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# Some new Opial-type inequalities involving higher order derivatives

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**Abstract.** We establish several new Opial-type inequalities involving different types of boundary conditions.

**1. Introduction.** In 1960 the Polish mathematician Z. Opial [Op] published an inequality involving integrals of a function and its derivative.

Theorem 1.1. *Let $x(\cdot) \in C^{(1)}[a,b]$ be such that $x(a) = x(b) = 0$ and $x(t) > 0$ in $(a,b)$. Then*

$$\int_a^b |x(t)x'(t)|\,dt \le \frac{1}{4}(b-a)\int_a^b (x'(t))^2\,dt. \tag{1.1}$$

*where the constant $\frac{1}{4}(b-a)$ is the best possible.*

An improved form of Opial's inequality was established by Willett [W].

Theorem 1.2. *Let $x(\cdot)$ be absolutely continuous on $[a,b]$, and $x(a) = 0$. Then*

$$\int_a^b |x(t)x'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b (x'(t))^2\,dt. \tag{1.2}$$

The first natural extension of Opial's inequality (1.2) involving higher order derivatives $x^{(n)}(t)$ $(n \ge 1)$ is also due to Willett [W].

Theorem 1.3. *Let $x(\cdot) \in C^{(n)}[a,b]$ be such that $x^{(i)}(a) = 0$ for $0 \le i \le n-1$ $(n \ge 1)$. Then*

$$\int_a^b |x(t)x^{(n)}(t)|\,dt \le \frac{1}{2}(b-a)^n\int_a^b |x^{(n)}(t)|^2\,dt. \tag{1.3}$$

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A sharp version of inequality (1.3) is due to Das [D]:

THEOREM 1.4. *Let $x(\cdot) \in C^{(n-1)}[a,b]$ be such that $x^{(i)}(a) = 0$ for $0 \le i \le n-1$ $(n \ge 1)$. Further, let $x^{(n-1)}(\cdot)$ be absolutely continuous and $\int_a^b |x^{(n)}(t)|^2\,dt < \infty$. Then*

$$\int_a^b |x(t)x^{(n)}(t)|\,dt \le \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{1/2}(b-a)^n \int_a^b |x^{(n)}(t)|^2\,dt. \tag{1.4}$$

This simple inequality has motivated a large number of research papers giving its successively simpler proofs, providing various generalizations, and exhibiting discrete analogs (see the reference list).

This type of inequalities has been used recently in studying the gaps between zeros of solutions of differential equations. The main purpose of the present paper is to establish some new Opial-type inequalities with different types of boundary conditions. To the best of our knowledge, no author has discussed these problems till now.

**2. Main results.** In this section, we give our main results and some corollaries.

THEOREM 2.1. *Let $x(\cdot) \in C^{(n)}[a,b]$ be such that $\alpha_i x^{(i)}(a) + \beta_i x^{(i)}(b) = 0$ with $\alpha_i, \beta_i > 0$ for $0 \le i \le n-1$ $(n \ge 1)$. Then*

$$\int_a^b |x(t)x^{(n)}(t)|\,dt \le \prod_{i=0}^{n-1} \frac{\sqrt{(\alpha_i^2+\beta_i^2)/2}}{\alpha_i+\beta_i}(b-a)^n \int_a^b |x^{(n)}(t)|^2\,dt. \tag{2.1}$$

*Proof.* Define

$$H_i(s,t) = \begin{cases} \dfrac{\alpha_i}{\alpha_i+\beta_i}, & a \le t \le s, \\ -\dfrac{\beta_i}{\alpha_i+\beta_i}, & s \le t \le b, \end{cases}$$

and

$$L_i(s) = \Big(\int_a^b |H_i(s,t)|^2\,dt\Big)^{1/2} = \left[\frac{\alpha_i^2(s-a)+\beta_i^2(b-s)}{(\alpha_i+\beta_i)^2}\right]^{1/2}.$$

Then, by the condition $\alpha_i x^{(i)}(a) + \beta_i x^{(i)}(b) = 0$ for $0 \le i \le n-1$, we have

$$x^{(i)}(s) = \int_a^b x^{(i+1)}(t)H_i(s,t)\,dt \qquad (a \le s \le b). \tag{2.2}$$

So, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}|x^{(i)}(s)| &\le \int_a^b |x^{(i+1)}(t)|\,|H_i(s,t)|\,dt\\ &\le \Big(\int_a^b |x^{(i+1)}(t)|^2\,dt\Big)^{1/2}\Big(\int_a^b |H_i(s,t)|^2\,dt\Big)^{1/2}\\ &= L_i(s)\cdot\Big(\int_a^b |x^{(i+1)}(t)|^2\,dt\Big)^{1/2}.\end{aligned}$$

Thus, for $0 \le i \le n-1$, we have

$$\Big(\int_a^b |x^{(i)}(t)|^2 dt\Big)^{1/2} \le \frac{\sqrt{(\alpha_i^2+\beta_i^2)/2}}{\alpha_i+\beta_i}\cdot(b-a)\Big(\int_a^b |x^{(i+1)}(t)|^2\,dt\Big)^{1/2}. \tag{2.3}$$

Using (2.3) inductively, we obtain

$$\begin{aligned}|x(t)| &\le L_0(t)\cdot\Big(\int_a^b |x'(t)|^2\,dt\Big)^{1/2}\\ &\le L_0(t)\cdot\frac{\sqrt{(\alpha_1^2+\beta_1^2)/2}}{\alpha_1+\beta_1}\cdot(b-a)\Big(\int_a^b |x''(t)|^2\,dt\Big)^{1/2} \le \cdots\\ &\le L_0(t)\cdot\prod_{i=1}^{n-1}\frac{\sqrt{(\alpha_i^2+\beta_i^2)/2}}{\alpha_i+\beta_i}(b-a)^{n-1}\Big(\int_a^b |x^{(n)}(t)|^2\,dt\Big)^{1/2}.\end{aligned} \tag{2.4}$$

Using the Cauchy–Schwarz inequality, we have

$$\int_a^b L_0(t)\cdot|x^{(n)}(t)|\,dt \le \frac{\sqrt{(\alpha_0^2+\beta_0^2)/2}}{\alpha_0+\beta_0}(b-a)\Big(\int_a^b |x^{(n)}(t)|^2\,dt\Big)^{1/2}. \tag{2.5}$$

Multiplying (2.4) by $|x^{(n)}(t)|$, and using (2.5), we obtain (2.1). ∎

Letting $\alpha_i = \beta_i = 1$ $(0 \le i \le n-1)$ in Theorem 2.1 yields the following result.

COROLLARY 2.2. *Let $x(\cdot) \in C^{(n)}[a,b]$ be such that $x^{(i)}(a) + x^{(i)}(b) = 0$ for $0 \le i \le n-1$ $(n \ge 1)$. Then*

$$\int_a^b |x(t)x^{(n)}(t)|\,dt \le \Big(\frac{b-a}{2}\Big)^n \int_a^b |x^{(n)}(t)|^2\,dt. \tag{2.6}$$

THEOREM 2.3. *Let $x(\cdot) \in C^{(2n)}[a,b]$ be such that $x^{(2i)}(a) = x^{(2i)}(b) = 0$ for $0 \le i \le n-1$ $(n \ge 1)$. Then*

$$\int_a^b |x(t)x^{(2n)}(t)|\,dt \le \left(\frac{4}{3\sqrt{10}}\right)^n \left(\frac{b-a}{2}\right)^{2n} \int_a^b |x^{(2n)}(t)|^2\,dt. \tag{2.7}$$

*Proof.* In view of the assumptions on $x(t)$, it is easy to check that

$$x^{(2i)}(t) = -\int_a^b T(t,s)x^{(2i+2)}(s)\,ds, \qquad 0 \le i \le n-1, \tag{2.8}$$

where

$$T(t,s) = \begin{cases} \dfrac{(s-a)(b-t)}{b-a}, & a \le s \le t \le b, \\[2mm] \dfrac{(t-a)(b-s)}{b-a}, & a \le t \le s \le b. \end{cases}$$

So by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |x^{(2i)}(t)| &\le \int_a^b |T(t,s)|\,|x^{(2i+2)}(s)|\,ds \\ &\le \Big(\int_a^b |T(t,s)|^2\,ds\Big)^{1/2} \Big(\int_a^b |x^{(2i+2)}(s)|^2\,ds\Big)^{1/2} \\ &= \frac{(b-t)(t-a)}{\sqrt{3(b-a)}} \Big(\int_a^b |x^{(2i+2)}(s)|^2\,ds\Big)^{1/2}. \end{aligned} \tag{2.9}$$

Thus, for $0 \le i \le n-1$, we get

$$\begin{aligned} &\Big(\int_a^b |x^{(2i)}(t)|^2\,dt\Big)^{1/2} \\ &\quad \le \frac{1}{\sqrt{3(b-a)}} \Big(\int_a^b (t-a)^2(b-t)^2\,dt\Big)^{1/2} \Big(\int_a^b |x^{(2i+2)}(t)|^2\,dt\Big)^{1/2} \\ &\quad = \frac{(b-a)^2}{3\sqrt{10}} \Big(\int_a^b |x^{(2i+2)}(t)|^2\,dt\Big)^{1/2}. \end{aligned} \tag{2.10}$$

Using (2.9) and (2.10) inductively, we obtain

$$\begin{aligned} |x(t)| &\le \int_a^b |T(t,s)|\,|x''(s)|\,ds \\ &\le \frac{(b-t)(t-a)}{\sqrt{3(b-a)}} \Big(\int_a^b |x''(s)|^2\,ds\Big)^{1/2} \end{aligned} \tag{2.11}$$

$$\le \frac{(b-t)(t-a)}{\sqrt{3(b-a)}} \cdot \frac{(b-a)^2}{3\sqrt{10}} \Big(\int_a^b |x^{(4)}(t)|^2\,dt\Big)^{1/2} \le \cdots$$
$$\le \frac{(b-t)(t-a)}{\sqrt{3(b-a)}} \cdot \frac{(b-a)^{2n-2}}{(3\sqrt{10})^{n-1}} \cdot \Big(\int_a^b |x^{(2n)}(t)|^2\,dt\Big)^{1/2}.$$

Using the Cauchy–Schwarz inequality, we have

$$\int_a^b \frac{(b-t)(t-a)}{\sqrt{3(b-a)}} |x^{(2n)}(t)|\,dt \le \frac{(b-a)^2}{3\sqrt{10}} \Big(\int_a^b |x^{(2n)}(t)|^2\,dt\Big)^{1/2}. \tag{2.12}$$

Multiplying (2.11) by $|x^{(2n)}(t)|$, and using (2.12), we obtain

$$\int_a^b |x(t)x^{(2n)}(t)|\,dt$$
$$\le \frac{(b-a)^{2n-2}}{(3\sqrt{10})^{n-1}} \Big(\int_a^b |x^{(2n)}(t)|^2\,dt\Big)^{1/2} \int_a^b \frac{(b-t)(t-a)}{\sqrt{3(b-a)}} |x^{(2n)}(t)|\,dt$$
$$\le \frac{(b-a)^{2n}}{(3\sqrt{10})^n} \int_a^b |x^{(2n)}(t)|^2\,dt.$$

The inequality (2.7) follows immediately. ■

THEOREM 2.4. *Let $x(\cdot) \in C^{(2n+1)}[a,b]$ be such that $x(a)+x(b)=0$ and $x^{(2i+1)}(a) = x^{(2i+1)}(b) = 0$ for $0 \le i \le n-1$ ($n \ge 1$). Then*

$$\int_a^b |x(t)x^{(2n+1)}(t)|\,dt \le \Big(\frac{4}{3\sqrt{10}}\Big)^n \Big(\frac{b-a}{2}\Big)^{2n+1} \int_a^b |x^{(2n+1)}(t)|^2\,dt. \tag{2.13}$$

*Proof.* By the assumption $x(a)+x(b)=0$, it is easy to check that

$$x(t) = \int_a^b K(t,s)x'(s)\,ds, \tag{2.14}$$

where

$$K(t,s) = \begin{cases} 1/2, & a \le s \le t \le b, \\ -1/2, & a \le t \le s \le b. \end{cases}$$

So by the Cauchy–Schwarz inequality, we obtain

$$|x(t)| \le \Big(\int_a^b |K(t,s)|^2\,ds\Big)^{1/2} \Big(\int_a^b |x'(s)|^2\,ds\Big)^{1/2} \tag{2.15}$$
$$= \Big(\frac{b-a}{4}\Big)^{1/2} \Big(\int_a^b |x'(s)|^2\,ds\Big)^{1/2}.$$

For $0 \le i \le n-1$, with the help of (2.10), we also have

$$\Big(\int_a^b |x^{(2i+1)}(t)|^2\,dt\Big)^{1/2} \le \frac{(b-a)^2}{3\sqrt{10}}\Big(\int_a^b |x^{(2i+3)}(t)|^2\,dt\Big)^{1/2}. \tag{2.16}$$

Using (2.15) and (2.16) inductively, we obtain

$$\begin{aligned} |x(t)| &\le \Big(\frac{b-a}{4}\Big)^{1/2}\Big(\int_a^b |x'(s)|^2\,ds\Big)^{1/2} \\ &\le \Big(\frac{b-a}{4}\Big)^{1/2}\cdot\frac{(b-a)^2}{3\sqrt{10}}\Big(\int_a^b |x^{(3)}(t)|^2\,dt\Big)^{1/2} \le \cdots \\ &\le \Big(\frac{b-a}{4}\Big)^{1/2}\cdot\frac{(b-a)^{2n}}{(3\sqrt{10})^n}\Big(\int_a^b |x^{(2n+1)}(t)|^2\,dt\Big)^{1/2}. \end{aligned} \tag{2.17}$$

Multiplying (2.17) by $|x^{(2n+1)}(t)|$, and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\int_a^b |x(t)x^{(2n+1)}(t)|\,dt \\ &\quad\le \Big(\frac{b-a}{4}\Big)^{1/2}\cdot\frac{(b-a)^{2n}}{(3\sqrt{10})^n}\Big(\int_a^b |x^{(2n+1)}(t)|^2\,dt\Big)^{1/2}\int_a^b |x^{(2n+1)}(t)|\,dt \\ &\quad\le \frac{(b-a)^{2n+1}}{2(3\sqrt{10})^n}\int_a^b |x^{(2n)}(t)|^2\,dt. \end{aligned}$$

The inequality (2.13) follows immediately. ∎

THEOREM 2.5. *Let $x(\cdot) \in C^{(2n)}[a,b]$ be such that $x^{(2i)}(a) = x^{(2i+1)}(b) = 0$ for $0 \le i \le n-1$ $(n \ge 1)$. Then*

$$\int_a^b |x(t)x^{(2n)}(t)|\,dt \le \Big(\frac{4}{\sqrt{6}}\Big)^n\Big(\frac{b-a}{2}\Big)^{2n}\int_a^b |x^{(2n)}(t)|^2\,dt. \tag{2.18}$$

*Proof.* In view of the assumptions on $x(t)$, it is easy to check that

$$x^{(2i)}(t) = -\int_a^b P(t,s)x^{(2i+2)}(s)\,ds, \qquad 0 \le i \le n-1, \tag{2.19}$$

where

$$P(t,s) = \begin{cases} s-a, & a \le s \le t \le b, \\ t-a, & a \le t \le s \le b. \end{cases}$$

Denote

$$Q(t)=\Big(\int_a^b |P(t,s)|^2\,ds\Big)^{1/2}=\frac{1}{\sqrt{3}}\sqrt{(t-a)^3+3(t-a)^2(b-t)}.$$

By the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}(2.20)\qquad |x^{(2i)}(t)| &\le \int_a^b |P(t,s)|\,|x^{(2i+2)}(s)|\,ds\\ &\le \Big(\int_a^b |P(t,s)|^2\,ds\Big)^{1/2}\Big(\int_a^b |x^{(2i+2)}(s)|^2\,ds\Big)^{1/2}\\ &= Q(t)\cdot\Big(\int_a^b |x^{(2i+2)}(s)|^2\,ds\Big)^{1/2}.\end{aligned}$$

Thus, for $0\le i\le n-1$, we get

$$\begin{aligned}(2.21)\qquad \Big(\int_a^b |x^{(2i)}(t)|^2\,dt\Big)^{1/2} &\le \Big(\int_a^b Q^2(t)\,dt\Big)^{1/2}\Big(\int_a^b |x^{(2i+2)}(t)|^2\,dt\Big)^{1/2}\\ &= \frac{(b-a)^2}{\sqrt{6}}\Big(\int_a^b |x^{(2i+2)}(t)|^2\,dt\Big)^{1/2}.\end{aligned}$$

Using (2.20) and (2.21) inductively, we obtain

$$\begin{aligned}(2.22)\qquad |x(t)| &\le \int_a^b |P(t,s)|\,|x''(s)|\,ds\\ &\le Q(t)\cdot\Big(\int_a^b |x''(s)|^2\,ds\Big)^{1/2}\\ &\le Q(t)\cdot\frac{(b-a)^2}{\sqrt{6}}\Big(\int_a^b |x^{(4)}(t)|^2\,dt\Big)^{1/2}\le\cdots\\ &\le Q(t)\cdot\frac{(b-a)^{2n-2}}{(\sqrt{6})^{n-1}}\Big(\int_a^b |x^{(2n)}(t)|^2\,dt\Big)^{1/2}.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we find that

$$\begin{aligned}(2.23)\qquad \int_a^b Q(t)|x^{(2n)}(t)|\,dt &\le \Big(\int_a^b Q^2(t)\,dt\Big)^{1/2}\Big(\int_a^b |x^{(2n)}(t)|^2\,dt\Big)^{1/2}\\ &= \frac{(b-a)^2}{\sqrt{6}}\Big(\int_a^b |x^{(2n)}(t)|^2\,dt\Big)^{1/2}.\end{aligned}$$

Multiplying (2.22) by $|x^{(2n)}(t)|$, and using (2.23), we conclude that

$$\int_a^b |x(t)x^{(2n)}(t)|\,dt \le \frac{(b-a)^{2n}}{(\sqrt{6})^n}\int_a^b |x^{(2n)}(t)|^2\,dt.$$

The inequality (2.18) follows immediately. ∎

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