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On the uniqueness problem for meromorphic mappings with truncated multiplicities

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Abstract. The purpose of this paper is twofold. The first is to weaken or omit the condition $\dim f^{-1}(H_i \cap H_j) \leq m-2$ for $i \neq j$ in some previous uniqueness theorems for meromorphic mappings. The second is to decrease the number q of hyperplanes H_j such that $f(z)=g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$, where $f,\ g$ are meromorphic mappings.

1. Introduction and main results. In 1975, the Nevanlinna "5IM" Theorem was generalized to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ by H. Fujimoto [3]. From then on, the study of the uniqueness problem for meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting a finite set of hyperplanes has been extended and deepened by many authors. At the same time, many outstanding results were derived (see H. Fujimoto [4], M. Ru [10]).

Suppose that f is a linearly non-degenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. For each hyperplane H we denote by $v_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ such that $v_{(f,H)}(a)$ $(a \in \mathbb{C}^m)$ is the intersection multiplicity of the image of f and H at a. Take q hyperplanes $H_1,\dots,H_q$ in $\mathbb{P}^n(\mathbb{C})$ in general position and a positive integer l_0.

Consider the family $\mathcal{F}(\{H_j\}_{j=1}^{q}, f, l_0)$ of all linearly non-degenerate meromorphic mappings $g: \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ satisfying the conditions:

(a) $\min\{v_{(g,H_j)}(z), l_0\} = \min\{v_{(f,H_j)}(z), l_0\}$ for all $j \in \{1,\dots,q\}$,
(b) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \leq m-2$ for all $1 \leq i < j \leq q$,
(c) $f(z) = g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

Denote by $\sharp S$ the cardinality of the set S. We use the standard notations $\overline{E}$ and $\overline{E}_{m_j)}$ as appearing in [2, 6].

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In 1983, L. Smiley [11] showed that

THEOREM A. *If* $q \geq 3n+2$, *then* $g_1 = g_2$ *for any* $g_1, g_2 \in \mathcal{F}(\{H_j\}_{j=1}^{q}, f, 1)$.

In 2009, Z. Chen and Q. Yan [1] proved the following theorem, which is an improvement of Theorem A.

THEOREM B. $\sharp\mathcal{F}(\{H_j\}_{j=1}^{2n+3}, f, 1) = 1$.

Recently, Z. Chen and Q. Yan [2] considered the uniqueness of meromorphic mappings partially sharing $2n+3$ hyperplanes and proved:

THEOREM C. *Let* f *and* g *be linearly non-degenerate meromorphic mappings of* $\mathbb{C}^m$ *into* $\mathbb{P}^n(\mathbb{C})$, *and let* H_j $(1 \leq j \leq q)$ *be* q *hyperplanes in general position such that* $\dim f^{-1}(H_i \cap H_j) \leq m-2$ *for* $i \neq j$. *Assume that*

$$\overline{E}(H_j, f) \subseteq \overline{E}(H_j, g), \qquad 1 \leq j \leq q,$$

and $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$. *If* $q = 2n+3$ *and*

$$\liminf_{r\to\infty} \sum_{j=1}^{2n+3} N^1_{(f,H_j)}(r) \Big/ \sum_{j=1}^{2n+3} N^1_{(g,H_j)}(r) > \frac{n}{n+1},$$

then $f = g$.

REMARK. In fact, the condition $\overline{E}(H_j, f) \subseteq \overline{E}(H_j, g)$ $(1 \leq j \leq q)$ can be deleted in Theorem C, because it can be easily deduced from the condition $f(z) = g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

In the previous results on the uniqueness problem with truncated multiplicity, the condition $\dim f^{-1}(H_i \cap H_j) \leq m-2$ for $i \neq j$ is always needed. So, it is of interest to omit or weaken this condition. Recently, H. Giang, L. Quynh and S. Quang [5] have done some work in this direction.

The first purpose of this paper is to generalize Theorem C by omitting the condition $\dim f^{-1}(H_i \cap H_j) \leq m-2$. In fact, we get a more general result:

THEOREM 1.1. *Let* f *and* g *be linearly non-degenerate meromorphic mappings of* $\mathbb{C}^m$ *into* $\mathbb{P}^n(\mathbb{C})$, *let* m_j $(\geq n)$ $(1 \leq j \leq q)$, k $(1 \leq k \leq n)$ *be integers, and let* H_j $(1 \leq j \leq q)$ *be hyperplanes in general position such that*

$$\dim f^{-1}\Big(\bigcap_{j=1}^{k+1} H_{i_j}\Big) \leq m-2 \quad \textit{for all } 1 \leq i_1 < \cdots < i_{k+1} \leq q. \tag{1.1}$$

Assume that $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q} \overline{E}_{m_j)}(H_j, f)$. *If* $q \geq 2(n+1) + \sum_{i=1}^{q} \frac{2n}{m_i+1}$ *and*

$$\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j),\leq m_j}(r) \Big/ \sum_{j=1}^{q} N^1_{(g,H_j),\leq m_j}(r) > \frac{nk}{q-n-k-1-\sum_{i=1}^{q} \frac{n}{m_i+1}},$$

then $f = g$.

REMARK. Obviously, Theorem C is a special case of the above theorem when $q = 2n+3$, $k = 1$ and $m_j = \infty$ for $1 \leq j \leq q$. When $k = 1$, Theorem 1.1 becomes [7, Theorem 1.1].

The condition (1.1) is always satisfied when $k = n$, since the family of hyperplanes is assumed to be in general position. So, the following result is a corollary of Theorem 1.1 when $k = n$.

COROLLARY 1.2. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, let m_j $(\geq n)$ $(1 \leq j \leq q)$ be integers and H_j $(1 \leq j \leq q)$ be hyperplanes in general position. Assume that $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \overline{E}_{m_j)}(H_j, f)$. If $q \geq 2(n+1) + \sum_{i=1}^{q} \frac{2n}{m_i+1}$ and*

$$\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j),\leq m_j}(r) / \sum_{j=1}^{q} N^1_{(g,H_j),\leq m_j}(r) > \frac{n^2}{q - 2n - 1 - \sum_{i=1}^{q} \frac{n}{m_i+1}},$$

then $f = g$.

REMARK. In Theorem 1.1 and [7, Theorem 1.1], the condition $q \geq 2(n+1) + \sum_{i=1}^{q} \frac{2n}{m_i+1}$ is needed. So, it is natural to ask what will happen if this condition is invalid. However, it seems that the problem is complicated. In the following, we consider the problem for the special case when $m_j = l$ for all $1 \leq j \leq q$.

THEOREM 1.3. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, let k $(1 \leq k \leq n)$ and l $(\geq n)$ be integers and H_j $(1 \leq j \leq q)$ be hyperplanes in general position such that*

$$\dim f^{-1}\Big(\bigcap_{j=1}^{k+1} H_{i_j}\Big) \leq m - 2 \quad \textit{for all } 1 \leq i_1 < \cdots < i_{k+1} \leq q.$$

Assume that $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \overline{E}_{l)}(H_j, f)$ and

$$\liminf_{r\to\infty} \sum_{j=1}^{q} N^1_{(f,H_j),\leq l}(r) / \sum_{j=1}^{q} N^1_{(g,H_j),\leq l}(r) = A.$$

Then $f = g$ if one of the following conditions holds:

(i) $q \geq 2(n+1) + \frac{2n(n+1)}{l+1-n}$ *and* $A > \frac{nk(l+1-n)}{(l+1-n)(q-k)-(l+1)(n+1)}$,

(ii) $q < 2(n+1) + \frac{2n(n+1)}{l+1-n}$, $A \geq \frac{2nk}{q-2k}$ *and*

$$qnk(l+1-n) < A[q(l+1-n)(q+nk-2k)-(q+2nk-2k)(l+1)(n+1)].$$

Now, we will give an application of the above theorem.

In 2011, S. Quang [9] considered the uniqueness of meromorphic mappings sharing hyperplanes with multiplicities. His result can be described as follows.

Theorem D. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, let l be an integer and H_j $(1 \le j \le q)$ be $q = 2n+3$ hyperplanes in general position such that* $\dim f^{-1}(H_i \cap H_j) \le m - 2$ *for $i \ne j$. Assume that*

(1) $\min\{v_{(f,H_j),\le l}(z), 1\} = \min\{v_{(g,H_j),\le l}(z), 1\}, \quad 1 \le j \le q,$

(2) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q} \overline{E}_{l)}(H_j, f)$.

If $l > \frac{n(4n^2+11n+4)}{3n+2} - 1$, *then* $f = g$.

We now apply Theorem 1.3 to prove Theorem D.

From the assumptions of Theorem D, it is obvious that $q = 2n+3$, $k = 1$ and $A = 1$.

In order to prove $f = g$, it suffices to show that $q,\ k,\ A$ satisfy condition (i) or (ii) in Theorem 1.3.

If $l + 1 \ge 2n^2 + 3n$, then a calculation shows that (i) holds.

If $l + 1 < 2n^2 + 3n$, then $q < 2(n+1) + \frac{2n(n+1)}{l+1-n}$. It follows from $A = 1$ that $1 = A \ge \frac{2nk}{q-2k} = \frac{2n}{2n+1}$. Furthermore, if $l > \frac{n(4n^2+8n+3)}{3n+2} - 1$, then

$$qnk(l+1-n) < A[q(l+1-n)(q+nk-2k) - (q+2nk-2k)(l+1)(n+1)].$$

Noting that $l > \frac{n(4n^2+11n+4)}{3n+2} - 1$ in Theorem D, we see that (ii) is valid.

This finishes the proof of Theorem D.

In the previous results, such as Theorem A and B, the condition (c) of the introduction is needed, that is, $f(z) = g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$. So, it is natural to ask whether this condition can be omitted or the number q can be replaced by a smaller one. The second purpose of the paper is to deal with this problem. Our result is:

Theorem 1.4. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and let H_j $(1 \le j \le q)$ be $q = 2n+3$ hyperplanes in general position. Assume that*

(1) $v^1_{(f,H_i)}(z) = v^1_{(g,H_i)}(z)$ $(1 \le i \le 2n+2)$ *and* $v^n_{(f,H_{2n+3})}(z) = v^n_{(g,H_{2n+3})}(z)$,

(2) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \le m - 2$ *for all* $1 \le i < j \le q$,

(3) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$.

Then $f = g$.

Remark. In Theorem 1.4, condition (3) is weaker than that of the previous theorems such as Theorems A and B, but condition (1) is stronger.

Actually, we obtain a more general result, of which Theorem 1.4 is an immediate consequence when $k = 1$.

Theorem 1.5. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, let k $(1 \le k \le n)$ be an integer and H_j*

$(1 \le j \le q)$ be $q = 2kn + 2k + 1$ hyperplanes in general position such that

$$\dim f^{-1}\Big(\bigcap_{j=1}^{k+1} H_{i_j}\Big) \le m-2 \quad \text{for all } 1 \le i_1 < \cdots < i_{k+1} \le q.$$

Assume that f is linearly non-degenerate and

(1) $v^1_{(f,H_i)}(z) = v^1_{(g,H_i)}(z)$ $(1 \le i \le 2n+2)$ *and* $v^n_{(f,H_i)}(z) = v^n_{(g,H_i)}(z)$ $(2n+3 \le i \le q)$,
(2) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$.

Then $f = g$.

When $k = n$ in Theorem 1.5, we have the following corollary.

Corollary 1.6. *Let f and g be linearly non-degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, let k $(1 \le k \le n)$ be an integer and H_j $(1 \le j \le q)$ be $q = 2n^2 + 2n + 1$ hyperplanes in general position. Assume that f is linearly non-degenerate and*

(1) $v^1_{(f,H_i)}(z) = v^1_{(g,H_i)}(z)$ $(1 \le i \le 2n+2)$ *and* $v^n_{(f,H_i)}(z) = v^n_{(g,H_i)}(z)$ $(2n+3 \le i \le q)$,
(2) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$.

Then $f = g$.

For a further study of this kind of problems, we pose two questions.

Question 1. In Theorem 1.1 and 1.3, the condition

$$\dim f^{-1}\Big(\bigcap_{j=1}^{k+1} H_{i_j}\Big) \le m-2 \quad \text{for all } 1 \le i_1 < \cdots < i_{k+1} \le q$$

is needed. We ask whether the results still hold or not if the above condition is weakened to

$$\dim\Big(\bigcap_{j=1}^{k+1} \overline{E}_{m_{i_j})}(H_{i_j}, f)\Big) \le m-2 \quad \text{for all } 1 \le i_1 < \cdots < i_{k+1} \le q.$$

Question 2. In Theorems 1.4 and 1.5, we assume that $f(z) = g(z)$ on $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$. We wonder whether the number $2n+2$ can be decreased or not if q does not change.

2. Preliminaries and some lemmas. Set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \dots, z_m)$ and define

$$B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) = \{z \in \mathbb{C}^m : \|z\| = r\} \quad (0 < r < \infty),$$

and

$$\upsilon_{m-1}(z) = (dd^c\|z\|^2)^{m-1}, \quad \sigma_m(z) = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}$$

on $\mathbb{C}^m \setminus \{0\}$.

Let f be a non-constant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. We take holomorphic functions $f_0, \ldots, f_n$ on $\mathbb{C}^m$ such that $\mathcal{I}_f = \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\}$ is of dimension at most $m-2$; then $f = \{f_0, \ldots, f_n\}$ is called a *reduced representation* of f. The *characteristic function* of f is defined as

$$T_f(r) = \int\limits_{S(r)} \log \|f\| \, \sigma_m - \int\limits_{S(1)} \log \|f\| \, \sigma_m.$$

Note that $T_f(r)$ is independent of the choice of the reduced representation of f.

For a divisor ν on $\mathbb{C}^m$ and positive integers $k,\ p$ (or $k,\ p = \infty$), we define some divisors as follows:

$$\nu^p(z) = \min\{p, \nu(z)\},$$
$$\nu^p_{\leq k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k, \\ \nu^p(z) & \text{if } \nu(z) \leq k, \end{cases}$$
$$\nu^p_{>k}(z) = \begin{cases} \nu^p(z) & \text{if } \nu(z) > k, \\ 0 & \text{if } \nu(z) \leq k. \end{cases}$$

Set

$$n(t) = \begin{cases} \int_{|\nu| \cap B(t)} \nu(z) \upsilon_{m-1} & \text{if } m \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, define $n^p(t)$, $n^p_{\leq k}(t)$, $n^p_{>k}(t)$. Define the counting function of ν as

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2n-1}} \, dt \qquad (1 < r < \infty).$$

Similarly, we also define $N(r, \nu^p)$, $N(r, \nu^p_{\leq k})$, $N(r, \nu^p_{>k})$ and denote them by $N^p(r, \nu)$, $N^p_{\leq k}(r, \nu)$, $N^p_{>k}(r, \nu)$, respectively.

Let $\phi : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. Define

$$N_\phi(r) = N(r, \nu_\phi), \qquad N^p_{\phi, \leq k}(r) = N^p_{\leq k}(r, \nu_\phi),$$
$$N^p_\phi(r) = N^p(r, \nu_\phi), \qquad N^p_{\phi, >k}(r) = N^p_{>k}(r, \nu_\phi).$$

In order to prove our results, we recall the second main theorem for meromorphic mappings.

Lemma 2.1 ([10]). *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic mapping and $H_1, \ldots, H_q$ be q ($\geq n+1$) hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then*

$$\| \; (q - n - 1) T_f(r) \leq \sum_{j=1}^{q} N^n_{(f, H_j)}(r) + o(T_f(r)).$$

As usual, the notation "$\| \; P$" means that the assertion P holds for all r in $[0, \infty)$ excluding a Borel subset $E \subset [0, \infty)$ with $\int_E dr < \infty$.

The following modification of the above lemma is essential to the proof of Theorem 1.3.

LEMMA 2.2 ([9]). *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic mapping, let $l \geq n$ be an integer and $H_1, \ldots, H_q$ be q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then*

$$\Big\| \ \frac{(l+1)(q-n-1)-nq}{l+1-n} T_f(r) \leq \sum_{j=1}^{q} N^n_{(f,H_j),\leq l}(r) + o(T_f(r)).$$

3. Proof of Theorem 1.1. Suppose that $f \neq g$. From the assumptions, we can easily deduce that

$$\| \ T_f(r) = O(T_g(r)) \quad \text{and} \quad \| \ T_g(r) = O(T_f(r)).$$

As in [1], we introduce an equivalence relation on $L := \{1, \ldots, q\}$ as follows: $i \sim j$ if and only if $(f,H_i)/(f,H_j) - (g,H_i)/(g,H_j) = 0$. Let $\{L_1, \ldots, L_s\} = L/\sim$. Since $f \neq g$ and $\{H_j\}_{j=1}^q$ are in general position, we have $\sharp L_k \leq n$ for all $k \in \{1, \ldots, s\}$. Without loss of generality, we assume that $L_k := \{i_{k-1}+1, \ldots, i_k\}$ $(k \in \{1, \ldots, s\})$ where $1 = i_0 < \cdots < i_s = q$.

Define $\sigma : \{1, \ldots, q\} \to \{1, \ldots, q\}$ by

$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \leq q, \\ i+n-q & \text{if } i+n > q. \end{cases}$$

It is easy to see that σ is bijective and $|\sigma(i) - i| \geq n$ (note that $q \geq 2n$). This implies that i and $\sigma(i)$ belong to distinct sets of $\{L_1, \ldots, L_s\}$ and

$$\frac{(f,H_i)}{(f,H_{\sigma(i)})} - \frac{(g,H_i)}{(g,H_{\sigma(i)})} \neq 0.$$

Let $P_i = (f,H_i)(g,H_{\sigma(i)}) - (g,H_i)(f,H_{\sigma(i)})$. Obviously, $P_i \neq 0$. With the Jensen formula, we obtain

$$N_{P_i}(r) \leq T_f(r) + T_g(r) + O(1) = T(r) + O(1),$$

where $T(r) = T_f(r) + T_g(r)$. Then $P = \prod_{i=1}^q P_i \neq 0$ and $N_P(r) \leq qT(r) + O(1)$. Let

$$S = \bigcup_{1 \leq i_1 < \cdots < i_{k+1} \leq q} f^{-1}\Big(\bigcap_{j=1}^{k+1} H_{i_j}\Big).$$

Then S is an analytic set of codimension at least 2. Take a point z not in $I(f) \cup I(g) \cup S$. We claim that

$$v_P(z) \geq 2\sum_{i=1}^{q} \min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} + \frac{q-2k}{k} \sum_{i=1}^{q} v^1_{(f,H_i),\leq m_i}(z).$$

Suppose that z is a zero of some function (f,H_i) $(1 \leq i \leq q)$. Let

$$I = \{1 \leq i \leq q : (f,H_i)(z) = 0\}, \qquad t = \sharp I.$$

It is clear that $t \leq k$. We now prove the claim by considering two cases.

Case 1: *There exists $l \in I$ such that $v_{(f,H_l)}(z) \leq m_l$.* Assume that $j \in \{1, \ldots, q\} \backslash (I \cup \sigma^{-1}(I))$. Noting that $f(\zeta) = g(\zeta)$ on $\bigcup_{j=1}^{q} \{\zeta \in \mathbb{C}^m : 0 < v_{(f,H_j)}(\zeta) \leq m_j\}$, we see that z is a zero of P_j of multiplicity at least 1. So $v_{P_j}(z) \geq 1$. Therefore,

$$\begin{aligned} v_p(z) &\geq 2 \sum_{i \in I} \min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} + q - 2t \\ &\geq 2 \sum_{i \in I} \min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} + q - 2k \\ &\geq 2 \sum_{i=1}^{q} \min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} + \frac{q - 2k}{k} \sum_{i=1}^{q} v^1_{(f,H_i), \leq m_i}(z). \end{aligned}$$

Thus, the claim holds.

Case 2: *$v_{(f,H_l)}(z) \leq m_l$ for no $l \in I$.* Then $\sum_{i=1}^{q} v^1_{(f,H_i), \leq m_i}(z) = 0$. Thus,

$$\begin{aligned} v_p(z) &\geq 2 \sum_{i \in I} \min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} \\ &\geq 2 \sum_{i=1}^{q} \min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} + \frac{q - 2k}{k} \sum_{i=1}^{q} v^1_{(f,H_i), \leq m_i}(z). \end{aligned}$$

So, the claim is also valid.

Since $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \{z \in \mathbb{C}^m : 0 < v_{(f,H_j)}(z) \leq m_j\}$, for $1 \leq i \leq q$ we get $v_{(g,H_i)}(z) > 0$ if $0 < v_{(f,H_i)}(z) \leq m_i$.

Furthermore, we have

$$\min\{v_{(f,H_i)}(z), v_{(g,H_i)}(z)\} \geq v^n_{(f,H_i), \leq m_i}(z) + v^n_{(g,H_i), \leq m_i}(z) - n v^1_{(g,H_i), \leq m_i}(z).$$

The claim and the above inequality yield

$$\begin{aligned} v_p(z) \geq\ & 2 \sum_{i=1}^{q} [v^n_{(f,H_i), \leq m_i}(z) + v^n_{(g,H_i), \leq m_i}(z) - n v^1_{(g,H_i), \leq m_i}(z)] \\ & + \frac{q - 2k}{k} \sum_{i=1}^{q} v^1_{(f,H_i), \leq m_i}(z). \end{aligned} \tag{3.1}$$

Moreover, it follows from Lemma 2.1 that

$$\begin{aligned}
\| \ (q-n-1)T_f(r) &\le \sum_{j=1}^{q} N^n_{(f,H_j)}(r) + o(T_f(r)) \\
&= \sum_{j=1}^{q} [N^n_{(f,H_j),\le m_j}(r) + N^n_{(f,H_j),>m_j}(r)] + o(T_f(r)) \\
&\le \sum_{j=1}^{q} \left[N^n_{(f,H_j),\le m_j}(r) + \frac{n}{m_j+1} N_{(f,H_j),>m_j}(r) \right] + o(T_f(r)) \\
&\le \sum_{j=1}^{q} \left[N^n_{(f,H_j),\le m_j}(r) + \frac{n}{m_j+1} T_f(r) \right] + o(T_f(r)),
\end{aligned}$$

which implies that

$$\left\| \ \left(q-n-1-\sum_{j=1}^{q} \frac{n}{m_j+1} \right) T_f(r) \le \sum_{j=1}^{q} N^n_{(f,H_j),\le m_j}(r) + o(T_f(r)). \right.$$

Integrating both sides of (3.1), we deduce

$$\begin{aligned}
qT(r) &\ge N_P(r) \\
&\ge 2\sum_{i=1}^{q} [N^n_{(f,H_i),\le m_i}(r) + N^n_{(g,H_i),\le m_i}(r) - nN^1_{(g,H_i),\le m_i}(r)] \\
&\quad + \frac{q-2k}{k} \sum_{i=1}^{q} N^1_{(f,H_i),\le m_i}(r) \\
&\ge 2\left(q-n-1-\sum_{j=1}^{q} \frac{n}{m_j+1} \right) T(r) \\
&\quad + \frac{q-2k}{k} \sum_{i=1}^{q} N^1_{(f,H_i),\le m_i}(r) - 2n\sum_{i=1}^{q} N^1_{(g,H_i),\le m_i}(r) + o(T_f(r)),
\end{aligned}$$

which leads to

$$\begin{aligned}
(3.2) \qquad & \left(q-2n-2-\sum_{j=1}^{q} \frac{2n}{m_j+1} \right) T(r) \\
& \le -\frac{q-2k}{k} \sum_{i=1}^{q} N^1_{(f,H_i),\le m_i}(r) + 2n\sum_{i=1}^{q} N^1_{(g,H_i),\le m_i}(r) + o(T_f(r)).
\end{aligned}$$

We know that, for some $1 \le j \le q$, there exists $c \in \mathbb{C}^{n+1} \backslash \{0\}$ such that

$$F_f^{H_j,c} - F_g^{H_j,c} = \frac{(f,H_j)}{(f,c)} - \frac{(g,H_j)}{(g,c)} \ne 0.$$

Since $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \{z \in \mathbb{C}^m : 0 < v_{(f,H_j)}(z) \le m_j\}$, we have

$$F_f^{H_j,c} - F_g^{H_j,c} = 0$$

on $\bigcup_{j=1}^{q}\{z\in\mathbb{C}^m : 0 < v_{(f,H_j)}(z)\le m_j\}$. Then

$$\left\|\ \sum_{i=1}^{q} N^1_{(f,H_i),\le m_i}(r) \le kN^1_{F_f^{H_j,c}-F_g^{H_j,c}}(r) \le kT(r, F_f^{H_j,c}-F_g^{H_j,c})+O(1) \le kT(r)+O(1). \right. \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\left\|\ \frac{1}{k}\left[q-2n-2-\sum_{j=1}^{q}\frac{2n}{m_j+1}+q-2k\right]\sum_{i=1}^{q}N^1_{(f,H_i),\le m_i}(r) \le 2n\sum_{i=1}^{q}N^1_{(g,H_i),\le m_i}(r)+o(T_f(r)). \right.$$

This can be rewritten as

$$\liminf_{r\to\infty}\sum_{j=1}^{q}N^1_{(f,H_j),\le m_j}(r)\Big/\sum_{j=1}^{q}N^1_{(g,H_j),\le m_j}(r)\le\frac{nk}{q-n-k-1-\sum_{i=1}^{q}\frac{n}{m_i+1}},$$

which contradicts the assumption.

4. The proof of Theorem 1.3. Suppose that $f\not\equiv g$. Then, with the same discussion as in Theorem 1.1, we can deduce that

$$v_p(z)\ge 2\sum_{i=1}^{q}[v^n_{(f,H_i),\le l}(z)+v^n_{(g,H_i),\le l}(z)-nv^1_{(g,H_i),\le l}(z)] + \frac{q-2k}{k}\sum_{i=1}^{q}v^1_{(f,H_i),\le l}(z). \tag{4.1}$$

By Lemma 2.2 integrating both sides of (4.1), we have

$$\begin{aligned} qT(r) &\ge 2\sum_{i=1}^{q}[N^n_{(f,H_i),\le l}(r)+N^n_{(g,H_i),\le l}(r)-nN^1_{(g,H_i),\le l}(r)] \\ &\quad + \frac{q-2k}{k}\sum_{i=1}^{q}N^1_{(f,H_i),\le l}(r) \\ &\ge 2\sum_{i=1}^{q}[N^n_{(f,H_i),\le l}(r)+N^n_{(g,H_i),\le l}(r)]+\frac{q-2k}{k}\sum_{i=1}^{q}N^1_{(f,H_i),\le l}(r) \\ &\quad -2n\sum_{i=1}^{q}N^1_{(g,H_i),\le l}(r) \end{aligned} \tag{4.2}$$

$$\geq 2\frac{(l+1)(q-n-1)-nq}{l+1-n}T(r)+\frac{q-2k}{k}\sum_{i=1}^{q}N^{1}_{(f,H_i),\leq l}(r)$$
$$-2n\sum_{i=1}^{q}N^{1}_{(g,H_i),\leq l}(r)+o(T_f(r)).$$

We consider two cases.

CASE 1: $q\leq 2\frac{(l+1)(q-n-1)-nq}{l+1-n}$. This is equivalent to

$$q>\frac{2(l+1)(n+1)}{l+1-n}=2(n+1)+\frac{2n(n+1)}{l+1-n}.$$

Then it follows from (4.2) that

$$\frac{(l+1-n)q-2(l+1)(n+1)}{l+1-n}T(r)\leq -\frac{q-2k}{k}\sum_{i=1}^{q}N^{1}_{(f,H_i),\leq l}(r)+2n\sum_{i=1}^{q}N^{1}_{(g,H_i),\leq l}(r). \tag{4.3}$$

As in the proof in Theorem 1.1, we have

$$\Big\|\ \sum_{i=1}^{q}N^{1}_{(f,H_i),\leq l}(r)\leq kT(r)+O(1). \tag{4.4}$$

Combining (4.3) and (4.4) yields

$$\Big\|\ \left[\frac{(l+1-n)q-2(l+1)(n+1)}{l+1-n}+q-2k\right]\sum_{i=1}^{q}N^{1}_{(f,H_i),\leq l}(r)\leq 2nk\sum_{i=1}^{q}N^{1}_{(g,H_i),\leq l}(r).$$

From the above inequality, we derive that

$$\liminf_{r\to\infty}\sum_{j=1}^{q}N^{1}_{(f,H_j),\leq l}(r)\Big/\sum_{j=1}^{q}N^{1}_{(g,H_j),\leq l}(r)\leq\frac{nk(l+1-n)}{(l+1-n)(q-k)-(l+1)(n+1)},$$

which contradicts the assumption.

CASE 2: $q<\frac{2(l+1)(n+1)}{l+1-n}=2(n+1)+\frac{2n(n+1)}{l+1-n}$. Since

$$\liminf_{r\to\infty}\sum_{j=1}^{q}N^{1}_{(f,H_j),\leq l}(r)\Big/\sum_{j=1}^{q}N^{1}_{(g,H_j),\leq l}(r)=A,$$

for any positive constant ε there exists a positive constant r_0 such that $\sum_{j=1}^{n+1} N^1_{(f,H_j),\leq l}(r) \geq (A-\varepsilon)\sum_{j=1}^{n+1} N^1_{(g,H_j),\leq l}(r)$ for $r \geq r_0$.

Using (4.2), we can deduce that

$$\begin{aligned} qT(r) &\geq 2\sum_{i=1}^{q}[N^n_{(f,H_i),\leq l}(r) + N^n_{(g,H_i),\leq l}(r)] - \frac{q-2k+2nk}{(1+A)k}\varepsilon\sum_{i=1}^{q} N^1_{(g,H_i),\leq l}(r) \\ &\quad + \frac{(q-2k)A-2nk}{(1+A)nk}\sum_{i=1}^{q} N^n_{(f,H_i),\leq l}(r) + N^n_{(g,H_i),\leq l}(r)] \\ &\geq \left[2+\frac{(q-2k)A-2nk}{(1+A)nk}\right]\frac{(l+1)(q-n-1)-nq}{l+1-n}T(r) + o(T_f(r)) \\ &\quad - \frac{q-2k+2nk}{(1+A)nk}\varepsilon\sum_{i=1}^{q} N^1_{(g,H_i),\leq l}(r), \end{aligned}$$

which indicates that

$$\begin{aligned} q\frac{q-2k+2nk}{(1+A)k}\varepsilon T(r) &\geq \frac{q-2k+2nk}{(1+A)k}\varepsilon\sum_{i=1}^{q} N^1_{(g,H_i),\leq l}(r) \\ &\geq \left\{\left[2+\frac{(q-2k)A-2nk}{(1+A)nk}\right]\frac{(l+1)(q-n-1)-nq}{l+1-n} - q\right\}T(r) + o(T_f(r)) \\ &\geq \frac{B}{(l+1-n)(1+A)nk}T(r) + o(T_f(r)), \end{aligned}$$

where $B = A[q(l+1-n)(q+nk-2k) - (q+2nk-2k)(l+1)(n+1)] - qnk(l+1-n)$.

Choosing ε small enough, we can easily obtain a contradiction from the above inequality.

5. The proof of Theorem 1.5. Suppose that $f \neq g$. We repeat verbatim the proof of Theorem 1.1 until the definition of the set S, which is again an analytic set of codimension at least 2.

Take a point $z \notin I(f) \cup I(g) \cup S$. We now claim that

$$\begin{aligned} v_P(z) &\geq 2\sum_{i=1}^{2n+2}[v^n_{(f,H_i)}(z) + v^n_{(g,H_i)}(z) - nv^1_{(f,H_i)}(z)] \\ &\quad + \sum_{i=2n+3}^{q}[v^n_{(f,H_i)}(z) + v^n_{(g,H_i)}(z)] + \frac{q-2k}{k}\sum_{i=1}^{2n+2} v^1_{(f,H_i)}(z). \end{aligned} \tag{5.1}$$

Assume that z is a zero of some function (f,H_i) $(1 \leq i \leq q)$. Let

$$I = \{1 \leq i \leq 2n+2 : (f,H_i)(z) = 0\}, \qquad t = \sharp I,$$
$$J = \{2n+3 \leq i \leq q : (f,H_i)(z) = 0\}, \qquad s = \sharp J.$$

Clearly, $1 \leq t+s \leq k$. We now prove the claim by distinguishing two cases.

CASE 1: $I=\emptyset$. Then

$$\sum_{i=1}^{2n+2} v^1_{(f,H_i)}(z)=0,\qquad 2\sum_{i=1}^{2n+2}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)-nv^1_{(f,H_i)}(z)]=0.$$

Furthermore, we have

$$v_p(z)\ge 2\sum_{i\in J} v^n_{(f,H_i)}(z)=\sum_{i=2n+3}^{q}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)].$$

Thus, the claim holds.

CASE 2: $I\neq\emptyset$. Suppose that $j\in\{1,\dots,q\}\setminus[I\cup J\cup\sigma^{-1}(I\cup J)]$. Since $f(\zeta)=g(\zeta)$ on $\bigcup_{j=1}^{2n+2} f^{-1}(H_j)$, z is a zero point of P_j with multiplicity at least 1. So $v_{P_j}(z)\ge 1$.

If $i\in\{1,\dots,2n+2\}$, then

$$\min\{v_{(f,H_i)}(z),v_{(g,H_i)}(z)\}\ge v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)-nv^1_{(f,H_i)}(z).$$

If $i\in\{2n+3,\dots,q\}$, then

$$\min\{v_{(f,H_i)}(z),v_{(g,H_i)}(z)\}\ge v^n_{(f,H_i)}(z).$$

It follows that

$$\begin{aligned}
v_p(z)&\ge 2\sum_{i\in I}\min\{v_{(f,H_i)}(z),v_{(g,H_i)}(z)\}+2\sum_{i\in J}\min\{v_{(f,H_i)}(z),v_{(g,H_i)}(z)\}\\
&\quad+q-2(l+s)\\
&\ge 2\sum_{i=1}^{2n+2}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)-nv^1_{(f,H_i)}(z)]+2\sum_{i\in J}v^n_{(f,H_i)}(z)+q-2k\\
&\ge 2\sum_{i=1}^{2n+2}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)-nv^1_{(f,H_i)}(z)]\\
&\quad+\sum_{i=2n+3}^{q}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)]+\frac{q-2k}{k}\sum_{i=1}^{2n+2}v^1_{(f,H_i)}(z).
\end{aligned}$$

Thus, the claim holds.

By the claim, we have

$$\begin{aligned}
v_P(z)\ge&\sum_{i=1}^{q}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)]+\sum_{i=1}^{2n+2}[v^n_{(f,H_i)}(z)+v^n_{(g,H_i)}(z)]\\
&+\frac{q-2k-2nk}{k}\sum_{i=1}^{2n+2}v^1_{(f,H_i)}(z).
\end{aligned}$$

By integrating both sides of the above inequality, we get

$$\begin{aligned} qT(r) &\ge N_p(r) \\ &\ge \sum_{i=1}^{q}[N^n_{(f,H_i)}(r) + N^n_{(g,H_i)}(r)] + \sum_{i=1}^{2n+2}[N^n_{(f,H_i)}(r) + N^n_{(g,H_i)}(r)] \\ &\quad + \frac{q-2k-2kn}{k}\sum_{i=1}^{2n+2} N^1_{(f,H_i)}(r) \\ &\ge (q-n-1)T(r) + (2n+2-n-1)T(r) \\ &\quad + \frac{q-2k-2kn}{k}\sum_{i=1}^{2n+2} N^1_{(f,H_i)}(r) + o(T_f(r)), \end{aligned}$$

which implies that

$$\sum_{i=1}^{2n+2} N^1_{(f,H_i)}(r) = o(T_f(r)). \tag{5.2}$$

By combining (5.2) and Lemma 2.1, we can easily deduce a contradiction.

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