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Finiteness problem for meromorphic mappings sharing $n+3$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$

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Abstract. We prove some finiteness theorems for differential nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which share $n+3$ hyperplanes.

1. Introduction. Using the Second Main Theorem of Value Distribution Theory and Borel's lemma, R. Nevanlinna [N] proved that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values then $f \equiv g$, and that g is a special type of linear fractional transformation of f if they have the same inverse images, counted with multiplicities, for four distinct values.

In 1981, Drouilhet considered the results of Nevanlinna for higher dimensions and differential nondegenerate meromorphic mappings. He proved the following uniqueness theorem.

Theorem 1.1 ([D, Theorem 4.2]). *Let $f, g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be differential nondegenerate meromorphic maps with $m \geq n$. Let A be a hypersurface of degree at least $n+4$ in $\mathbb{P}^n(\mathbb{C})$ having normal crossings. Suppose $f^{-1}(A) = g^{-1}(A)$ as point sets and f and g agree at all points of $f^{-1}(A)$ lying in their common domain of determinacy. Suppose either $M = \mathbb{C}^m$ or f and g are transcendental. Then $f = g$.*

Then a question arises naturally: *What about the case where the degree of A is $n+3$?*

We emphasize that for the case of linearly nondegenerate meromorphic mappings, in the best results available at present, given by Chen–Yan [CY] and Quang [Q], the authors just considered the case where the hypersurface A is a union of $2n+3$ hyperplanes in general position. Also their techniques of proof do not work for less than $2n+3$ hyperplanes.

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The purpose of this paper is to give a positive answer to the above question in a particular case where the hypersurface A is a union of $n+3$ hyperplanes.

Let f be a differential nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(m \geq n)$ and let $H_1, \dots, H_q$ be q hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Let d be a positive integer. We denote by $\mathcal{G}(f, \{H_i\}_{i=1}^q, d)$ the set of all differential nondegenerate meromorphic mappings g of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which satisfy the following two conditions:

(i) $\min\{\nu^0_{(f,H_i)}(z), d\} = \min\{\nu^0_{(g,H_i)}(z), d\}$ for all $1 \leq i \leq q$, $z \in \mathbb{C}^m$,

(ii) $f = g$ on $\bigcup_{i=1}^q f^{-1}(H_i)$.

We will prove the following.

Theorem 1.2. *Let f be a differential nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(m \geq n)$ and let $H_1, \dots, H_{n+3}$ be $n+3$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Then the set $\mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 2)$ contains at most two elements.*

Theorem 1.3. *Let f and $H_1, \dots, H_{n+3}$ be as in Theorem 1.2. Assume that*

$$\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2 \quad \textit{for all } 1 \leq i < j \leq n+3.$$

If $n \geq 2$, then the set $\mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$ contains at most two elements.

Theorem 1.4. *Let f and $H_1, \dots, H_{n+3}$ be as in Theorem 1.2. Let f_1, f_2, f_3 be in $\mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$. Assume that $\dim f^{-1}(H_1 \cap \bigcup_{i=2}^{n+3} H_i) \leq m-2$ and $\min\{\nu_{(f_s,H_1)}(z), 2\} = \min\{\nu_{(f_t,H_1)}(z), 2\}$ for all $1 \leq s,\ t \leq 3$ and $z \in f^{-1}(H_1)$. Then $f_1 = f_2$ or $f_2 = f_3$ or $f_3 = f_1$.*

2. Preliminaries

(a) For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, we set $\|z\| = (\sum_{j=1}^m |z_j|^2)^{1/2}$ and define

$$B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}, \quad \Gamma(r) = \{z \in \mathbb{C}^m : \|z\| = r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \sigma = (dd^c\|z\|^2)^{m-1},$$

$$\eta = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|)^{m-1}.$$

Denote by $\mathrm{Mer}(\mathbb{C}^m)$ the set of all meromorphic functions on $\mathbb{C}^m$. A divisor E on $\mathbb{C}^m$ is given by a formal sum $E = \sum \mu_\nu X_\nu$, with $\{X_\nu\}$ is a locally family of distinct irreducible analytic hypersurfaces in $\mathbb{C}^m$ and $\mu_\nu \in \mathbb{Z}$. We define the *support* of E by $\mathrm{Supp}(E) = \bigcup_{\nu \neq 0} X_\nu$. Sometimes we identify the divisor E with the function $E(z)$ from $\mathbb{C}^m$ into $\mathbb{Z}$ defined by $E(z) := \sum_{X_\nu \ni z} \mu_\nu$.

Let k be a positive integer or $+\infty$. We define the divisor $E^{[k]}_{>t}$ by

$$E^{[k]}_{>t} := \sum_{\mu_\nu > t} \min\{\mu_\nu, k\} X_\nu.$$

and the *truncated counting function to level k of E* by

$$N^{[k]}_{>t}(r, E) := \int_1^r \frac{n^{[k]}_{>t}(t, E)}{t^{2m-1}}\, dt \qquad (1 < r < \infty),$$

where

$$n^{[k]}_{>t}(t, E) := \begin{cases} \displaystyle\int_{\mathrm{Supp}(E)\cap B(t)} E^{[k]}_{>t} \cdot \sigma & \text{if } m \geq 2, \\ \displaystyle\sum_{|z|\leq t} E^{[k]}_{>t}(z) & \text{if } m = 2. \end{cases}$$

We omit $^{[k]}$ (resp. $_{>t}$) if $k = +\infty$ (resp. $t = 0$).

An analytic hypersurface E of $\mathbb{C}^m$ may be considered as a reduced divisor; we then denote by $N(r, E)$ its counting function.

For two divisors E_1, E_2, we define the divisor $\min\{E_1, E_2\}$ by setting

$$\min\{E_1, E_2\}(z) = \min\{E_1(z), E_2(z)\}.$$

(b) Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\mathcal{D}^\alpha F = \partial^{|\alpha|} F / \partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m$. We define the *zero divisor* of F as follows:

$$\nu^0_F(a) = \max\{p : \mathcal{D}^\alpha F(a) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\}.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = F/G$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m - 2$ and we define the *zero* (*resp. pole*) *divisor* of φ by $\nu^0_\varphi(a) = \nu^0_F(a)$ (resp. $\nu^\infty_\varphi(a) = \nu^0_G(a)$) and $\nu_\varphi(a) = \nu^0_\varphi(a) - \nu^\infty_\varphi(a)$.

We have the following *Jensen formula*:

$$N(r, \nu^0_\varphi) - N(r, \nu^\infty_\varphi) = \int_{\Gamma(r)} \log|\varphi|\eta - \int_{\Gamma(1)} \log|\varphi|\eta. \tag{2.1}$$

For convenience, we will write $N_\varphi(r)$ and $N^{[k]}_{\varphi,>t}(r)$ for $N(r, \nu^0_\varphi)$ and $N^{[k]}_{>t}(r, \nu^0_\varphi)$ respectively.

We denote by $\mathcal{M}_{\mathbb{C}^m}$ the field of all meromorphic functions on $\mathbb{C}^m$.

(c) Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, $(m \geq n)$. We say that f is *differential nondegenerate* if *df* has maximal rank. For any homogeneous coordinates $(w_0 : \cdots : w_n)$ of $\mathbb{P}^n(\mathbb{C})$, we take a *reduced representation* $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $I(f) := \{z : f_0(z) = \cdots = f_n(z) = 0\}$ of codimension ≥ 2.

Denote by Ω the Fubini–Study form of $\mathbb{P}^n(\mathbb{C})$. The *characteristic function* of f (with respect to Ω) is defined by

$$T_f(r) := \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} f^*\Omega \wedge \sigma, \qquad 1 < r < \infty.$$

By Jensen's formula we have

$$T_f(r) = \int_{\Gamma(r)} \log \|f\| \, \eta + O(1), \tag{2.2}$$

where $\|f\| = \max\{|f_0|, \ldots, |f_n|\}$.

(d) For a meromorphic function φ on $\mathbb{C}^m$, the *proximity function* $m(r, \varphi)$ is defined by

$$m(r,\varphi) = \int_{\Gamma(r)} \log^+ |\varphi| \, \eta,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$. The *Nevanlinna characteristic function* is defined by

$$T(r,\varphi) = N(r, \nu_\varphi^\infty) + m(r,\varphi).$$

If we regard φ as a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^1(\mathbb{C})$, then

$$T_\varphi(r) = T(r,\varphi) + O(1).$$

(e) Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ given by $H = \{a_0\omega_0 + \cdots + a_n\omega_n = 0\}$, where $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$. We set $(f, H) = \sum_{i=0}^n a_i f_i$. We define the *proximity function* of f *with respect to* H by

$$m_f(r,H) = \int_{\Gamma(r)} \log \frac{\|f\| \cdot \|H\|}{|(f,H)|} \eta - \int_{\Gamma(1)} \log \frac{\|f\| \cdot \|H\|}{|(f,H)|} \eta,$$

where $\|H\| = (\sum_{i=0}^n |a_i|^2)^{1/2}$.

Theorem 2.1 (The first main theorem). *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping and H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Assume that $f(\mathbb{C}^m) \not\subset H$. Then*

$$T_f(r) = N(r, \nu^0_{(f,H)}) + m_f(r,H) + O(1) \qquad (r > 1). \tag{2.3}$$

Theorem 2.2 (Lemma on logarithmic derivative). *Let f be a nonzero meromorphic function on $\mathbb{C}^m$. Then*

$$\Big\| \quad m\bigg(r, \frac{\mathcal{D}^\alpha(f)}{f}\bigg) = O(\log^+ T_f(r)) \qquad (\alpha \in \mathbb{Z}^m_+). \tag{2.4}$$

As usual, "$\|$ P" means the assertion P holds for all $r \in (1,\infty)$ off a finite Lebesgue measure subset of $(1, \infty)$.

Let $\{H_i\}_{i=1}^q$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$. They are said to be *in general position* if $\bigcap_{j=0}^n H_{i_j} = \emptyset$ for any $1 \leq i_0 < \cdots < i_n \leq q$.

We now state the known result on the Second Main Theorem for differential nondegenerate meromorphic mappings.

Theorem 2.3 (Carlson–Griffiths [CG], Shiffman [Sh], Noguchi [Ng], Drouilhet [D]). *Let f be a differential nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(m \geq n)$. Let $\{H_i\}_{i=1}^q$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then*

$$\| \ (q-n-1)T_f(r) \leq N^{[1]}(r, f^*A) + o(T_f(r)),$$

where A is the divisor $\sum_{i=1}^q H_i$.

3. Some lemmas. Let f be a differential nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ $(m \geq n)$ and let $H_1, \dots, H_q$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. For each $g \in \mathcal{G}(f, \{H_i\}_{i=1}^q, 1)$, it is easy to see that if $q \geq n+2$ then

$$I(g) = \bigcap_{i=1}^{q} g^{-1}(H_i) = \bigcap_{i=1}^{q} f^{-1}(H_i) = I(f).$$

Let $f_1, f_2, f_3 \in \mathcal{G}(f, \{H_i\}_{i=1}^q, 1)$. Assume that each f_k has a reduced representation

$$f_k := (f_{k0} : \cdots : f_{kn}) \quad (1 \leq k \leq 3).$$

We now introduce some notations which will be used throughout this paper.

We denote by A_{ij}, B_{ij} the hypersurfaces in $\mathbb{C}^m$ defined by

$$A_{ij} = \bigcup \{\alpha\text{: irreducible hypersurface} \subset f^{-1}(H_i) \cap f^{-1}(H_j)\},$$
$$B_{ij} = \bigcup \{\alpha\text{: irreducible hypersurface} \subset \overline{f^{-1}(H_i) \setminus f^{-1}(H_j)}\}.$$

We set $T(r) := \sum_{k=1}^3 T_{f_k}(r)$.

For each $c = (c_0, \dots, c_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, we denote by H_c the hyperplane $\{c_0\omega_0 + \cdots + c_n\omega_n = 0\}$ and put

$$(f_k, H_c) := \sum_{i=0}^{n} c_i f_{ki} \quad (1 \leq k \leq 3).$$

For $i \in \{1, \dots, q\}$, let

$$V_i = ((f_1, H_i), (f_2, H_i), (f_3, H_i)) \in \mathcal{M}^3_{\mathbb{C}^m}.$$

We write

- $V_i \approx V_j$ if $\dfrac{(f_1, H_i)}{(f_1, H_j)} = \dfrac{(f_2, H_i)}{(f_2, H_j)} = \dfrac{(f_3, H_i)}{(f_3, H_j)}$,

  otherwise we write $V_i \not\approx V_j$,

- $V_i \sim V_j$ if there exists a permutation $\{k,t,s\}$ of $\{1,2,3\}$ so that $\dfrac{(f_k,H_i)}{(f_k,H_j)} = \dfrac{(f_t,H_i)}{(f_t,H_j)} \neq \dfrac{(f_s,H_i)}{(f_s,H_j)}$,
- $V_i \not\sim V_j$ if $\dfrac{(f_1,H_i)}{(f_1,H_j)} \neq \dfrac{(f_2,H_i)}{(f_2,H_j)} \neq \dfrac{(f_3,H_i)}{(f_3,H_j)} \neq \dfrac{(f_1,H_i)}{(f_1,H_j)}$.

We decompose the set of indices $\{1,\dots,q\}$ into disjoint sets as follows:

$$\text{(3.1)}\qquad \begin{aligned} &\text{(i) } \{1,\dots,q\} = I_1 \cup \dots \cup I_k,\\ &\text{(ii) } V_i \approx V_j \text{ for all } i,j \in I_t\ (1 \le t \le k),\\ &\text{(iii) } V_i \not\approx V_j \text{ for all } i \in I_t,\ j \in I_s\ (1 \le t < s \le k). \end{aligned}$$

We set $I(\{H_i\}_{i=1}^q; f_1,f_2,f_3) = k$, the number of sets in the above partition of $\{1,\dots,q\}$.

LEMMA 3.1. *If $q \ge n+2$ and $I(f_1,f_2,f_3) \le 2$ then $f_1 = f_2 = f_3$.*

Proof. If $I(f_1,f_2,f_3) = 1$, then the conclusion is clear. Now suppose that $I(f_1,f_2,f_3) = 2$ and $f_1 \neq f_2$.

Let I_1, I_2 be two disjoint sets in the partition of $\{1,\dots,q\}$ as in (3.1). By changing the indices if necessary, we may assume that $I_1 = \{1,\dots,l\}$ and $I_2 = \{l+1,\dots,q\}$, where $l \le q-1$. If $\sharp I_1 = l \ge n+1$ then $f_1 = f_2$, contrary to assumption. Therefore $l \le n$.

Without loss of generality, we may assume that the hyperplanes H_i ($1 \le i \le n+2$) are given by $H_i = \{\omega_{i-1} = 0\}$ ($1 \le i \le n+1$) and $H_{n+2} = \{-\omega_0 - \dots - \omega_n = 0\}$. Then

$$\text{(3.2)}\qquad \sum_{i=1}^{n+2} (f_s,H_i) = 0 \qquad (s = 1,2).$$

We set

$$h = \frac{(f_1,H_1)}{(f_2,H_1)} \quad \text{and} \quad g = \frac{(f_1,H_{l+1})}{(f_2,H_{l+1})}.$$

Since $f_1 \neq f_2$, we have $h \neq g$. From (3.2), it follows that

$$h(f_2,H_1) + \dots + h(f_2,H_l) + g(f_2,H_{l+1}) + \dots + g(f_2,H_{n+2}) = 0.$$

Thus

$$(h-g)((f_2,H_1) + \dots + (f_2,H_l)) = 0,$$

and so

$$f_{20} + f_{21} + \dots + f_{2(l-1)} = 0.$$

This contradicts the differential nondegeneracy of f_2. Hence $f_1 = f_2$.

Similarly, we have $f_1 = f_2 = f_3$. ■

Denote by $\mathcal{C}$ the set of all $c \in \mathbb{C}^{n+1} \setminus \{0\}$ such that

(i) $\dim\{z \in \mathbb{C}^m : (f_k, H_i)(z) = (f_k, H_c)(z) = 0\} \le m-2$ $(1 \le i \le q,$ $1 \le k \le 3)$,
(ii) $\{H_1, \dots, H_q, H_c\}$ are in general position.

Lemma 3.2. *$\mathcal{C}$ is dense in $\mathbb{C}^{n+1}$.*

Proof. Denote by $\mathcal{C}_1$ the set of all $c \in \mathbb{C}^{n+1} \setminus \{0\}$ which satisfy (i). Then, by [J, Lemma 5.1], $\mathcal{C}_1$ is dense in $\mathbb{C}^{n+1}$.

For $I = \{i_1, \dots, i_n\} \subset \{1, \dots, q\}$ with $\sharp I = n$, define a holomorphic function T_I on $\mathbb{C}^{n+1}$ by setting $T_I(c) := \det(a_{i_j k}, c_k)$, where $1 \le j \le n$, $0 \le k \le n$. It is easy to see that $T_I \not\equiv 0$. Thus, $S = \bigcap_I T_I^{-1}\{0\}$ is an analytic set of codimension one in $\mathbb{C}^{n+1}$.

Therefore, $\mathcal{C} = \mathcal{C}_1 \setminus S$ is dense in $\mathbb{C}^{n+1}$. ∎

Lemma 3.3 (see [Fu98]). *For each $c \in \mathbb{C}^n \setminus \{0\}$, set $F_c^{is} = (f_s, H_i)/(f_s, c)$ (we will write F_j^{is} for F_c^{is} if $H_c = H_j$). Then $T_{F_c^{is}}(r) \le T_{f_s}(r) + o(T(r))$.*

Lemma 3.4. *Let $q = n+3$ and let f_1, f_2 be as above. Suppose that $f_1 \ne f_2$. Then:*

(i) $\| \ N^{[1]}_{(f,H_i)}(r) = N(r, \min\{\nu^0_{(f_1,H_i)}, \nu^0_{(f_2,H_i)}\}) + o(T(r))$ $(1 \le i \le n+3)$,
(ii) $\| \ T_{f_1}(r) = T_{f_2}(r) + o(T(r)) = \frac{1}{2} N^{[1]}(r, f^*A) + o(T(r))$, *where* $A = H_1 + \dots + H_q$,
(iii) *if* $\| \ \frac{(f_1,H_i)}{(f_1,H_j)} = \frac{(f_2,H_i)}{(f_2,H_j)}$ *then* $\| \ N^{[1]}(r, B_{ij}) = N^{[1]}(r, B_{ji}) + o(T(r)) = o(T(r))$,
(iv) *if* $\| \ \frac{(f_1,H_i)}{(f_1,H_j)} \ne \frac{(f_2,H_i)}{(f_2,H_j)}$ *then* $\| \ N^{[1]}(r, A_{ij}) = o(T(r))$.

Proof. (i)–(iii) Fix $i \in \{1, \dots, n+3\}$. Since $f \ne g$, there exists $c = (c_0, \dots, c_n) \in \mathcal{C}$ such that

$$P_{ic} := (f_1, H_i)(f_2, H_c) - (f_1, H_c)(f_2, H_i) \ne 0.$$

For $z \in \mathbb{C}^m \setminus I(f)$, it is easy to see that

- if $z \in \operatorname{Supp} f^*A \setminus f^{-1}(H_i)$ then $\nu^0_{P_{ic}}(z) \ge 1$, since $f_1(z) = f_2(z)$,
- if $z \in f^{-1}(H_i)$ then $\nu^0_{P_{ic}}(z) \ge \min\{\nu^0_{(f_1,H_i)}(z), \nu^0_{(f_2,H_i)}(z)\}$.

This yields

$$\nu^0_{P_{ic}}(z) \ge \min\{\nu^0_{(f_1,H_i)}(z), \nu^0_{(f_2,H_i)}(z)\} + \min\{f^*A(z), 1\} - \min\{\nu^0_{(f,H_i)}(z), 1\}.$$

Integrating, we obtain

$$N_{P_{ic}}(z) \ge N(r, \min\{\nu^0_{(f_1,H_i)}, \nu^0_{(f_2,H_i)}\}) + N^{[1]}(r, f^*A) - N^{[1]}_{(f,H_i)}(r). \tag{3.3}$$

On the other hand, by Jensen's formula and the definition of the characteristic function,

$$
\begin{aligned}
(3.4)\qquad N_{P_{ic}}(r) &= \int\limits_{\Gamma(r)} \log|P_{ic}|\,\eta + O(1)\\
&\le \int\limits_{\Gamma(r)} \big(\log(|(f_1,c)|^2+|(f_1,H_i)|^2)^{1/2}\\
&\qquad\qquad + \log(|(f_2,c)|^2+|(f_2,H_i)|^2)^{1/2}\big)\eta + O(1)\\
&\le T_{f_1}(r) + T_{f_2}(r) + o(T(r)).
\end{aligned}
$$

By the Second Main Theorem, we also have

$$
(3.5)\qquad \| \ T_{f_s}(r) \le \tfrac{1}{2} N^{[1]}(r, f^*A) + o(T(r)) \quad (1\le s\le 2).
$$

Combining (3.3)–(3.5) with $N(r,\min\{\nu^0_{(f_1,H_i)},\nu^0_{(f_2,H_i)}\}) \ge N^{[1]}_{(f,H_i)}(r)$, we obtain

$$
(3.6)\qquad
\begin{aligned}
&\| \ N(r,\min\{\nu^0_{(f_1,H_i)},\nu^0_{(f_2,H_i)}\}) = N^{[1]}_{(f,H_i)}(r) + o(T(r)),\\
&\| \ T_{f_s}(r) = \tfrac{1}{2} N_{P_{ic}}(r) + o(T(r)) = \tfrac{1}{2} N^{[1]}(r, f^*A) \quad (1\le s\le 2).
\end{aligned}
$$

Thus (i) and (ii) are proved.

If there is an index j such that $\| \ \frac{(f_1,H_i)}{(f_1,H_j)} = \frac{(f_2,H_i)}{(f_2,H_j)}$, then we see that:

- If $z \in \operatorname{Supp} f^*A \setminus B_{ij}$ then $\nu^0_{P_{ic}}(z) \ge 1$, since $f(z)=g(z)$.
- If $z \in B_{ij} \setminus f^{-1}(H_j)$, we rewrite P_{ic} as follows:

$$
P_{ic} = \frac{(f_1,H_i)}{(f_1,H_j)}\big((f_1,H_j)(f_2,H_c) - (f_1,H_c)(f_2,H_j)\big).
$$

  Then $\nu^0_{P_{ic}}(z) \ge 2$.

This yields

$$
\nu^0_{P_{ic}}(z) \ge (\min\{f^*A(z),1\} - B_{ij}(z)) + 2B_{ij}(z) = \min\{f^*A(z),1\} + B_{ij}(z).
$$

Integrating, we get

$$
N_{P_{ic}}(z) \ge N^{[1]}(r, f^*A) + N^{[1]}(r, B_{ij}).
$$

Combining this with (3.6), we have $\| \ N^{[1]}(r,B_{ij}) = o(T(r))$.

Similarly, $\| \ N^{[1]}(r,B_{ij}) = N^{[1]}(r,B_{ji}) + o(T(r)) = o(T(r))$, proving (iii).

(iv) Suppose that $\| \ \frac{(f_1,H_i)}{(f_1,H_j)} \ne \frac{(f_2,H_i)}{(f_2,H_j)}$. We consider the holomorphic function

$$
P_{ij} := (f_1,H_i)(f_2,H_j) - (f_1,H_j)(f_2,H_i) \ne 0.
$$

Similarly to the argument in (i)–(iii), we see that:

- If $z \in \operatorname{Supp} f^*A \setminus A_{ij}$ then $\nu^0_{P_{ic}}(z) \ge 1$, since $f(z)=g(z)$.
- If $z \in A_{ij}$, then $\nu^0_{P_{ic}}(z) \ge \min\{\nu^0_{(f,H_i)}(z),1\} + \min\{\nu^0_{(f,H_j)}(z),1\} = 2$.

Hence $\nu^0_{P_{ic}}(z) \geq (\min\{f^*A(z),1\} - A_{ij}(z)) + 2A_{ij}(z) = \min\{f^*A(z),1\} + A_{ij}(z)$. It follows that

$$N_{P_{ij}}(r) \geq N^{[1]}(r, f^*A) + N^{[1]}(r, A_{ij}).$$

Combining this with (3.6), we have $\| \ N^{[1]}(r, A_{ij}) = o(T(r))$, and (iv) is proved. ■

REMARK 3.5. 1) Lemma 3.4 is also valid for any distinct maps h, g in $\mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$.

2) If $q = n+3$ and f_1, f_2, f_3 are distinct maps in $\mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$, then:

(a) For $i, j \in \{1, \dots, n+3\}$ with $V_i \sim V_j$, we have $\| \ N(r, A_{ij}) = o(T(r))$ and $\| \ N(r, B_{ij}) = N(r, B_{ji}) + o(T(r)) = o(T(r))$. From Lemma 3.4(iii)–(iv), it follows that

$$\| \ N^{[1]}_{(f,H_i)}(r) = N(r, A_{ij}) + N(r, B_{ij}) = o(T(r)),$$
$$\| \ N^{[1]}_{(f,H_j)}(r) = N(r, A_{ij}) + N(r, B_{ji}) = o(T(r)).$$

(b) Take a partition $I_1 \cup \dots \cup I_k$ as in (3.1). Denote by A_i the the hypersurface defined by

$$A_i = \bigcup \Big\{\alpha\text{: irreducible hypersurface} \subset \overline{\bigcap_{j\in I_i} f^{-1}(H_j) \setminus \bigcup_{j\notin I_i} f^{-1}(H_j)}\Big\}.$$

From the above remark, we see that for any irreducible hypersurface $\alpha \subset f^{-1}(H_j)$ in $\mathbb{C}^m$ such that $\alpha \not\subset f^{-1}(H_{j'})$ whenever $V_{j'} \approx V_j$ or $\alpha \subset f^{-1}(H_{j''})$ whenever $V_{j''} \not\approx V_j$ we have $\| \ N(r, \alpha) = o(T(r))$. Therefore,

- $\dim(A_{i_1} \cap A_{i_2}) \leq m-2$ for all $\leq i_1 < i_2 \leq k$,
- $A_i \subset f^{-1}(H_j)$ and $\| \ N^{[1]}_{(f,H_j)}(r) = N(r, A_i) + o(T(r))$ for all $j \in I_i$, $1 \leq i \leq k$,
- $\| \ T_{f_s}(r) = \frac{1}{2}\sum_{i=1}^k N(r, A_i) + o(T(r))$.

DEFINITION 3.6 (see [Fu98]). Let F_0, F_1, F_2 be meromorphic functions on $\mathbb{C}^m$. Write $\alpha := (\alpha^1, \dots, \alpha^m)$ where α^k are nonnegative integers, and set $|\alpha| = |\alpha^1| + \dots + |\alpha^m|$. We define *Cartan's auxiliary function* by

$$\Phi^\alpha(F_0, F_1, F_2) := F_0F_1F_2 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1/F_0 & 1/F_1 & 1/F_2 \\ \mathcal{D}^\alpha(1/F_0) & \mathcal{D}^\alpha(1/F_1) & \mathcal{D}^\alpha(1/F_2) \end{vmatrix}.$$

LEMMA 3.7 ([Fu98, Proposition 3.4]). *If $\Phi^\alpha(F, G, H) = 0$ and $\Phi^\alpha(1/F, 1/G, 1/H) = 0$ for all α with $|\alpha| \leq 1$, then one of the following assertions holds:*

(i) *$F = G$, $G = H$ or $H = F$.*

(ii) *There exist $\alpha, \beta \notin \{0, 1\}$ with $\alpha \neq \beta$ such that $F = \alpha G = \beta H$.*

For $f_1, f_2, f_3 \in \mathcal{G}(f, \{H_i\}_{i=1}^q, 1)$ and each j $(1 \leq j \leq q)$, we define a divisor D_j by

$$D_j(z) = \begin{cases} \nu^0_{(f_1,H_j)}(z) & \text{if } \nu^0_{(f_1,H_j)}(z) = \nu^0_{(f_2,H_j)}(z) = \nu^0_{(f_3,H_j)}(z), \\ 0 & \text{otherwise.} \end{cases}$$

We now prove the following lemma, which is an improvement of the lemma on Cartan's auxiliary function of Fujimoto [Fu98].

LEMMA 3.8. *Let $f_1, f_2, f_3 \in \mathcal{G}(f, \{H_i\}_{i=1}^q, 1)$. Assume that $\Phi^\alpha := \Phi^\alpha(F_c^{i_0 1}, F_c^{i_0 2}, F_c^{i_0 3}) \neq 0$ for some $c \in \mathcal{C}$ and α with $|\alpha| = 1$. Then:*

(i) $\| \ N(r, D_{i_0}) + 2(N^{[1]}(r, f^*A) - N^{[1]}_{(f,H_{i_0})}(r)) \leq N_{\Phi^\alpha}(r) \leq T(r) + o(T(r))$, *where A is the divisor $\sum_{i=1}^q H_i$ on $\mathbb{P}^n(\mathbb{C})$.*

(ii) *If $q = n+3$ then* $\| \ 2N^{[1]}_{(f,H_{i_0})}(r) \geq N(r, D_{i_0}) + T_{f_s}(r) + o(T(r))$ $(1 \leq s \leq 3)$.

Proof. Set $I := I(f) \cup \bigcup_{1 \leq i \leq n+3} (f^{-1}(H_i) \cap f^{-1}(H_c))$. Then I is either an analytic subset of codimension at least two in $\mathbb{C}^m$ or an empty set.

Assume that a is a zero of some (f, H_i), $i \neq i_0$, such that $a \notin I$ and $a \notin (f, H_{i_0})^{-1}\{0\}$. Let Γ be an irreducible component of the zero divisor of the function (f, H_i) which contains a. We take a holomorphic function h on $\mathbb{C}^m$ satisfying $\nu^0_h = \nu(L)$, where $\nu(L)$ denotes the reduced divisor with support L.

The function

$$\varphi_s := \frac{1}{hF_c^{i_0 s}} - \frac{1}{hF_c^{j_0 3}}$$

is holomorphic on a neighbourhood U of a for all $1 \leq s \leq 2$. Since $|\alpha| = 1$, we have

$$\Phi^\alpha := h^2 F_c^{i_0 1} F_c^{i_0 2} F_c^{i_0 3} \cdot \begin{vmatrix} \varphi_1 & \varphi_2 \\ \mathcal{D}^\alpha \varphi_1 & \mathcal{D}^\alpha \varphi_2 \end{vmatrix}.$$

This implies that

$$\nu^0_{\Phi^\alpha}(a) \geq 2. \tag{3.7}$$

Assume that b is a zero of (f, H_{i_0}) such that $b \notin I$. We write

$$\Phi^\alpha = \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) F_c^{i_0 1} F_c^{i_0 2} F_c^{i_0 3} \cdot \frac{1}{F_c^{i_0 \sigma(2)}} \cdot \mathcal{D}^\alpha \bigg(\frac{1}{F_c^{i_0 \sigma(3)}} \bigg).$$

This implies that

$$\begin{aligned} \nu_{\Phi^\alpha}(b) &\geq \min_{\sigma \in S_3} \Big(\sum_{s=1}^3 \nu_{F_c^{i_0 s}}(b) - \nu^0_{F_c^{i_0 \sigma(2)}}(b) - \nu^0_{F_c^{i_0 \sigma(3)}}(b) - 1 \Big) \\ &= \min_{\sigma \in S_3} (\nu^0_{F_c^{i_0 \sigma(1)}}(b) - 1) \geq 0. \end{aligned} \tag{3.8}$$

If $b \in \operatorname{Supp} D_{i_0}$ then $\nu_{(f_1,H_{i_0})}(b) = \nu_{(f_2,H_{i_0})}(b) = \nu_{(f_3,H_{i_0})}(b) = D_{i_0}(b)$. There exists a holomorphic function h on an open neighbourhood U of b such that $\nu_h = D_{i_0|_U}$. We write

$$\Phi^\alpha = h^{-2}F_c^{i_01}F_c^{i_02}F_c^{i_03} \cdot \begin{vmatrix} h/F_c^{i_01} - h/F_c^{i_03} & h/F_c^{i_02} - h/F_c^{i_03} \\ \mathcal{D}^\alpha(h/F_c^{i_01}) - D^\alpha(h/F_c^{i_03}) & \mathcal{D}^\alpha(h/F_c^{i_02}) - D^\alpha(h/F_c^{i_03}) \end{vmatrix}.$$

Then

$$\nu_{\Phi^\alpha}(b) \geq \nu_h(b) = D_{i_0}(b). \tag{3.9}$$

From (3.7)–(3.9), we have

$$D_{i_0}(z) + 2\big(\min\{f^*(A)(z), 1\} - \min\{\nu^0_{(f,H_{i_0})}(z)\}\big) \leq \nu^0_{\Phi^\alpha}(z)$$

for all z outside an analytic subset of codimension at least two. This immediately implies the first inequality of (i).

It is easy to see that a pole of Φ^α is either a zero or a pole of some $F_c^{i_0s}$. By (3.7)–(3.9) we see that Φ^α is holomorphic at all zeros of $F_c^{i_0s}$ $(1 \leq s \leq 3)$. Then

$$N_{1/\Phi^\alpha}(r) \leq \sum_{s=1}^{3} N_{1/F_c^{i_0s}}(r).$$

On the other hand, it is easy to see that

$$\begin{aligned} m(r,\Phi^\alpha) &\leq \sum_{s=1}^{3} m(r, F_c^{i_0s}) + O\bigg(\sum m\bigg(r, \frac{\mathcal{D}^\alpha(\varphi_c^{i_0s})}{\varphi_c^{i_0s}}\bigg)\bigg) + O(1) \\ &\leq \sum_{s=1}^{3} m(r, F_c^{i_0s}) + o(T(r)), \end{aligned}$$

where $\varphi_c^{i_0s} = 1/F_c^{i_0s}$. Hence,

$$\begin{aligned} N_{\Phi^\alpha}(r) \leq T_{\Phi^\alpha}(r) + O(1) &\leq m(r,\Phi^\alpha) + N_{1/\Phi^\alpha}(r) + O(1) \\ &\leq \sum_{s=1}^{3}(N_{1/F_c^{i_0s}}(r) + m(r, F_c^{i_0s})) + o(T(r)) \\ &= \sum_{s=1}^{3} T_{F_c^{i_0s}}(r) + o(T(r)) \leq T(r) + o(T(r)), \end{aligned}$$

proving the second inequality of (i).

Finally, the second assertion of the lemma immediately follows from the first assertion and Lemma 3.4(ii). ∎

From now on, we will denote by $Q(\{H_i\}_{i=1}^q; f_1, f_2, f_3)$ the set of all indices $j \in \{1,\ldots,q\}$ such that $\Phi^\alpha(F_c^{j1}, F_c^{j2}, F_c^{j3}) = 0$ for all $c \in \mathcal{C}$ and α with $|\alpha| = 1$.

Lemma 3.9. *Let $f_1, f_2, f_3 \in \mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$. Then there do not exist $i_0, j_0 \in \{1, \dots, n+3\}$ and $\alpha, \beta \notin \{0,1\}, \alpha \neq \beta$, such that*

$$\frac{(f_1, H_{i_0})}{(f_1, H_{j_0})} = \alpha \frac{(f_2, H_{i_0})}{(f_2, H_{j_0})} = \beta \frac{(f_3, H_{i_0})}{(f_3, H_{j_0})}. \tag{3.10}$$

Proof. Suppose that such i_0, j_0 and α, β exist. Then f_1, f_2, f_3 must be pairwise distinct. Take a partition $I_1 \cup \dots \cup I_k$ of $\{1, \dots, n+3\}$ as in (3.1). By Lemma 3.1, we see that $k = I(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3) \geq 3$. Without loss of generality, we may assume that $i_0 \in I_1$ and $j_0 \in I_2$.

For each $3 \leq t \leq k$, if there exists $z \in A_t$, then $\frac{(f_1,H_{i_0})(z)}{(f_1,H_{j_0})(z)} = \alpha \frac{(f_2,H_{i_0})(z)}{(f_2,H_{j_0})(z)}$ and hence $1 = \alpha$, since $f_1(z) = f_2(z)$. This is a contradiction. Therefore, $A_t = \emptyset$ for all $3 \leq t \leq k$. By Remark 3.5(2b), we have

$$\| \ N^{[1]}_{(f,H_j)} = o(T(r)) \quad \forall j \in I_t,\ i \geq 3. \tag{3.11}$$

Then from Lemma 3.8(ii) and (3.11), we see that $j \in \mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$ for all $j \in I_t$ and $i \geq 3$.

By Remark 3.5(2b), for $1 \leq s \leq 3$ we have

$$\begin{aligned} \| \ T_{f_s}(r) &= \tfrac{1}{2} \sum_{i=1}^{k} N(r, A_i) + o(T(r)) = \tfrac{1}{2}(N(r, A_1) + N(r, A_2)) + o(T(r)) \\ &= \tfrac{1}{2}(N^{[1]}_{(f_s,H_{i_0})}(r) + N^{[1]}_{(f_s,H_{j_0})}(r)) + o(T(r)). \end{aligned}$$

This easily implies that

$$\| \ T_{f_s}(r) = N_{(f_s,H_{i_0})}(r) + o(T(r)) = N^{[1]}_{(f_s,H_{i_0})}(r) + o(T(r)). \tag{3.12}$$

Then

$$\| \ N^{[1]}_{(f_s,H_{i_0})}(r) = N_{(f_s,H_{i_0})}(r) + o(T(r)) \geq N^{[1]}_{(f_s,H_{i_0})}(r) + \tfrac{1}{2} N^{[1]}_{(f_s,H_{i_0}),>1}(r).$$

Therefore, $\| \ N^{[1]}_{(f_s,H_{i_0}),>1}(r) = o(T(r))$. Similarly,

$$\| \ T_{f_s}(r) = N^{[1]}_{(f_s,H_{j_0})}(r) + o(T(r)) \quad \text{and} \quad \| \ N^{[1]}_{(f_s,H_{j_0}),>1}(r) = o(T(r)). \tag{3.13}$$

From (3.11)–(3.13) and Remark 3.5(2a), we see that $V_i \not\sim V_{i_0}$ and $V_i \not\sim V_{j_0}$ for all $i \in I_t$ and $t \geq 3$.

Taking $i_1 \in I_3$, by the density of $\mathcal{C}$, we have

$$\Phi^\alpha(F_c^{i_1 1}, F_c^{i_1 2}, F_c^{i_1 3}) = 0$$

for all $c \in \mathbb{C}^n$ and α with $|\alpha| = 1$. In particular,

$$\Phi^\alpha(F_{i_0}^{i_1 1}, F_{i_0}^{i_1 2}, F_{i_0}^{i_1 3}) = 0,$$

i.e.,

$$\begin{vmatrix} F_{i_1}^{i_0 1} - F_{i_1}^{i_0 2} & F_{i_1}^{i_0 1} - F_{i_1}^{i_0 3} \\ \mathcal{D}^\alpha(F_{i_1}^{i_0 1} - F_{i_1}^{i_0 2}) & \mathcal{D}^\alpha(F_{i_1}^{i_0 1} - F_{i_1}^{i_0 3}) \end{vmatrix} = 0$$

for all α with $|\alpha| = 1$. Since the last determinant is a Wronskian, there exist constants α_1 and β_1, not both zero, such that

$$\alpha_1(F_{i_1}^{i_0 1} - F_{i_1}^{i_0 2}) = \beta_1(F_{i_1}^{i_0 1} - F_{i_1}^{i_0 3}).$$

Thus

$$(\alpha_1 - \beta_1)F_{i_1}^{i_0 1} - \alpha_1 F_{i_1}^{i_0 2} + \beta_1 F_{i_1}^{i_0 3} = 0. \tag{3.14}$$

Because $V_i \not\sim V_{i_0}$, we have $\alpha_1, \beta_1 \notin \{0, 1\}$ and $\alpha_1 \neq \beta_1$. We consider the meromorphic mapping $F : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ with reduced representation $F = (hF_{i_1}^{i_0 1} : hF_{i_1}^{i_0 2})$, where h is a meromorphic function on $\mathbb{C}^m$. We distinguish the following two cases.

Case 1: $F =$ const. Then there exist constants α_2 and β_2 such that

$$F_{i_1}^{i_0 1} = \alpha_2 F_{i_1}^{i_0 1} = \beta_2 F_{i_1}^{i_0 1}. \tag{3.15}$$

Since $V_i \not\sim V_{i_0}$, we have $\alpha_2, \beta_2 \notin \{0, 1\}$ and $\alpha_2 \neq \beta_2$. Repeating the same argument as above, we get the following estimate, similar to (3.11):

$$N^{[1]}_{(f_s, H_{j_0})}(r) = o(T(r)).$$

This contradicts (3.13).

Case 2: $F \neq$ constant. We see that a zero of some $hF_{i_1}^{i_0 s}$ $(1 \leq s \leq 3)$ must be a zero of (f, H_{i_0}) or a zero of (f, H_{i_1}).

Take a regular point z_0 of A_{i_1} with $z_0 \notin A_{i_0}$. From (3.15), there exists a permutation $\{s_1, s_2, s_3\}$ of $\{1, 2, 3\}$ such that $\nu^0_{(f_{s_1}, H_{i_1})}(z_0) \leq \nu^0_{(f_{s_2}, H_{i_1})}(z_0) = \nu^0_{(f_{s_3}, H_{i_1})}(z_0)$. This yields $\nu^0_h(z_0) = \nu^0_{(f_{s_2}, H_{i_1})}(z_0)$. Thus

$$\begin{aligned} \sum_{s=1}^{3} \min\{\nu^0_{hF_{i_1}^{i_0 s}}(z_0), 1\} \\ &= \nu^0_{(f_{s_2}, H_{i_1})}(z_0) - \nu^0_{(f_{s_1}, H_{i_1})}(z_0) \\ &= \min\{\nu^0_{(f_{s_2}, H_{i_1})}(z_0), \nu^0_{(f_{s_3}, H_{i_1})}(z_0)\} - \min\{\nu^0_{(f, H_{i_1})}(z_0), 1\} \\ &\leq \sum_{1 \leq s < t \leq 3} \big(\min\{\nu^0_{(f_s, H_{i_1})}(z_0), \nu^0_{(f_t, H_{i_1})}(z_0)\} - \nu^0_{(f, H_{i_1})}(z_0)\big). \end{aligned} \tag{3.16}$$

Now take a regular point z_0 of A_{i_0} with $z_0 \notin A_{i_1}$. Again by (3.15), there exists a permutation $\{s_1, s_2, s_3\}$ of $\{1, 2, 3\}$ such that $\nu^0_{(f_{s_1}, H_{i_0})}(z_0) = \nu^0_{(f_{s_2}, H_{i_0})}(z_0) \leq \nu^0_{(f_{s_3}, H_{i_0})}(z_0)$. This yields $\nu^\infty_h(z_0) = \nu^0_{(f_{s_1}, H_{i_1})}(z_0)$. Thus

$$\begin{aligned} \sum_{s=1}^{3} \min\{\nu^0_{hF_{i_1}^{i_0 s}}(z_0), 1\} &= \min\{\nu^0_{(f_{s_3}, H_{i_0})}(z_0) - \nu^0_{(f_{s_1}, H_{i_0})}(z_0), 1\} \\ &\leq 1 - D^{[1]}_{i_0}(z_0). \end{aligned} \tag{3.17}$$

Combining (3.16), (3.17) and Lemma 3.4(i), we obtain

$$\Big\| \sum_{s=1}^{3} N^{[1]}_{hF^{i_0 s}_{i_1}}(r) \le \sum_{1\le s<t\le 3} \big(N(r,\min\{\nu^0_{(f_s,H_{i_1})},\nu^0_{(f_t,H_{i_1})}\}) - N^{[1]}_{(f,H_{i_1})}(r)\big)$$
$$+ N^{[1]}_{(f,H_{i_0})} - N^{[1]}(r,D_{i_0}) + o(T(r))$$
$$= N^{[1]}_{(f,H_{i_0})} - N^{[1]}(r,D_{i_0}) + o(T(r)).$$

Since $f(z) = g(z)$ for all $z \in A_2$ we obtain $(hF^{i_0 1}_{i_1} - hF^{i_0 2}_{i_1})(z) = 0$ for all $z \in A_2 \setminus (A_1 \cup A_3)$. Then we have

$$\text{(3.18)} \quad \| \ N^{[1]}_{(f,H_{j_0})}(r)$$
$$= N(r,A_2) + o(T(r)) \le N^{[1]}_{hF^{i_0 1}_{i_1} - hF^{i_0 2}_{i_1}}(r) + o(T(r))$$
$$\le T_F(r) + o(T(r))$$
$$\le \sum_{s=1}^{3} N^{[1]}_{hF^{i_0 s}_{i_1}}(r) + o(T(r)) \le N^{[1]}_{(f,H_{i_0})} - N^{[1]}(r,D_{i_0}) + o(T(r))$$
$$\le \sum_{s=1}^{2} N^{[1]}_{(f_s,H_{i_0}),>1} + o(T(r)) = o(T(r)).$$

This contradicts $N^{[1]}_{(f,H_{j_0})}(r) = T_{f_s}(r) + o(T(r))$ $(1 \le s \le 3)$. ∎

From Lemmas 3.7 and 3.9, we immediately get

Lemma 3.10. *Let $f_1, f_2, f_3 \in \mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$. Suppose that $i_0, j_0 \in Q(\{H_i\}_{i=1}^{3}; f_1, f_2, f_3)$. Then $V_{i_0} \approx V_{j_0}$ or $V_{i_0} \sim V_{j_0}$.*

Proof. By the density of $\mathcal{C}$, we have

$$\Phi^\alpha(F^{i_0 1}_c, F^{i_0 2}_c, F^{i_0 3}_c) = \Phi^\alpha(F^{j_0 1}_c, F^{j_0 2}_c, F^{j_0 3}_c) = 0$$

for all $c \in \mathcal{C}$ and α with $|\alpha| = 1$. In particular,

$$\Phi^\alpha(F^{i_0 1}_{j_0}, F^{i_0 2}_{j_0}, F^{i_0 3}_{j_0}) = \Phi^\alpha(F^{j_0 1}_{i_0}, F^{j_0 2}_{i_0}, F^{j_0 3}_{i_0}) = 0.$$

By Lemma 3.7, one of the following two assertions holds:

- $F^{i_0 1}_{j_0} = F^{i_0 2}_{j_0}$ or $F^{i_0 2}_{j_0} = F^{i_0 3}_{j_0}$ or $F^{i_0 3}_{j_0} = F^{i_0 1}_{j_0}$,
- there exist $\alpha, \beta \notin \{0,1\}, \alpha \ne \beta$, such that $F^{i_0 1}_{j_0} = \alpha F^{i_0 2}_{j_0} = \beta F^{i_0 3}_{j_0}$.

Lemma 3.9 shows that the second assertion cannot be true. Thus the first must hold. Hence $V_{i_0} \approx V_{j_0}$ or $V_{i_0} \sim V_{j_0}$. ∎

4. Proofs of main theorems. We need the following lemma.

Lemma 4.1. *Let $f_1, f_2, f_3 \in \mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$ be distinct and $i_0 \in Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$. Then there is a partition $I_1 \cup I_2 \cup I_3$ of $\{1, \dots, n+3\}$ as in* (3.1) *satisfying:*

(i) $i \in Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$ *if and only if* $i \in I_1$,

(ii) $\| \; N(r, A_2) = N(r, A_3) + o(T(r))$ *and* $\| \; N(r, D_i) = o(T(r))$ *for all* $i \in I_2 \cup I_3$.

Proof. Since f_1, f_2, f_3 are distinct, $I(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3) \geq 3$. We take a partition $I_1 \cup \cdots \cup I_l \cup I_{l+1} \cup \cdots \cup I_{l+t}$ (changing the indices if necessary) of $\{1, \ldots, n+3\}$ as in (3.1), where $l + t \geq 3$, so that

$$i_0 \in I_1, \quad V_i \sim V_{i_0} \quad \forall i \in \bigcup_{1<i\leq l} I_i, \quad V_i \nsim V_{i_0} \quad \forall i \in \bigcup_{l<i\leq t+l} I_i.$$

(i) If $V_i \nsim V_{i_0} \Leftrightarrow i \in I_{l+1} \cup \cdots \cup I_{l+t}$ then $i \notin Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$, by Lemma 3.10. Therefore, to prove (i), it is sufficient to show that $l = 1$ and $t = 2$.

Indeed, suppose that $t > 2$. For each $i \in \{l+1, \ldots, l+t\}$ we pick $j_i \in I_i$. By Lemma 3.8(ii),

$$\| \; 2N^{[1]}_{(f,H_{j_i})}(r) \geq N(r, D_{j_i}) + T_{f_s}(r) + o(T(r)). \tag{4.1}$$

Since $i_0 \in Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$, we have $\Phi^\alpha(F^i_{i_0}, F^i_{i_0}, F^i_{i_0}) = 0$ for all $|\alpha| = 1$. Repeating the same argument in the proof of Lemma 3.9, similarly to (3.14), there exist $\alpha_1, \beta_1 \notin \{0,1\}, \alpha_1 \neq \beta_1$, such that

$$(\alpha_1 - \beta_1)F^{i_0 1}_{j_i} - \alpha_1 F^{i_0 2}_{j_i} + \beta_1 F^{i_0 3}_{j_i} = 0.$$

We consider the meromorphic mapping $F : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$ with representation $F = (F^{i_0 1}_{i_1} : F^{i_0 2}_{i_1})$.

If $F = \text{const}$, then there exist constants α_2 and β_2 such that

$$F^{i_0 1}_{j_i} = \alpha_2 F^{i_0 1}_{j_i} = \beta_2 F^{i_0 1}_{j_i}.$$

Since $V_{j_i} \nsim V_{i_0}$, we have $\alpha_2, \beta_2 \notin \{0,1\}$ and $\alpha_2 \neq \beta_2$. Since $F^{i_0 1}_{i_1}(z) = F^{i_0 1}_{i_1}(z) \notin \{1, \infty\}$ for all $z \in A_v \setminus (A_i \cup A_1)$ with $v \notin \{1, i\}$, it follows that $A_v = \emptyset$ for all $v \notin \{1, i\}$. In particular,

$$\| \; T_{f_s}(r) \leq N(r, A_v) + o(T(r)) = o(T(r)), \qquad \forall\; 1 \leq s \leq 3,\; l+1 \leq v \leq l+t,\; v \neq i.$$

This contradicts (4.1).

Thus $F \not\equiv \text{const}$. Repeating the same argument as in Case 2 of Lemma 3.9, we have the following inequality, similar to (3.18):

$$\Big\| \sum_{\substack{v\neq i\\ v=l+1}}^{l+t} N(r, A_v) \leq N^{[1]}_{(F^{i_0 1}_{j_i}/F^{i_0 2}_{j_i})-1}(r) + o(T(r)) \leq T_F(r) + o(T(r)) \tag{4.2}$$
$$\leq N^{[1]}_{(f,H_{i_0})} - N^{[1]}(r, D_{j_i}) + o(T(r)).$$

Summing over all $i = l+1, \dots, l+t$, we get

$$(4.3)\qquad (t-2)\sum_{i=l+1}^{l+t} N(r, A_i) + \sum_{i=l+1}^{l+t} N(r, A_i)N^{[1]}(r, D_{j_i}) \le o(T(r)).$$

This is a contradiction.

Therefore, $t \le 2$.

Suppose that $t = 1$. We have $l + t \ge 3 \Leftrightarrow l \ge 2$, so by Remark 3.5(2a), $\| \ N(r, A_i) = o(T(r))$ for all $1 \le i \le l$. Therefore, from Remark 3.5(2b) it follows that

$$\begin{aligned}\| \ T_{f_s}(r) &\le \tfrac{1}{2}\sum_{i=1}^{l+1} N(r, A_i) + o(T(r)) \le \tfrac{1}{2}N(r, A_{l+1}) + o(T(r))\\ &\le \tfrac{1}{2}T_{f_s}(r) + o(T(r)),\end{aligned}$$

a contradiction. Hence $t = 2$.

We now prove $l = 1$. Suppose that $l \ge 2$. Similarly to the above, we have $\| \ N(r, A_i) = o(T(r))$ for all $1 \le i \le l$ and

$$\begin{aligned}\| \ T_{f_s}(r) &\le \tfrac{1}{2}\sum_{i=1}^{l+2} N(r, A_i) + o(T(r))\\ &\le \tfrac{1}{2}(N(r, A_{l+1}) + N(r, A_{l+2})) + o(T(r))\\ &\le T_{f_s}(r) + o(T(r)).\end{aligned}$$

This yields

$$(4.4)\qquad \begin{aligned}\| \ T_{f_s}(r) &= N(r, A_{l+1}) + o(T(r))\\ &= N(r, A_{l+2}) + o(T(r)) \quad (1 \le s \le 3).\end{aligned}$$

Then for $l + 1 \le i \le l + 2$, we have

$$\begin{aligned}\| \ N^{[1]}_{(f_s,H_{j_i}),>2}(r) &\le N_{(f_s,H_{j_i})}(r) - N^{[1]}_{(f_s,H_{j_i})}(r)\\ &\le T_{f_s}(r) - N^{[1]}_{(f_s,H_{j_i})}(r) + o(T(r))\\ &= T_{f_s}(r) - N(r, A_i)(r) + o(T(r)) = o(T(r)).\end{aligned}$$

This implies that

$$\begin{aligned}\| \ N(r, D_{j_i}) &\ge N^{[1]}_{(f,H_{j_i})}(r) - \sum_{v=1}^{3} N^{[1]}_{(f_v,H_{j_i}),>2}(r)\\ &= N(r, A_i) + o(T(r)) = T_{f_s}(r) + o(T(r)) \quad (1 \le s \le 3, i > l),\end{aligned}$$

contradicting (4.3). Therefore, $l = 1$. The first assertion of the lemma is proved.

(ii) On the other hand, the inequality (4.3) implies that

$$\| \ N(r,A_2) \le N(r,A_3) - N(r,D_i) + o(T(r)) \qquad \forall i \in I_3,$$
$$\| \ N(r,A_3) \le N(r,A_2) - N(r,D_i) + o(T(r)) \qquad \forall i \in I_2.$$

Thus $\| \ N(r,A_2) = N(r,A_3) + o(T(r))$ and $N(r,D_i) = o(T(r))$ for all $i \in I_2 \cup I_3$. The second assertion of the lemma is proved. ∎

Proof of Theorem 1.2. Suppose that there exist three distinct mappings $f^0, f^1, f^2 \in \mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 2)$. Then, by Lemma 3.4(i),

$$\begin{aligned}(4.5)\qquad \| \ N^{[1]}_{(f_s,H_i),>2}(r) &= N^{[2]}_{(f_s,H_i)}(r) - N^{[1]}_{(f_s,H_i)}(r)\\ &\le N(r, \min\{\nu^0_{(f_s,H_i)}, \nu^0_{(f_t,H_i)}\}) - N^{[1]}_{(f_s,H_i)}(r)\\ &= o(T(r)) \qquad (1 \le s \ne t \le 3).\end{aligned}$$

Suppose that there exists $i_0 \in Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$. We take a partition $I_1 \cup I_2 \cup I_3$ as in Lemma 4.1 with $i_0 \in I_1$.

Then for each $i \in I_2 \cup I_3$, we have $\| \ N(r,D_i) = o(T(r))$. Since $i \notin Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$, combining Lemma 3.8(ii) and (4.5), we also have

$$\begin{aligned}(4.6)\quad \| \ N(r,D_i) &\ge N^{[1]}_{(f,H_i)}(r) - \sum_{s=1}^{3} N^{[1]}_{(f_s,H_i),>2}(r) = N^{[1]}_{(f,H_i)}(r) + o(T(r))\\ &\ge \tfrac{1}{2}(N(r,D_i) + T_{f_s}(r)) + o(T(r)) \qquad (1 \le s \le 3).\end{aligned}$$

It follows that $\| \ N(r,D_i) = T_{f_s}(r) + o(T(r))$. This contradicts $\| \ N(r,D_i) = o(T(r))$.

Hence $i \notin Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$ for all $1 \le i \le n+3$. Similarly to (4.6), we have $\| \ N(r,D_i) = T_{f_s}(r) + o(T(r))$. By Lemma 3.8(ii), it follows that

$$\| \ 2N^{[1]}_{(f_s,H_i)}(r) \ge N(r,D_i) + T_{f_s}(r) + o(T(r)) = 2T_{f_s}(r) + o(T(r)).$$

Take a partition $I_1 \cup \cdots \cup I_k$ of $\{1, \ldots, n+3\}$ as in (3.1). Then

$$k = I(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3) \ge 3.$$

From Remark 3.5(2b), we have

$$\begin{aligned}\| \ T_{f_s}(r) &= \tfrac{1}{2} \sum_{i=1}^{k} N(r,A_i) + o(T(r))\\ &= \tfrac{1}{2} \sum_{i=1}^{k} T_{f_s}(r) + o(T(r)) = \frac{k}{2} T_{f_s}(r) + o(T(r)).\end{aligned}$$

Letting $r \to \infty$, we get $1 = k/2$, a contradiction. ∎

Proof of Theorem 1.3. Suppose that there exist three distinct mappings $f^0, f^1, f^2 \in \mathcal{G}(f, \{H_i\}_{i=1}^{n+3}, 1)$. Firstly, we notice that for $i \ne j$, $\dim A_{ij} \le$

$\dim(f^{-1}H_i \cap H_1) \le m-2$ (by the assumption of the theorem), and hence $A_{ij} = \emptyset$ and $\| \ N(r, A_{ij}) = 0$. Therefore if $V_i \approx V_j$, then

$$\| \ N^{[1]}_{(f,H_i)}(r) = N^{[1]}_{(f,H_i)}(r) + o(T(r)) = o(T(r)). \tag{4.7}$$

Suppose that there exists $i_0 \in Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$. We take a partition $I_1 \cup I_2 \cup I_3$ as in Lemma 4.1 with $i_0 \in I_1$.

Then for each $i \in I_2 \cup I_3$, by Lemma 3.8(ii) we have

$$N^{[1]}_{(f_s,H_i)}(r) \ge \tfrac{1}{2}T_{f_s}(r)) + o(T(r)) \quad (1 \le s \le 3).$$

From this and (4.7), it is easy to see that $\sharp I_2 = \sharp I_3 = 1$. Then $\sharp I_1 = n+1$. It follows that $f_1 = f_2 = f_3$, a contradiction.

Therefore, $i \notin Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$ for all $1 \le i \le n+3$. By Lemma 3.8(ii), we have

$$\| \ N^{[1]}_{(f_s,H_i)}(r) \ge \tfrac{1}{2}T_{f_s}(r) + o(T(r)). \tag{4.8}$$

Take a partition $I_1 \cup \cdots \cup I_k$ of $\{1, \ldots, n+3\}$ as in (3.1). As above, by (4.7) and (4.8) we easily see that $\sharp I_i = 1$ for $1 \le i \le k$. Therefore $k = n+3$.

From Remark 3.5(2b), we have

$$\begin{aligned} \| \ T_{f_s}(r) &= \tfrac{1}{2}\sum_{i=1}^{n+3} N(r, A_i) + o(T(r)) \\ &\ge \tfrac{1}{4}\sum_{i=1}^{n+3} T_{f_s}(r) + o(T(r)) = \frac{n+3}{4}T_{f_s}(r) + o(T(r)). \end{aligned}$$

Letting $r \to \infty$, we get $1 \ge (n+3)/4$, a contradiction. ∎

Proof of Theorem 1.4. Suppose that f^0, f^1, f^2 are distinct. By Lemma 3.4(i),

$$\begin{aligned} \| \ N^{[1]}_{(f_s,H_1),>2}(r) &= N^{[2]}_{(f_s,H_1)}(r) - N^{[1]}_{(f_s,H_1)}(r) \\ &\le N(r, \min\{\nu^0_{(f_s,H_1)}, \nu^0_{(f_t,H_1)}\}) - N^{[1]}_{(f_s,H_1)}(r) \\ &= o(T(r)) \quad (1 \le s \ne t \le 3). \end{aligned} \tag{4.9}$$

We also notice that for $i \ne 1$, $\dim A_{i1} \le \dim(f^{-1}H_i \cap H_1) \le m-2$ (by assumption), so $A_{i1} = \emptyset$ and $\| \ N(r, A_{i1}) = 0$. This shows that if $V_i \approx V_1$ then

$$\| \ N^{[1]}_{(f,H_1)}(r) = N^{[1]}_{(f,H_i)}(r) + o(T(r)) = o(T(r)). \tag{4.10}$$

Suppose that there exists $i_0 \in Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3)$. We take a partition $I_1 \cup I_2 \cup I_3$ as in Lemma 4.1 with $i_0 \in I_1$.

If $\sharp I_1 = 1$ then $I(\{H_i\}_{i=2}^{n+3}; f_1, f_2, f_3) = 2$. By Lemma 3.1, we have $f_1 = f_2 = f_3$, a contradiction. Therefore $\sharp I_1 \ge 2$.

We distinguish the following two cases.

CASE 1: $1 \in I_1$. There exists $v \in I_1, v \neq 1$. By (4.10), one gets

$$N(r, A_i) = N^{[1]}_{(f,H_1)}(r) + o(T(r)) = o(T(r)) \quad \text{(because } V_v \approx V_1).$$

Therefore,

$$\begin{aligned} T_{f_s}(r) &= \tfrac{1}{2}(N(r, A_1) + N(r, A_2) + N(r, A_3)) + o(T(r)) \\ &= \tfrac{1}{2}(N(r, A_2) + N(r, A_3)) + o(T(r)). \end{aligned}$$

This yields $\| \ N(r, A_2) = N(r, A_3) + o(T(r)) = T_{f_s}(r) + o(T(r))$.

Taking $i \in I_2$, we have

$$\begin{aligned} \| \ N^{[1]}_{(f_s,H_i),>2}(r) &\leq N^{[2]}_{(f_s,H_i)}(r) - N^{[1]}_{(f_s,H_i)}(r) \\ &\leq T_{f_s}(r) - N^{[1]}_{(f_s,H_i)}(r) + o(T(r)) = o(T(r)) \quad (1 \leq s \leq 3). \end{aligned}$$

It follows that

$$\begin{aligned} \| \ N(r, D_i) &\geq N^{[1]}_{(f,H_i)}(r) - \sum_{s=1}^{3} N^{[1]}_{(f_s,H_i),>2}(r) \\ &= N(r, A_2) + o(T(r)) = T_{f_s}(r) + o(T(r)). \end{aligned}$$

This contradicts $\| \ N(r, D_i) = o(T(r))$ (because $i \in I_2$).

CASE 2: $1 \notin I_1$. We may assume that $1 \in I_2$. By Lemma 3.8(ii), we have $\| \ N^{[1]}_{(f,H_1)}(r) \geq \frac{1}{2}T_{f_s}(r) + o(T(r))$. Suppose that there exists $i \in I_2 \setminus \{1\}$. Then $V_i \approx V_1$ and (4.10) implies that $\| \ N^{[1]}_{(f,H_1)}(r) = o(T(r))$, a contradiction.

Therefore $I_2 = \{1\}$. Hence $I(\{H_i\}_{i=2}^{n+3}; f_2, f_2, f_3) = 2$. Then $f_1 = f_2 = f_3$, by Lemma 3.1, a contradiction.

Therefore, $Q(\{H_i\}_{i=1}^{n+3}; f_1, f_2, f_3) = \emptyset$. Now Lemma 3.8(ii) yields

$$\| \ N^{[1]}_{(f_s,H_i)}(r) \geq \tfrac{1}{2}T_{f_s}(r) + o(T(r)) \quad (1 \leq s \leq 3). \tag{4.11}$$

We take a partition $I_1 \cup \cdots \cup I_k$ as in (3.1). We may assume that $1 \in I_1$. By repeating the same argument as in Case 2, we have $I_1 = \{1\}$. Then $k - 1 = I(\{H_i\}_{i=2}^{n+3}; f_1, f_2, f_3) \geq 3 \Leftrightarrow k \geq 4$, by Lemma 3.1.

On the other hand, it follows from Lemma 3.8(ii) that

$$\begin{aligned} \| \ 2N^{[1]}_{(f_s,H_1)}(r) &\geq N(r, D_1) + T_{f_s}(r) + o(T(r)) \\ &\geq N^{[1]}_{(f_s,H_1)}(r) - \sum_{v=1}^{3} N^{[1]}_{(f_s,H_1),>2}(r) + T_{f_s}(r) + o(T(r)) \\ &= N^{[1]}_{(f_s,H_1)}(r) + T_{f_s}(r) + o(T(r)) \quad (1 \leq s \leq 3). \end{aligned}$$

Thus

$$N^{[1]}_{(f_s,H_1)}(r) = T_{f_s}(r) + o(T(r)) \quad (1 \le s \le 3). \tag{4.12}$$

Combining Remark 3.5(2b), (4.11) and (4.12) we get

$$\begin{aligned} 2T_{f_s}(r) &= \sum_{i=1}^{k} N(r, A_i) + o(T(r)) \\ &\ge T_{f_s}(r) + \sum_{i=2}^{k} \tfrac{1}{2} T_{f_s}(r) + o(T(r)) \\ &= \frac{k+1}{2} T_{f_s}(r) + o(T(r)). \end{aligned}$$

Letting $r \to \infty$, we get $2 \ge (k+1)/2$, that is, $k \le 3$. This contradicts $k \ge 4$. ■

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