T^*, \\ \dfrac{d\eta}{dt} = F'(Z(t, \tau)z_0, t)\eta, \quad \tau < t < T^*, \\ \eta(\tau) = \eta_0. \end{cases}$$

We make the assumption that

$$\eta(t; \tau, z_0, \eta_0) = L(t; \tau, z_0)\eta_0 \quad \text{for all } \tau \le t \le T^*, \ z_0, \eta_0 \in A(\tau). \tag{2.14}$$

Let us write, for $m = 1, 2, \ldots,$

$$\tilde{q}_m = \lim_{T\to\infty} \sup_{\tau \le T^*} \sup_{z_0 \in A(\tau - T)} \frac{1}{T} \int_{\tau-T}^{\tau} \mathrm{Tr}_m\big(F'(Z(s, \tau - T)z_0, s)\big)\, ds,$$

where

$$\mathrm{Tr}_m\big(F'(Z(s,\tau)z_0, s)\big) = \sup_{\eta_0^i \in H,\, |\eta_0^i| \le 1,\, i \le m} \sum_{i=1}^{m} \big\langle F'(Z(s,\tau)z_0, s)\varphi_i, \varphi_i\big\rangle,$$

$\{\varphi_i\}_{i=1,\ldots,m}$ being an orthonormal basis of the subspace in H spanned by

$$\eta(s; \tau, z_0, \eta_0^1), \ldots, \eta(s; \tau, z_0, \eta_0^m).$$

THEOREM 2.2 ([LŁR, Theorem 2.2]). *Under the assumptions above, suppose that*

$$\bigcup_{\tau \le T^*} A(\tau) \text{ is relatively compact in } H,$$

and there exist q_m, $m = 1, 2, \ldots,$ such that

$$\begin{aligned} &\tilde{q}_m \le q_m && \text{for any } m \ge 1,\\ &q_{n_0} \ge 0,\ q_{n_0+1} < 0 && \text{for some } n_0 \ge 1,\\ &q_m \le q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - m) && \text{for all } m = 1, 2, \ldots. \end{aligned}$$

Then

$$d_{\mathrm{F}}(A(\tau)) \le d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}} \quad \text{for all } \tau \le T^*.$$

3. Existence and uniqueness of a weak solution. In this section, we assume that

$$f \in L^2_{\mathrm{loc}}(\mathbb{R}; V_1').$$

Then $\Psi = (f, 0) \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ and

$$\langle \Psi, z\rangle_{V',V} = \langle f, u\rangle_{V_1',V_1} \quad \text{for a.e. } t \in \mathbb{R}.$$

We define $e : V \to \mathbb{R}$ by $e(z) = \langle \Psi, z\rangle_{V',V}$. It is obvious that

$$|e(z)| = |\langle \Psi, z\rangle| \le \|\Psi\|_{V'}\|z\|.$$

Taking the inner product of the first equation of (1.1) with $v \in V_1$, we obtain

$$\frac{d}{dt}(u, v) + \frac{1}{R_e}((u, v))_1 + b(u, u, v) - Sb(B, B, v) = \langle f, v\rangle.$$

We take the inner product of the second equation of (1.1) with SC ($C \in V_2$) to obtain

$$S\frac{d}{dt}(B, C) + \frac{S}{R_m}((B, C))_2 + Sb(u, B, C) - Sb(B, u, C) = 0.$$

This suggests the following weak formulation of problem (1.1)–(1.3).

PROBLEM. For $z_0 = (u_0, B_0) \in H$ given, find $z = (u, B)$ such that

$$
(3.1) \qquad \begin{cases} z \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H), \quad \forall T > \tau, \\ \dfrac{d}{dt}(z, \tilde{z}) + a(z, \tilde{z}) + \mathbb{B}_0(z, z, \tilde{z}) = e(\tilde{z}), \quad \forall \tilde{z} \in V, \forall t > \tau, \\ z(\tau) = z_0. \end{cases}
$$

Equation (3.1) is equivalent to the following functional equation in V':

$$
(3.2) \qquad \begin{cases} z \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H), \\ z' + Az + \mathbb{B}(z) = \Psi \quad \text{in } V', \forall t > \tau, \\ z(\tau) = z_0, \end{cases}
$$

where $z' = (du/dt, dB/dt)$, and $\Psi = (f, 0)$.

We are now ready to prove the existence of a weak solution to problem (3.1).

THEOREM 3.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V_1')$. Then for any $z_0 \in H$, $\tau \in \mathbb{R}$, and $T > \tau$, there exists a unique solution $z \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ to problem* (3.1) *(hence* (3.2)*). Since $z \in L^2(\tau, T; V)$, equation* (3.2) *implies that $z' \in L^2(\tau, T; V')$ for all $T > \tau$. Hence $z \in C([\tau, T]; H)$.*

Proof. Existence. The proof of existence of a weak solution to problem (3.1) in (τ, T) is based on Galerkin approximations, *a priori* estimates, and the compactness method. As it is standard and similar to the case of the Navier–Stokes equations [T79], we only provide some basic *a priori* estimates.

We define a symmetric bilinear form $[\cdot, \cdot] : V \times V \to \mathbb{R}$ by

$$
(3.3) \qquad [z, \tilde{z}] = \langle Az, \tilde{z} \rangle - \frac{\zeta \min(\lambda_1, c_0)}{2} (z, \tilde{z}), \quad \forall z, \tilde{z} \in V,
$$

where ζ is defined as

$$
\zeta = \min\left(\frac{1}{R_e}, \frac{1}{R_m}\right).
$$

From the definition of A we have

$$
[z, z] + \frac{\zeta \min(\lambda_1, c_0)}{2} |z|^2 = \langle Az, z \rangle \leq \max\left(\frac{1}{R_e}, \frac{1}{R_m}\right) \|z\|^2.
$$

Thus,

$$
(3.4) \qquad [z]^2 \equiv [z, z] \leq \max\left(\frac{1}{R_e}, \frac{1}{R_m}\right) \|z\|^2.
$$

Let $z = (u, B)$. From the definition of $|z|$ and (2.1), we have

$$
\begin{aligned}
\frac{\zeta \min(\lambda_1, c_0)}{2} |z|^2 &= \frac{\zeta \min(\lambda_1, c_0)}{2} (|u|^2 + S|B|^2) \\
&\leq \frac{\zeta}{2} (\lambda_1 |u|^2 + S c_0 |B|^2) \leq \frac{\zeta}{2} (\|u\|_1^2 + S\|B\|_2^2) = \frac{\zeta}{2} \|z\|^2.
\end{aligned}
$$

Using this and (2.2) we obtain

$$[z]^2 \geq \zeta\|z\|^2 - \frac{\zeta\lambda_1}{2}|z|^2 \geq \frac{\zeta}{2}\|z\|^2. \tag{3.5}$$

Putting together (3.4) and (3.5) leads to

$$\frac{\zeta}{2}\|z\|^2 \leq [z]^2 \leq \max\left(\frac{1}{R_e}, \frac{1}{R_m}\right)\|z\|^2, \quad \forall z \in V. \tag{3.6}$$

Thus, $[\cdot,\cdot]$ defines an inner product in V with norm $[\cdot] = [\cdot,\cdot]^{1/2}$ equivalent to $\|\cdot\|$.

Now let $z(t) = (u(t), B(t))$ be a solution given by Theorem 3.1. Since $z = (u, B) \in L^2(\tau, T; V)$ and $z' = (u', B') \in L^2(\tau, T; V')$, we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 = \langle u', u\rangle_{V_1', V_1} \quad \text{and} \quad \frac{1}{2}\frac{d}{dt}|B|^2 = \langle B', B\rangle_{V_2', V_2}.$$

It is easy to see that

$$\frac{1}{2}\frac{d}{dt}|z|^2 = \frac{1}{2}\frac{d}{dt}(|u|^2 + S|B|^2) = \langle u', u\rangle_{V_1', V_1} + S\langle B', B\rangle_{V_2', V_2} = \langle z', z\rangle_{V', V}.$$

So from (3.2) and (2.9) we have

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \langle Az, z\rangle = \langle \varPsi, z\rangle.$$

From the definition (3.3) of the norm $[\cdot]$ and letting $\sigma = \zeta\min(\lambda_1, c_0)$, we deduce that

$$\frac{d}{dt}|z|^2 + \sigma|z|^2 + 2[z]^2 = 2\langle \varPsi, z\rangle. \tag{3.7}$$

Using the equivalence (3.6) of norms and the Cauchy inequality, we obtain

$$\frac{d}{dt}|z|^2 + \sigma|z|^2 + \zeta\|z\|^2 \leq \frac{2}{\zeta}\|\varPsi\|_{V'}^2 + \frac{\zeta}{2}\|z\|^2,$$

and hence

$$\frac{d}{dt}|z|^2 + \frac{\zeta}{2}\|z\|^2 \leq \frac{2}{\zeta}\|\varPsi\|_{V'}^2.$$

Integrating both sides of the above inequality on $[\tau, t]$, $\tau \leq t \leq T$, we get

$$|z(t)|^2 + \frac{\zeta}{2}\int_\tau^t \|z(s)\|^2\,ds \leq |z_0|^2 + \frac{2}{\zeta}\|\varPsi\|_{L^2(\tau,T;V')}^2, \quad \forall t \in [\tau, T].$$

This implies the estimates of z in the space $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$.

Uniqueness and continuous dependence. Assume that z^1 and z^2 are two weak solutions of (3.1) with initial data z_0^1 and z_0^2, respectively. Set $w = z^1 - z^2$. Then $w \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ and w satisfies

$$\begin{cases} \dfrac{d}{dt}w + Aw = \mathbb{B}(z^2) - \mathbb{B}(z^1), \\ w(\tau) = z_0^1 - z_0^2. \end{cases}$$

Hence

$$\frac{d}{dt}|w|^2 + 2[w]^2 = -\sigma|w|^2 + 2\mathbb{B}_0(z^2, z^2, w) - 2\mathbb{B}_0(z^1, z^1, w)$$
$$= -\sigma|w|^2 - 2\mathbb{B}_0(w, z^1, w).$$

By Lemma 2.3, we get

$$|-2\mathbb{B}_0(w, z^1, w)| \le 2C|w|\,\|z^1\|\,\|w\| \le \zeta\|w\|^2 + \frac{C^2}{\zeta}|w|^2\|z^1\|^2.$$

Therefore,

$$\frac{d}{dt}|w|^2 \le \left(\sigma + \frac{C^2}{\zeta}\|z^1\|^2\right)|w|^2,$$

or

$$|w(t)|^2 \le |w(\tau)|^2\left(\int_\tau^t\left(\sigma + \frac{C^2}{\zeta}\|z^1(s)\|^2\right)ds\right).$$

The last estimate implies the uniqueness (if $z_0^1 = z_0^2$) and the continuous dependence of solutions on the initial data. ∎

4. Existence of a pullback $\mathcal{D}_\sigma$-attractor. Thanks to Theorem 3.1, we can define a continuous process $Z(t,\tau) : H \to H$ by

$$Z(t,\tau)z_0 = z(t;\tau,z_0), \qquad \tau \le t,\ z_0 \in H,$$

where $z(t) = z(t;\tau,z_0)$ is the unique weak solution to problem (3.2) with the initial datum $z(\tau) = z_0$. The following lemma shows the weak continuity of the process $Z(t,\tau)$, which is needed to prove the pullback asymptotic compactness of the process by using the energy equation method.

LEMMA 4.1. *Let $\{z_{0_n}\}_n$ be a sequence in H converging weakly in H to an element $z_0 \in H$. Then*

(4.1) $$Z(t,\tau)z_{0_n} \rightharpoonup Z(t,\tau)z_0 \quad \textit{weakly in } H \textit{ for all } t \ge \tau.$$

(4.2) $$Z(\cdot,\tau)z_{0_n} \rightharpoonup Z(\cdot,\tau)z_0 \quad \textit{weakly in } L^2(\tau,T;V) \textit{ for all } T > \tau.$$

Proof. The proof is very similar to that of Lemma 2.1 in [Ros], so we omit it here. ∎

Now, in order to prove the existence of a pullback attractor for the process $Z(t,\tau)$, we furthermore assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V_1')$ satisfies

(4.3) $$\int_{-\infty}^t e^{\sigma s}\|f(s)\|^2_{V_1'}\,ds < \infty \qquad \text{for all } t \in \mathbb{R},$$

where $\sigma = \zeta\min(\lambda_1, c_0)$ with $\zeta = \min(1/R_e, 1/R_m)$ and c_0 is the constant in (2.1).

Let $\mathcal{R}_\sigma$ be the set of all functions $r : \mathbb{R} \to (0;\infty)$ such that

$$\lim_{t\to-\infty} e^{\sigma t} r^2(t) = 0, \tag{4.4}$$

and denote by $\mathcal{D}_\sigma$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(H)$ such that $D(t) \subset B(0, \hat{r}(t))$ for some $\hat{r}(t) \in \mathcal{R}_\sigma$, where $B(0, r)$ denotes the closed ball in H, centered at zero with radius r.

Now, we can prove one of the main results of the paper.

THEOREM 4.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V_1')$ satisfy* (4.3). *Then there exists a unique pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ for the process $Z(t,\tau)$ associated to problem* (3.1).

Proof. Let $\tau \in \mathbb{R}$ and $z_0 \in H$ be fixed, and denote

$$z(t) = z(t; \tau, z_0) = Z(t,\tau)z_0 \quad \text{for all } t \geq \tau.$$

To apply Theorem 2.1, we will check the two conditions in the abstract theorem.

(i) *The process $Z(t,\tau)$ has a family $\hat{\mathcal{B}}$ of pullback $\mathcal{D}_\sigma$-absorbing sets.* Let $\hat{\mathcal{D}} \in \mathcal{D}_\sigma$. First, we prove that

$$|z(t)|^2 \leq e^{-\sigma(t-\tau)}|z_0|^2 + \frac{e^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s}\|f(s)\|^2_{V_1'}\, ds. \tag{4.5}$$

Indeed, applying the Cauchy inequality to (3.7) we get

$$\frac{d}{dt}|z|^2 + \sigma|z|^2 + \zeta\|z\|^2 \leq \frac{1}{\zeta}\|\Psi\|^2_{V'} + \zeta\|z\|^2.$$

By the Gronwall inequality, we obtain (4.5). From (4.5) we have

$$|Z(t,\tau)z_0|^2 \leq e^{-\sigma(t-\tau)}\hat{r}^2(\tau) + \frac{e^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s}\|\Psi(s)\|^2_{V'}\, ds \tag{4.6}$$

for all $z_0 \in D(\tau)$ and all $t \geq \tau$. Define $R_\sigma(t) \in \mathcal{R}_\sigma$ by

$$R^2_\sigma(t) = \frac{2e^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s}\|\Psi(s)\|^2_{V'}\, ds, \tag{4.7}$$

and consider the family $\hat{\mathcal{B}}_\sigma$ of closed balls in H defined by $B_\sigma(t) = B(0, R_\sigma(t))$. It is straightforward to check that $\hat{\mathcal{B}}_\sigma \in \mathcal{D}_\sigma$, and moreover, by (4.6) and (4.4), the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing for the process $Z(t,\tau)$.

(ii) *$Z(t,\tau)$ is pullback $\mathcal{D}_\sigma$-asymptotically compact.* Fix $\hat{\mathcal{D}} \in \mathcal{D}_\sigma$, a sequence $\tau_n \to -\infty$, a sequence $z_{0_n} \in D(\tau_n)$, and $t \in \mathbb{R}$. We must prove that from the sequence $\{Z(t,\tau_n)z_{0_n}\}_n$ we can extract a subsequence that converges in H.

As the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing, for each integer $k \geq 0$, there exists a $\tau_{\hat{D}}(k) \leq t-k$ such that

$$Z(t-k,\tau)D(\tau) \subset B_\sigma(t-k) \quad \text{for all } \tau \leq \tau_{\hat{D}}(k), \tag{4.8}$$

so that for $\tau_n \leq \tau_{\hat{D}}(k)$,

$$Z(t-k,\tau_n)z_{0_n} \subset B_\sigma(t-k).$$

Thus, $\{Z(t-k,\tau_n)z_{0_n}\}_n$ is weakly precompact in H, and since $B_\sigma(t-k)$ is closed and convex, there exist a subsequence $\{\tau_{n'}, z_{0_{n'}}\}_{n'} \subset \{\tau_n, z_{0_n}\}_n$ and a sequence $\{w_k\}_k \subset H$ such that for all $k \geq 0$, $w_k \in B_\sigma(t-k)$ and

$$Z(t-k,\tau_{n'})z_{0_{n'}} \rightharpoonup w_k \quad \text{weakly in } H. \tag{4.9}$$

Note that from the weak continuity of $Z(t,\tau)$ established in Lemma 4.1, we get

$$\begin{aligned} w_0 &= \lim_{n'\to\infty}{}^{H_w} Z(t,\tau_{n'})z_{0_{n'}} = \lim_{n'\to\infty}{}^{H_w} Z(t,t-k)Z(t-k,\tau_{n'})z_{0_{n'}} \\ &= Z(t,t-k) \lim_{n'\to\infty}{}^{H_w} Z(t-k,\tau_{n'})z_{0_{n'}} = Z(t,t-k)w_k, \end{aligned}$$

where $\lim^{H_w}$ denotes the limit taken in the weak topology of H. Thus,

$$Z(t,t-k)w_k = w_0 \quad \text{for all } k \geq 0. \tag{4.10}$$

Now, from (4.9), by the lower semicontinuity of the norm, we have

$$|w_0| \leq \liminf_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}|.$$

If we now prove that also

$$\limsup_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}| \leq |w_0|,$$

then we will have

$$\lim_{n'\to\infty} |Z(t,\tau_{n'})z_{0_{n'}}| = |w_0|,$$

and this, together with the weak convergence, will imply the strong convergence in H of $Z(t,\tau_{n'})z_{0_{n'}}$ to w_0.

From (3.7) we get

$$|z(t)|^2 \leq e^{-\sigma(t-\tau)}|z_0|^2 + 2\int_\tau^t e^{-\sigma(t-s)}\big(\langle \Psi(s), z(s)\rangle - [z(s)]^2\big)\,ds,$$

which can be written as

$$|Z(t,\tau)z_0|^2 \leq e^{\sigma(\tau-t)}|z_0|^2 + 2\int_\tau^t e^{\sigma(s-t)}\big(\langle \Psi(s), z(s)\rangle - [z(s)]^2\big)\,ds \tag{4.11}$$

for all $\tau \le t$ and all $z_0 \in H$. Thus, for all $k \ge 0$ and all $\tau_{n'} \le t-k$,

$$
\begin{aligned}
(4.12)\qquad |Z(t,\tau_{n'})z_{0_{n'}}|^2 &= |Z(t,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}|^2\\
&\le e^{-\sigma k}|Z(t-k,\tau_{n'})z_{0_{n'}}|^2\\
&\quad + 2\int_{t-k}^{t} e^{\sigma(s-t)}\langle \Psi(s), Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}\rangle\, ds\\
&\quad - 2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\, ds.
\end{aligned}
$$

We now estimate each of the three terms above. By (4.8), we have $Z(t-k,\tau_{n'})z_{0_{n'}} \in B_\sigma(t-k)$ for all $\tau_{n'} \le \tau_{\hat{D}}(k)$, $k \ge 0$, and so

$$
(4.13)\qquad \limsup_{n'\to\infty}\big(e^{-\sigma k}|Z(t,\tau_{n'})z_{0_{n'}}|^2\big) \le e^{-\sigma k}R_\sigma^2(t-k), \qquad k \ge 0.
$$

This takes care of the first term in (4.12).

As $Z(t-k,\tau_{n'})z_{0_{n'}} \rightharpoonup w_k$ weakly in H, from Lemma 4.1 we obtain

$$
(4.14)\quad Z(\cdot,t-k)Z(t-k,\tau_{n'})z_{0_{n'}} \rightharpoonup Z(\cdot,t-k)w_k \text{ weakly in } L^2(t-k,t;V).
$$

Since, in particular, $e^{\sigma(s-t)}\Psi(s) \in L^2(t-k,t;V')$, from (4.14) we get

$$
\begin{aligned}
(4.15)\qquad \lim_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}\big\langle \Psi(s), Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}\big\rangle\, ds\\
= \int_{t-k}^{t} e^{\sigma(s-t)}\langle \Psi(s), Z(s,t-k)w_k\rangle\, ds.
\end{aligned}
$$

This takes care of the second term in (4.12).

By (3.6), the norm $[\cdot]$ is equivalent to $\|\cdot\|$ in V. Also

$$
0 < e^{-\sigma k} \le e^{\sigma(s-t)} \le 1, \qquad \forall s \in [t-k,t],
$$

and therefore

$$
\Big(\int_{t-k}^{t} e^{-\sigma(t-s)}[\cdot]^2\, ds\Big)^{1/2}
$$

is a norm in $L^2(t-k,t;V)$ equivalent to the usual norm. Hence from (4.14) we deduce that

$$
\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)w_k]^2\, ds \le \liminf_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\, ds.
$$

Hence

$$(4.16)\qquad \limsup_{n'\to\infty}\Big(-2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\,ds\Big)$$
$$= -\liminf_{n'\to\infty} 2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)Z(t-k,\tau_{n'})z_{0_{n'}}]^2\,ds$$
$$\le -2\int_{t-k}^{t} e^{\sigma(s-t)}[Z(s,t-k)w_k]^2\,ds.$$

This takes care of the last term in (4.12).

We can now pass to the lim sup as $n'\to\infty$ in (4.12), and take (4.13), (4.15), and (4.16) into account to obtain

$$(4.17)\qquad \limsup_{n'\to\infty}|Z(t,\tau_{n'})z_{0_{n'}}|^2 \le e^{-\sigma k}R_\sigma^2(t-k)$$
$$+ 2\int_{t-k}^{t} e^{\sigma(s-t)}\big(\langle\Psi(s),Z(s,t-k)w_k\rangle - [Z(s,t-k)w_k]^2\big)\,ds.$$

On the other hand, applying (4.11) in (4.10) we find that

$$(4.18)\qquad |w_0|^2 = |Z(t,t-k)w_k|^2$$
$$= |w_k|^2e^{-\sigma k} + 2\int_{t-k}^{t} e^{\sigma(s-t)}\big(\langle\Psi(s),Z(s,t-k)w_k\rangle - [Z(s,t-k)w_k]^2\big)\,ds.$$

From (4.17) and (4.18), we have

$$\limsup_{n'\to\infty}|Z(t,\tau_{n'})z_{0_{n'}}|^2 \le e^{-\sigma k}R_\sigma^2(t-k) + |w_0|^2 - |w_k|^2e^{-\sigma k}$$
$$\le e^{-\sigma k}R_\sigma^2(t-k) + |w_0|^2,$$

and thus, taking into account that

$$e^{-\sigma k}R_\sigma^2(t-k) = \frac{e^{-\sigma t}}{\zeta}\int_{-\infty}^{t-k} e^{\sigma s}\|\Psi(s)\|_{V'}^2\,ds \to 0 \quad \text{as } k\to\infty,$$

we easily obtain (4.12) from the last inequality above. ∎

5. Fractal dimension estimates of the pullback $\mathcal{D}_\sigma$-attractor. Observe that problem (3.2) can be written in the form (2.11) by taking

$$F(z,t) = -Az(t) - \mathbb{B}z(t) + \Psi(t).$$

Then it follows immediately that for all $t\in\mathbb{R}$, the mapping $F(\cdot,t)$ is Gateaux differentiable in V with

$$F'(z,t)\eta = -A\eta - \mathbb{B}(z,\eta) - \mathbb{B}(\eta,z),\qquad z,\eta\in V,$$

and the mapping $F' : (z,t)\in V\times\mathbb{R}\mapsto F'(z,t)\in\mathcal{L}(V;V')$ is continuous.

Evidently, for any $\tau \in \mathbb{R}$ and $z_0, \eta_0 \in H$, there exists a unique solution $\eta(t) = \eta(t; \tau, z_0, \eta_0)$ of the problem

$$
\begin{cases} \eta \in L^2(\tau, T; V) \cap C([\tau; T]; H), \\ \dfrac{d\eta}{dt} = -A\eta - \mathbb{B}(Z(t,\tau)z_0, \eta) - \mathbb{B}(\eta, Z(t,\tau)z_0), \\ \eta(\tau) = \eta_0. \end{cases} \tag{5.1}
$$

To estimate the dimension of the pullback $\mathcal{D}_\sigma$-attractor, we need two conditions:

(C_1) $f \in L^\infty(-\infty, T^*; V_1')$ for some $T^* \in \mathbb{R}$;

(C_2) $\mathbb{R}^2 \setminus \overline{\Omega}$ contains a semicone.

Notice that from (C_1) we have $\Psi = (f, 0) \in L^\infty(-\infty, T^*; V')$, and ($C_2$) ensures that we can use the generalized Lieb–Thirring inequality in the general case (see [GMT]) because in our problem, in contrast to the case of 2D Navier–Stokes equations with the homogeneous Dirichlet boundary condition [LŁR], the component B of the solution $z = (u, B)$ does not vanish on the boundary $\partial\Omega$.

LEMMA 5.1. *Assume the conditions of Theorem* 4.1 *and* (C_1)–(C_2) *hold. Then the pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}}$ obtained in Theorem* 4.1 *satisfies*

$$
\bigcup_{\tau \le T^*} A(\tau) \text{ is relatively compact in } H. \tag{5.2}
$$

Proof. Denoting $M = \|\Psi\|^2_{L^\infty(-\infty, T^*; V')}$, from (4.7) we have

$$
R_\sigma^2(t) \le \frac{2Me^{-\sigma t}}{\zeta} \int_{-\infty}^{t} e^{\sigma s}\, ds = \frac{2M}{\zeta\sigma},
$$

and consequently

$$
B^* := \bigcup_{\tau \le T^*} B_\sigma(\tau) \text{ is bounded in } H,
$$

where $B_\sigma(\tau) = B(0, R_\sigma(\tau))$.

Denote by $\mathcal{M}$ the set of all $y \in H$ such that there exist a sequence $\{(t_n, \tau_n)\}_n \subset \mathbb{R}^2$ satisfying

$$
\tau_n \le t_n \le T^*, \quad n \ge 1, \quad \lim_{n\to\infty} (t_n - \tau_n) = \infty,
$$

and a sequence $\{z_{0_n}\}_n \subset B^*$ such that $\lim_{n\to\infty} |Z(t, \tau_n) z_{0_n} - y| = 0$.

It is easy to see that $A(t) \subset \mathcal{M}$ for all $t \le T^*$. If we prove that $\mathcal{M}$ is relatively compact in H, then (5.2) follows immediately.

Let $\{y_k\}_k \subset \mathcal{M}$. For each $k \ge 1$, we can take $(t_k, \tau_k) \in \mathbb{R}^2$ and an element $z_{0_k} \in B^*$ such that $t_k \le T^*$, $t_k - \tau_k \ge k$, and $|Z(t_k, \tau_k) z_{0_k} - y_k| \le 1/k$. Using ($C_1$), by arguments as in [LŁR, Proposition 3.4], we can extract from $\{y_k\}_k$ a subsequence that converges in H. ∎

LEMMA 5.2. *Assume the conditions of Theorem* 4.1 *and* (C_1)–(C_2) *hold. Then the process* $Z(t,\tau)$ *associated to problem* (3.1) *has the quasidifferentiability properties* (2.12)–(2.14) *with* $\eta(t) = \eta(t;\tau,z_0,\eta_0)$ *defined by* (5.1).

Proof. By condition (C_1) and Lemma 5.1, there exists a constant $c_1 > 1$ such that

$$(5.3)\qquad \|\Psi\|^2_{L^\infty(-\infty,T^*;V')} \le c_1\zeta/2, \qquad |z_0|^2 \le c_1 \quad \text{for all } z_0 \in \bigcup_{\tau\le T^*} A(\tau).$$

Fix $\tau \le T^*$, $z_0, \bar{z}_0 \in A(\tau)$, denote $z(t) = Z(t,\tau)z_0$, $\bar{z}(t) = Z(t,\tau)\bar{z}_0$, and let $\eta(t)$ be the solution of (5.1) with $\eta_0 = \bar{z}_0 - z_0$. From (3.7) we easily find that

$$(5.4)\qquad |z(t)|^2 + \frac{\zeta}{2}\int_\tau^t \|z(s)\|^2\,ds \le |z_0|^2 + \frac{2}{\zeta}\int_\tau^t \|\Psi(s)\|^2_{V'}\,ds.$$

Taking into account (5.3), we easily deduce from (5.4) that

$$(5.5)\qquad \int_\tau^t \|z(s)\|^2\,ds \le \frac{2c_1}{\zeta}(1+t-\tau) \quad \text{for all } \tau \le t \le T^*.$$

Writing

$$w(t) = \bar{z}(t) - z(t), \qquad \tau \le t,$$

we get

$$\begin{aligned}\frac{d}{dt}|w|^2 + 2[w]^2 &= -\sigma|w|^2 + 2\mathbb{B}_0(z,z,w) - 2\mathbb{B}_0(\bar{z},\bar{z},w)\\ &= -\sigma|w|^2 - 2\mathbb{B}_0(w,z,w).\end{aligned}$$

By Lemma 2.3, we have

$$|-2\mathbb{B}_0(w,z,w)| \le 2C|w|\,\|z\|\,\|w\| \le \frac{\zeta}{2}\|w\|^2 + \frac{2C^2}{\zeta}|w|^2\|z\|^2.$$

Hence

$$(5.6)\qquad \frac{d}{dt}|w|^2 + \frac{\zeta}{2}\|w\|^2 \le \left(\sigma + \frac{2C^2}{\zeta}\|z\|^2\right)|w|^2.$$

In particular,

$$|w(t)|^2 \le |w(\tau)|^2 \exp\left(\int_\tau^t \left(\sigma + \frac{2C^2}{\zeta}\|z(s)\|^2\right)ds\right).$$

Thus, using (5.5), we obtain

$$(5.7)\qquad |w(t)|^2 \le |w(\tau)|^2 \exp(\tilde{K}(1+t-\tau)) \quad \text{for all } \tau \le t \le T^*,$$

where $\tilde{K} = \max(4C^2 c_1/\zeta^2 + \sigma, 1)$. Now from (5.6) and (5.7) we have

$$\begin{aligned}\frac{\zeta}{2}\int_\tau^t \|w(s)\|^2\,ds &\le |w(\tau)|^2 + \int_\tau^t \Big(\sigma + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)|w(s)|^2\,ds \\ &\le |w(\tau)|^2 + \int_\tau^t \Big(\sigma + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)|w(\tau)|^2 \exp[\tilde{K}(1+s-\tau)]\,ds \\ &\le |w(\tau)|^2\bigg[1 + \exp[\tilde{K}(1+t-\tau)]\int_\tau^t \Big(\sigma + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)\,ds\bigg].\end{aligned}$$

Hence

$$\begin{aligned}\frac{\zeta}{2}\int_\tau^t \|w(s)\|^2\,ds &\le |w(\tau)|^2\big[1 + \tilde{K}(1+t-\tau)\exp[\tilde{K}(1+t-\tau)]\big] \\ &\le |w(\tau)|^2[1 + \tilde{K}(1+t-\tau)]\exp[\tilde{K}(1+t-\tau)] \\ &\le |w(\tau)|^2 \exp[2\tilde{K}(1+t-\tau)].\end{aligned} \tag{5.8}$$

Let $\omega(t)$ be defined by

$$\omega(t) = \bar{z}(t) - z(t) - \eta(t) = w(t) - \eta(t), \qquad t \ge \tau.$$

Evidently, $\omega(t)$ satisfies

$$\begin{cases}\omega \in L^2(\tau, T; V) \cap C([\tau, T]; H) \quad \text{for all } T > \tau, \\ \dfrac{d\omega}{dt} = -A\omega - \mathbb{B}(\bar{z}, \bar{z}) + \mathbb{B}(z, z) + \mathbb{B}(z, \eta) + \mathbb{B}(\eta, z), \quad t > \tau, \\ \omega(\tau) = 0.\end{cases}$$

It is easy to see that

$$-\mathbb{B}(\bar{z}, \bar{z}) + \mathbb{B}(z, z) + \mathbb{B}(z, \eta) + \mathbb{B}(\eta, z) = -\mathbb{B}(z, \omega) - \mathbb{B}(\omega, z) - \mathbb{B}(w, w),$$

and consequently, for all $t > \tau$,

$$\begin{aligned}\frac{d}{dt}|\omega|^2 + \zeta\|\omega\|^2 &= -\sigma|\omega|^2 - 2\mathbb{B}_0(\omega, z, \omega) - 2\mathbb{B}_0(w, w, \omega) \\ &\le \sigma|\omega|^2 + 2C|\omega|\,\|z\|\,\|\omega\| + 2C|w|\,\|w\|\,\|\omega\| \\ &\le \sigma|\omega|^2 + \frac{2C^2}{\zeta}|\omega|^2\|z\|^2 + \frac{\zeta}{2}\|\omega\|^2 + \frac{2C^2}{\zeta}|w|^2\|w\|^2 + \frac{\zeta}{2}\|\omega\|^2 \\ &= \zeta\|\omega\|^2 + \Big(\sigma + \frac{2C^2}{\zeta}\|z\|^2\Big)|\omega|^2 + \frac{2C^2}{\zeta}|w|^2\|w\|^2.\end{aligned} \tag{5.9}$$

Integrating (5.9) from τ to t, and using the fact that $\omega(\tau) = 0$, we get

$$|\omega(t)|^2 \le \frac{2C^2}{\zeta}\int_\tau^t |w(s)|^2\|w(s)\|^2\,ds + \int_\tau^t \Big(\sigma + \frac{2C^2}{\zeta}\|z(s)\|^2\Big)|\omega(s)|^2\,ds$$

for all $t \ge \tau$, and consequently, by the Gronwall inequality,

$$|\omega(t)|^2 \le \exp\biggl[\int_\tau^t \Bigl(\sigma + \frac{2C^2}{\zeta}\|z(s)\|^2\Bigr)\,ds\biggr] \int_\tau^t \frac{2C^2}{\zeta}|w(s)|^2\|w(s)\|^2\,ds.$$

From (5.7) we obtain

$$|\omega(t)|^2 \le \frac{2C^2}{\zeta}|w(\tau)|^2 \exp[2\tilde{K}(1+t-\tau)] \int_\tau^t \|w(s)\|^2\,ds.$$

Plugging (5.8) into the last estimate, we get

$$|\omega(t)|^2 \le \frac{4C^2}{\zeta^2}|w(\tau)|^4 \exp[4\tilde{K}(1+t-\tau)],$$

i.e., (2.12)–(2.14) hold with $\chi(s,r) = (2r/\zeta)\exp[2\tilde{K}(1+s)]$, where $\tilde{K} \ge 1$. ∎

We now prove the main result of this section.

THEOREM 5.1. *Assume that the conditions of Theorem* 4.1 *and* (C_1)–(C_2) *hold. Then the pullback $\mathcal{D}_\sigma$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ satisfies*

$$d_{\mathrm{F}}(A(\tau)) \le \max\{1, K/L\},$$

where

$$K = \mu\bigl[(\sqrt{R_e} + \sqrt{R_m})^2 R_e^2 + (R_e R_m)^{3/2}\bigr]\|f\|^2_{L^\infty(-\infty,T^*;V_1')},$$
$$L = \frac{1}{4}\biggl(\frac{\lambda_1}{R_e} + \frac{c_0}{R_m}\biggr).$$

Here μ is the constant in the Lieb–Thirring inequality, and c_0 is the constant in (2.1).

Proof. To estimate the number $\tilde{q}_m$, let $z_0 \in \hat{\mathcal{A}}$ and $\xi_1, \ldots, \xi_m \in H$. Set $z(t) = Z(t,\tau)z_0$ and $\eta_i(t) = L(t;\tau,z_0)\xi_i$, $t \ge \tau$. Let

$$\{(\tilde{\phi}_i(t), \tilde{C}_i(t))\}_{i=1,\ldots,m}, \qquad t \ge \tau,$$

be a basis for $\operatorname{span}\{\eta_1(t), \ldots, \eta_m(t)\}$ such that $\{\tilde{\phi}_i(t)\}_{i=1,\ldots,m}$ is orthonormal in H_1 and $\{\tilde{C}_i(t)\}_{i=1,\ldots,m}$ is orthonormal in H_2. Set $\varphi_i = (\phi_i, C_i) = (\tilde{\phi}_i/\sqrt{2}, \tilde{C}_i/\sqrt{2S})$. An easy computation shows that $\{\varphi_i\}_{i=1,\ldots,m}$ is orthonormal in H. Since $\eta_i(t) \in V$ for a.e. $t \ge \tau$, we can assume that $\varphi_i(t) \in V$ for a.e. $t \ge \tau$ (by the Gram–Schmidt orthogonalization procedure).

From (5.1), (2.8), and the definition of A, for a.e. $s \ge \tau$ we have

$$\operatorname{Tr}_m\bigl(F'(Z(s,\tau)z_0, s)\bigr) = \sum_{i=1}^m \bigl\langle F'(Z(s,\tau)z_0, s)\varphi_i, \varphi_i\bigr\rangle_{V',V} \tag{5.10}$$

$$= \sum_{i=1}^{m} \langle -A\varphi_i - \mathbb{B}(z,\varphi_i) - \mathbb{B}(\varphi_i, z), \varphi_i \rangle_{V',V}$$
$$\leq \sum_{i=1}^{m} -\Big(\frac{1}{R_e} \|\phi_i\|_1^2 + S\frac{1}{R_m}\|C_i\|_2^2 \Big) + |\mathbb{B}_0(\varphi_i, z, \varphi_i)|.$$

Now let

$$\rho(x) = \sum_{i=1}^{m} \Big(\frac{1}{\sqrt{R_e}} |\phi_i(x)|^2 + \frac{S}{\sqrt{R_m}} |C_i(x)|^2 \Big).$$

A standard computation (see e.g. [Ros]) and the definition of ρ yield

$$\Big|\sum_{i=1}^{m} b(\phi_i, u, \phi_i)\Big| \leq \sqrt{R_e}\, \|u\|_1 \Big| \frac{1}{\sqrt{R_e}} \sum_{i=1}^{m} |\phi_i|^2 \Big|_{L^2} \leq \sqrt{R_e}\, \|u\|_1 |\rho|_{L^2}. \tag{5.11}$$

Similarly,

$$\Big|\sum_{i=1}^{m} Sb(C_i, u, C_i)\Big| \leq \sqrt{R_m}\, \|u\|_1 \Big| \frac{S}{\sqrt{R_m}} \sum_{i=1}^{m} |C_i|^2 \Big|_{L^2} \leq \sqrt{R_m}\, \|u\|_1 |\rho|_{L^2}. \tag{5.12}$$

Applying Cauchy's and Young's inequalities, we obtain

$$S|C_i(x)\cdot\nabla B(x)||\phi_i(x)| \leq S|\nabla B(x)|\,|\phi_i(x)|\,|C_i(x)|$$
$$\leq \frac{S^{1/2}(R_eR_m)^{1/4}}{2} |\nabla B(x)| \Big(\frac{1}{\sqrt{R_e}} |\phi_i(x)|^2 + \frac{S}{\sqrt{R_m}} |C_i(x)|^2 \Big).$$

Integrating this expression in x, summing it over i from 1 up to m, and using the definition of ρ we get

$$S\Big|\sum_{i=1}^{m} b(\phi_i, B, C_i)\Big| = S\Big|\sum_{i=1}^{m} \int_{\Omega} \phi_i(x)\cdot\nabla B(x)\cdot C_i(x)\Big|$$
$$\leq \frac{S^{1/2}(R_eR_m)^{1/4}}{2} \int_{\Omega} |\nabla B(x)| \Big(\frac{1}{R_e} |\phi_i(x)|^2 + \frac{S}{\sqrt{R_m}} |C_i(x)|^2 \Big)\, dx$$
$$= \frac{S^{1/2}(R_eR_m)^{1/4}}{2} |\nabla B|\, |\rho|_{L^2}.$$

Therefore,

$$S\Big|\sum_{i=1}^{m} b(\phi_i, B, C_i)\Big| \leq \frac{S^{1/2}(R_eR_m)^{1/4}}{2} |\nabla B|\, |\rho|_{L^2}. \tag{5.13}$$

Similarly,

$$S\Big|\sum_{i=1}^{m} b(C_i, B, \phi_i)\Big| \leq \frac{S^{1/2}(R_eR_m)^{1/4}}{2} |\nabla B|\, |\rho|_{L^2}. \tag{5.14}$$

Hence from (5.11)–(5.14), we get

$$(5.15)\quad \Big|\sum_{i=1}^m \mathbb{B}_0(\varphi_i, z, \varphi_i)\Big| \\ = \big|b(\phi_i, u, \phi_i) - Sb(C_i, B, \phi_i) + Sb(\phi_i, B, C_i) - Sb(C_i, u, C_i)\big| \\ \le |\rho|_{L^2}\big[(\sqrt{R_e} + \sqrt{R_m})\|u\|_1 + S^{1/2}(R_eR_m)^{1/4}|\nabla B|\big] \\ \le |\rho|_{L^2}\big[(\sqrt{R_e} + \sqrt{R_m})\|u\|_1 + S^{1/2}(R_eR_m)^{1/4}\|B\|_2\big].$$

From the definition of $\rho, \tilde{\phi}_i$, and $\tilde{C}_i$, we observe that

$$\rho(x) = \frac{1}{2}\sum_{i=1}^m\bigg(\frac{1}{\sqrt{R_e}}|\tilde{\phi}_i(x)|^2 + \frac{1}{\sqrt{R_m}}|\tilde{C}_i(x)|^2\bigg).$$

The generalized Lieb–Thirring inequality (see [GMT]) can be applied to the orthonormal finite families $\{\tilde{\phi}_i\}_i$ and $\{\tilde{C}_i\}_i$ (by condition (C_2)). This guarantees the existence of a constant μ independent of the number of functions m (but depending on the shape of Ω) such that

$$(5.16)\quad |\rho|_{L^2}^2 \le \frac{1}{2}\bigg(\frac{1}{R_e}\Big|\sum_{i=1}^m(\tilde{\phi}_i)^2\Big|_{L^2}^2 + \frac{1}{R_m}\Big|\sum_{i=1}^m(\tilde{C}_i)^2\Big|_{L^2}^2\bigg) \\ \le \frac{\mu}{2}\sum_{i=1}^m\bigg(\frac{1}{R_e}\|\tilde{\phi}_i\|_1^2 + \frac{1}{R_m}\|\tilde{C}_i\|_{\mathbb{H}^1}^2\bigg) \\ \le \mu\sum_{i=1}^m\bigg(\frac{1}{R_e}\|\phi_i\|_1^2 + \frac{S}{R_m}\|C_i\|_2^2\bigg).$$

Inserting (5.16) into (5.15) and using Young's inequality, we obtain

$$\Big|\sum_{i=1}^m \mathbb{B}_0(\varphi_i, z, \varphi_i)\Big| \le \mu[(\sqrt{R_e} + \sqrt{R_m})^2\|u\|_1^2 + S(R_eR_m)^{1/2}\|B\|_2^2] \\ + \frac{1}{2}\sum_{i=1}^m\bigg(\frac{1}{R_e}\|\phi_i\|_1^2 + \frac{S}{R_m}\|C_i\|_2^2\bigg).$$

We recall that the dependence on s has been omitted, and in fact $z = z(s, x)$, $\rho = \rho(s, x)$, etc. Using this inequality in (5.10) we obtain

$$(5.17)\quad \mathrm{Tr}_m\big(F'(Z(s,\tau)z_0, s)\big) \\ \le \mu\big[(\sqrt{R_e} + \sqrt{R_m})^2\|u\|_1^2 + S(R_eR_m)^{1/2}\|B\|_2^2\big] \\ - \frac{1}{2}\sum_{i=1}^m\bigg(\frac{1}{R_e}\|\phi_i\|_1^2 + \frac{S}{R_m}\|C_i\|_2^2\bigg).$$

Since $\{\varphi_i\}_{i=1,\dots,m}$ is orthonormal in H, we see that $|\phi_i|^2 = S|C_i|^2 = 1/2$.

Using this and (2.1) in (5.17) we obtain

$$\begin{aligned}&\mathrm{Tr}_m\big(F'(Z(s,\tau)z_0,s)\big)\\&\qquad\le \mu\big[(\sqrt{R_e}+\sqrt{R_m})^2\|u\|_1^2+S(R_eR_m)^{1/2}\|B\|_2^2\big]-m\frac{1}{4}\bigg(\frac{\lambda_1}{R_e}+\frac{c_0}{R_m}\bigg).\end{aligned}$$

Hence, for all $m\in\mathbb{N}$, we have

$$\begin{aligned}\tilde{q}_m&=\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\mathrm{Tr}_m\big(F'(Z(s,\tau-T)z_0,s)\big)\,ds\\&\le\mu\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\big[(\sqrt{R_e}+\sqrt{R_m})^2\|u\|_1^2+S(R_eR_m)^{1/2}\|B\|_2^2\big]\,ds\\&\quad-m\frac{1}{4}\bigg(\frac{\lambda_1}{R_e}+\frac{c_0}{R_m}\bigg).\end{aligned}$$

Let us now estimate the last term of the inequality above. From (1.1) and using (2.9) (orthogonality of $\mathbb{B}$), we obtain the following energy estimates:

$$\frac{d}{dt}|u|^2+\frac{1}{R_e}\|u\|_1^2-Sb(B,B,u)\le R_e\|f\|_{V_1'}^2, \tag{5.18}$$

$$\frac{d}{dt}|B|^2+\frac{1}{R_m}\|B\|_2^2-b(B,u,B)=0. \tag{5.19}$$

Multiplying (5.19) by S, adding up with (5.18), and using (2.3) we obtain

$$\frac{d}{dt}|z|^2+\frac{1}{R_e}\|u\|_1^2+\frac{S}{R_m}\|B\|_2^2\le R_e\|f\|_{V_1'}^2.$$

It follows that

$$\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\|u\|_1^2\,ds\le R_e^2\|f\|^2_{L^\infty(-\infty,T^*;V_1')},$$

$$\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{S}{T}\int_{\tau-T}^{\tau}\|B\|_2^2\,ds\le R_eR_m\|f\|^2_{L^\infty(-\infty,T^*;V_1')}.$$

Therefore,

$$\begin{aligned}&\limsup_{T\to\infty}\sup_{z_0\in A(\tau-T)}\frac{1}{T}\int_{\tau-T}^{\tau}\big[(\sqrt{R_e}+\sqrt{R_m})^2\|u\|_1^2+S(R_eR_m)^{1/2}\|B\|_2^2\big]\,ds\\&\qquad\le\big[(\sqrt{R_e}+\sqrt{R_m})^2R_e^2+(R_eR_m)^{3/2}\big]\|f\|^2_{L^\infty(-\infty,T^*;V_1')}.\end{aligned}$$

Hence

$$\tilde{q}_m\le -mL+K,$$

where K and L are as in the statement of the theorem. We now consider two cases:

CASE 1: $K < L$. Taking

$$q_m = -L(m-1), \quad m = 1, 2, \ldots, \quad n_0 = 1,$$

we can apply Theorem 2.2 to obtain

$$d_{\mathrm{F}}(A(\tau)) \leq 1 \quad \text{for all } \tau \leq T^*.$$

CASE 2: $K \geq L$. Taking

$$q_m = -mL + K, \quad m = 1, 2, \ldots, \quad n_0 = 1 + [K/L - 1],$$

where $[r]$ denotes the integer part of a real number r, we obtain

$$d_{\mathrm{F}}(A(\tau)) \leq K/L \quad \text{for all } \tau \leq T^*.$$

Finally, since $Z(t, \tau)$ is Lipschitz in $A(\tau)$, it follows from [Rob, Proposition 13.9] that $d_{\mathrm{F}}(A(t))$ is bounded for every $t \geq \tau$ with the same bound. ■

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