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Cutting description of trivial 1-cohomology

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Abstract. A characterisation of trivial 1-cohomology, in terms of some connectedness condition, is presented for a broad class of metric spaces.

1. Introduction. We will establish the following theorem:

Theorem 1.1. *A connected and locally connected metric space X has a trivial first Čech cohomology group if and only if every connected open subset U leaves $X \setminus U$ disconnected, provided it has a disconnected boundary.*

Consider the following conditions:

(1) X is connected and locally connected;
(2) $H^1(X) = 0$;
(3) ∂U is disconnected;
(4) $X \setminus U$ is disconneced;

Then Theorem 1.1 accounts for all nontrivial implications in

Theorem 1.2. $(1) \Rightarrow ((2) \Leftrightarrow (\forall U$ *open and connected,* $(3) \Leftrightarrow (4)))$.

That (4) always implies (3) is an exercise on normality of metric spaces. We dub the innermost parentheses the "cutting condition".

Of course, our theorems apply to the manifold category, and we can state one corollary in terms of de Rham cohomology, thus solving a PDE:

Corollary 1.3. *If every open domain U of a manifold M with ∂U disconnected leaves $M \setminus U$ disconnected, then every equation*

$$df = \alpha$$

has a solution, provided the 1*-form* α *is closed.*

Throughout this paper, $H^i(X)$ stands for the ith reduced Čech cohomology group with constant $\mathbb{Z}$ coefficients.

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This paper is a matured version of [CLK] and, as mentioned there, observations of this kind have (minor) applications to complex analysis, concerning boundary of domains of holomorphy. Apart from the proof, we will give examples to show that local connectedness cannot be ommited.

2. The proof. All the algebraic topology material used here is classic and can be found in any popular textbook on the subject. Recall that the $0-$th group $H^0(A)$ is always free and in the locally connected setting its rank is equal to the number of connected components of A minus 1. It is natural to apply the Mayer–Vietoris sequence in any problem concerning decompositions of a space and cohomology. However, some care is needed. Recall that for every pair of open sets A and B covering the space X, we have an exact sequence

$$H^0(X) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \xrightarrow{\partial_*} H^1(X) \to \cdots .$$

We label only the so-called connecting homomorphism for future reference. Now consider small open neighbourhoods of $X \setminus U$, the closure and the boundary of U: $X \setminus U \subset A$, $\overline{U} \subset B$, $\partial U \subset C$, respectively. Hence we have

$$H^0(X) \to H^0(A) \oplus H^0(B) \to H^0(C) \xrightarrow{\partial_*} H^1(X) \to \cdots .$$

The directed system of such neighbourhoods converges to our initial sets, and this is reflected by convergence in cohomology, by rigidity of Čech cohomology in metric spaces. Thus we have an exact sequence

$$H^0(X) \to H^0(X \setminus U) \oplus H^0(\overline{U}) \to H^0(\partial U) \xrightarrow{\partial_*} H^1(X) \to \cdots .$$

Assume now that X is connected, U is a domain and $H^1(X)$ is trivial. The sequence takes the form

$$0 \to H^0(X \setminus U) \to H^0(\partial U) \to 0$$

This establishes a bijection between the components of the boundary and of the complement, and thus one implication in our theorem.

Remark 2.1. Dropping the assumption that U is connected, we still get an exact sequence

$$H^0(X) \to H^0(X \setminus U) \oplus H^0(U) \to H^0(\partial U) \xrightarrow{\partial_*} \operatorname{Im} \partial_* \to 0.$$

Note that $H^1(X)$, and thus also $\operatorname{Im} \partial_*$, are free groups. Exactness means that the alternating sum of the ranks of the groups in the sequence (its Euler characteristic) is zero:

$$\operatorname{rk} H^0(X) - \operatorname{rk} H^0(X \setminus U) - \operatorname{rk} H^0(U) + \operatorname{rk} H^0(\partial U) + \operatorname{rk} \operatorname{Im} \partial_* = 0.$$

Translating that into the number of connected components (we write $\#A$ for the number of connected components of A), when X is connected, we get

$$-1 \leq \#\partial U - \#X \setminus U - \#U \leq -1 + \operatorname{rk} H^1(X).$$

Note that for a broad class of spaces (spaces with "good" coverings in the sense of homotopy theory, manifolds for example) $\operatorname{rk} H^1(X)$ is bounded by $\operatorname{rk} \pi_1(X)$.

For the other implication, assume that $H^1(X)$ is nontrivial. We will find a domain with a connected complement and disconnected boundary.

We have $H^1(X) = \varinjlim H^1\mathcal{U}$, the injective limit with respect to the directed system of all open coverings of X—without loss of generality, coverings by connected sets. Hence a nontrivial class in $H^1(X)$ arises as a nontrivial class in some $H^1(\mathcal{V})$ (and in all of its refinements). $H^1(\mathcal{V})$ is in turn equal to $H^1_S(N\mathcal{V})$, the singular cohomology of the nerve of $\mathcal{V}$, which is a simplicial complex. We can assume that $N\mathcal{V}$ is truncated over dimension 2, since we are interested only in the first cohomology group. For any simplicial complex K, there is a 1:1 corespondence between $H^1_S(K)$ and $[K, \mathbb{S}^1]$, the homotopy classes of continuous maps from K to the circle. Therefore a nontrivial class in $H^1_S(N\mathcal{V})$ is represented by a map θ from $N\mathcal{V}$ to $\mathbb{S}^1$. This map can be chosen simplicial (for a sufficiently fine simplicial structure on the circle; note that a simplicial circle has at least three vertices) and without local extrema (a point x is a local extremum of $\theta : N\mathcal{V} \to \mathbb{S}^1$ if it is a genuine local extremum in a neighbourhood V_x of $\theta|_{V_x} \to B(\theta(x), \epsilon) \subset \mathbb{S}^1$, a small ball in $\mathbb{S}^1$ identified with an interval in $\mathbb{R}$).

Starting from any vertex, enumerate the vertices in the circle clockwise. Pick any vertex $v_n \in \mathbb{S}^1$. The vertices a_i in $\theta^{-1}(v)$ are open sets in the covering $\mathcal{V}$. Any connected component of $\bigcup a_i$ must have disconnected boundary (with disjoint open sets $\theta^{-1}(v_{n-1})$ and $\theta^{-1}(v_{n+1})$). Moreover, there exists at least one connected component $A \subset \bigcup a_i$ whose complement has a connected component, say B, meeting both $\theta^{-1}(v_{n-1})$ and $\theta^{-1}(v_{n+1})$ (otherwise θ would be nullhomotopic). The domain $U = \bigcup\{v \text{ vertex in } N\mathcal{V} \mid v \cap B = \emptyset\}$ has a disconnected boundary and a connected complement. This proves the other implication in our theorem.

3. Counterexamples. As for the counterexample concerning local connectedness, consider the "rational Hawaiian earring", a dense subspace of a ball in $\mathbb{R}^2$:

$$\mathcal{H}_{\mathbb{Q}} = \bigcup_{q \in \mathbb{Q} \cap [0,1]} \partial\mathbb{B}((0, q), q).$$

This connected space obviously has nontrivial 1-cohomology, and all of its connected and open subsets must contain the point $(0, 0)$. Such subsets have a connected boundary only when the boundary is equal to the complement and is contained in one of the circles. Thus, in terms of Theorem 1.2, the cutting condition does not imply (1) without local connectedness.

We note however that trivial 1-cohomology always implies bijection between quasi-components of the complement and of the boundary of a domain, because the sequence

$$0 \to H^0(X \setminus U) \to H^0(\partial U) \to 0$$

remains exact, and—without assuming local connectedness—the ranks of the groups now equal the number of quasi-components minus 1.

To finish, we note that the last remaining one-way implication in Theorem 1.2 cannot be reversed by the following counterexample.

The Knaster–Kuratowski fan (a cone over the rationals) is a contractible space and satisfies the cutting condition (for reasons similar to the case of the Hawaiian earring), but is not locally connected. Observe, however, that this space is not locally homogeneous (the vertex is topologically different from other points).

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References

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