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A note on the regularity of the degenerate complex Monge–Ampère equation

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Abstract. We prove the almost $\mathcal{C}^{1,1}$ regularity of the degenerate complex Monge–Ampère equation in a special case.

If u is a smooth plurisubharmonic function, the Monge–Ampère operator is given by $(dd^cu)^n = \det(u_{p\bar{q}})d\mathcal{L}$, where $u_{p\bar{q}} = \partial^2 u/\partial z_p \partial \bar{z}_q$ and $\mathcal{L}$ is a $2n$-dimensional Lebesgue measure. For an arbitrary continuous plurisubharmonic function u one can define $(dd^cu)^n$ to be a regular Borel measure. Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ (throughout the note we always assume $n \geq 2$). Then for any nonnegative f which is continuous in Ω, and φ continuous on $\partial\Omega$, the Dirichlet problem

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\bar{\Omega}), \\ (dd^cu)^n = f d\mathcal{L} \quad \text{in } \Omega, \\ u = \varphi \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

has a unique continuous plurisubharmonic solution u (see [B-T]).

We say that Ω is a strictly pseudoconvex domain with a $\mathcal{C}^{2,1}$ boundary if $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ where ρ is a strictly plurisubharmonic function of class $\mathcal{C}^{2,1}$ on a neighbourhood of $\bar{\Omega}$ such that $\nabla\rho \neq 0$ on $\partial\Omega$. We say that a function u is *almost* $\mathcal{C}^{1,1}$ if the function Δu is bounded. Now we can formulate the main theorem.

THEOREM 1. *Let Ω be a strictly pseudoconvex domain with a $\mathcal{C}^{2,1}$ boundary, and f be a nonnegative function on $\bar{\Omega}$ such that $f^{1/(n-1)} \in \mathcal{C}^{1,1}(\bar{\Omega})$ and $f^{1/n} \in \mathcal{C}^{0,1}(\bar{\Omega})$. Then the Monge–Ampère equation* (1) *with the boundary condition $\varphi \equiv 0$ has a unique almost $\mathcal{C}^{1,1}$ solution.*

REMARK 2. There are similar theorems for the degenerate real Monge–Ampère equation in [G] and [G-T-W] (with nonzero $\varphi \in \mathcal{C}^{3,1}(\bar{\Omega})$). In the

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complex case Krylov proved (see [K1, K2]) that if Ω is of class $\mathcal{C}^{3,1}$, $\varphi \in \mathcal{C}^{3,1}(\bar{\Omega})$, $f^{1/n} \in \mathcal{C}^{1,1}(\bar{\Omega})$, then $u \in \mathcal{C}^{1,1}(\bar{\Omega})$.

Remark 3. If $f^{1/(n-1)}$ is nonnegative and $\mathcal{C}^{1,1}$ on some neighbourhood of $\bar{\Omega}$ then $f^{1/n} \in \mathcal{C}^{0,1}(\bar{\Omega})$.

Remark 4. It is shown in [P] that the exponent $1/(n-1)$ in Theorem 1 is optimal.

Remark 5. If a function u is almost $\mathcal{C}^{1,1}$, then u is $\mathcal{C}^{1,\alpha}$ for all $\alpha < 1$. A plurisubharmonic function is almost $\mathcal{C}^{1,1}$ if and only if the mixed complex derivatives $u_{p\bar{q}}$ are bounded.

If $\Omega = B$ is the unit ball then we can omit the assumption that $f^{1/n} \in \mathcal{C}^{0,1}(\bar{\Omega})$.

Theorem 6. *Let $\Omega = B$ and let f be a nonnegative function on $\bar{\Omega}$ such that $f^{1/(n-1)} \in \mathcal{C}^{1,1}(\bar{\Omega})$. Then the Monge–Ampère equation* (1) *with the boundary condition $\varphi \equiv 0$ has a unique almost $\mathcal{C}^{1,1}$ solution.*

In all the lemmas below we assume that $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ where ρ is a strictly plurisubharmonic function of class $\mathcal{C}^\infty$ on a neighbourhood of $\bar{\Omega}$ such that $\nabla\rho \neq 0$ on $\partial\Omega$ and $f \in \mathcal{C}^\infty(\bar{\Omega})$ is positive. In that case the solution of (1) is of class $\mathcal{C}^\infty$ on $\bar{\Omega}$.

Lemma 7. *We have $\|u\|_{L^\infty(\Omega)} \le C$, where $C = C(\Omega, \|f\|_{L^\infty(\Omega)})$.*

Proof. From the comparison principle and the maximum principle we have $A|z|^2 - B \le u \le 0$ for $A, B \in \mathbb{R}$ large enough. ∎

The proof of the next lemma is similar to the proof of Lemma 11 in [G] (see also the proof of Theorem 3.1 in [B1]).

Lemma 8. *We have $\sup_\Omega \Delta u \le C(\sup_{\partial\Omega} \Delta u + 1)$, where $C = C(\Omega, \|f^{1/(n-1)}\|_{\mathcal{C}^{1,1}(\Omega)})$.*

Proof. For $k = 1, \ldots, n$, we have

$$(\log f)_k = u^{p\bar{q}} u_{kp\bar{q}}, \tag{2}$$

$$(\log f)_{k\bar{k}} = u^{p\bar{q}} u_{k\bar{k}p\bar{q}} - u^{p\bar{j}} u^{i\bar{q}} u_{ki\bar{j}} u_{\bar{k}p\bar{q}}, \tag{3}$$

where $(u^{p\bar{q}})$ is the inverse of the matrix $(\overline{u_{p\bar{q}}})$.

Let

$$h = \log(\max_{i\in\{1,\ldots,n\}} \lambda_i) + 2A|z|^2,$$

where λ_i are the eigenvalues of the matrix $(u_{p\bar{q}})$, and $A = (2\sup_{z\in\Omega} |z|)^{-2}$. We can assume that h attains its maximum at some point $z_0 \in \Omega$. We can also assume (after a change of variables) that at z_0 the matrix $(u_{p\bar{q}})$ is diagonal and $u_{1\bar{1}} = \max_{i\in\{1,\ldots,n\}} \lambda_i$. Then also the function $\tilde{h} = \log(u_{1\bar{1}}) + 2A|z|^2$ attains its maximum at z_0.

From now on, all formulas are assumed to hold at z_0. We have

$$0 = \tilde{h}_k = \frac{u_{1\bar{1}k}}{u_{1\bar{1}}} + 2A\bar{z}_k \tag{4}$$

and $\tilde{h}_{k\bar{k}} \le 0$. Using this and (3) we can compute

$$\begin{aligned} 0 \ge u^{k\bar{k}}\tilde{h}_{k\bar{k}} &= 2A\sum_k \frac{1}{u_{k\bar{k}}} + u^{k\bar{k}}(\log(u_{1\bar{1}}))_{k\bar{k}} \\ &= 2A\sum_k \frac{1}{u_{k\bar{k}}} + u^{k\bar{k}}\left(\frac{(u_{1\bar{1}})_{k\bar{k}}}{u_{1\bar{1}}} - \frac{u_{1\bar{1}k}u_{1\bar{1}\bar{k}}}{(u_{1\bar{1}})^2}\right) \\ &= 2A\sum_k \frac{1}{u_{k\bar{k}}} + \frac{(\log f)_{1\bar{1}}}{u_{1\bar{1}}} + \frac{1}{u_{1\bar{1}}}u^{p\bar{p}}u^{q\bar{q}}u_{1q\bar{p}}u_{\bar{1}p\bar{q}} - 4A^2\sum_k \frac{|z_k|^2}{u_{k\bar{k}}}. \end{aligned}$$

By the definition of A, we obtain

$$0 \ge A\sum_k \frac{1}{u_{k\bar{k}}} + \frac{(\log f)_{1\bar{1}}}{u_{1\bar{1}}} + \frac{1}{u_{1\bar{1}}}\sum_k u^{p\bar{p}}u^{q\bar{q}}u_{kq\bar{p}}u_{\bar{k}p\bar{q}}. \tag{5}$$

By the inequality between the arithmetic and geometric means, we have

$$\sum_k \frac{1}{u_{k\bar{k}}} \ge \left(\frac{u_{1\bar{1}}}{f}\right)^{1/(n-1)}. \tag{6}$$

The inequality between the root-mean square and the geometric mean, and (2), give us

$$\sum_k u^{p\bar{p}}u^{q\bar{q}}u_{kq\bar{p}}u_{\bar{k}p\bar{q}} \ge \sum_{k\ge 2} |u^{p\bar{p}}u_{kp\bar{p}}|^2 \ge \frac{1}{n-1}\,|(\log f)_1 - u^{1\bar{1}}u_{1\bar{1}1}|^2. \tag{7}$$

Using (5), (6), (7), and (4) we infer

$$0 \ge A\left(\frac{u_{1\bar{1}}}{f}\right)^{1/(n-1)} + (n-1)\frac{(f^{1/(n-1)})_{1\bar{1}}}{f^{1/(n-1)}u_{1\bar{1}}} + 4A\frac{\mathrm{Re}((f^{1/(n-1)})_1 z_1)}{f^{1/(n-1)}u_{1\bar{1}}}.$$

Multiplying both sides of the last inequality by $f^{1/(n-1)}u_{1\bar{1}}$, we get

$$0 \ge A(u_{1\bar{1}})^{n/(n-1)} + (n-1)(f^{1/(n-1)})_{1\bar{1}} + 4A\,\mathrm{Re}((f^{1/(n-1)})_1 z_1)$$

and the lemma follows. ∎

Remark 9. Using (5) and (6) we can prove the above lemma for $C = C(\Omega, -f^{1/(n-1)}\Delta\log f)$ (cf. Theorem 2 in [B2]).

In the two next lemmas we shall prove an *a prori* estimate for first derivatives.

Lemma 10. *We have*

$$\|u\|_{C^{0,1}(\partial\Omega)} \le C,$$

where $C = C(\Omega, \|f\|_{L^\infty(\Omega)})$.

Proof. By the comparison principle $A\rho \le u \le 0$ for A large enough. So on the boundary we have $|\nabla u| \le |\nabla(A\rho)|$. ∎

LEMMA 11. *We have*

$$\|u\|_{\mathcal{C}^{0,1}(\Omega)} \le C, \tag{8}$$

where $C = C(\Omega, \|f^{1/n}\|_{\mathcal{C}^{0,1}}, \|f\|_{L^\infty(\Omega)})$.

Proof. Let $L = u^{p\bar{q}}\partial_{p\bar{q}}$. Since u is strictly plurisubharmonic, the operator L is elliptic. Consider the function $w_i = \pm u_{x_i} + A|z|^2$ (we use the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $z = (z_1, \dots, z_n) = (x_1, \dots, x_{2n})$). Then for A large enough, by (2) and the inequality between the arithmetic and geometric means we have

$$Lw_i = \pm\frac{f_{x_i}}{f} + A\sum u^{p\bar{p}} \ge \frac{\pm n(f^{1/n})_{x_i} + A}{f^{1/n}} \ge 0.$$

The maximum principle and Lemma 10 give us (8). ∎

To obtain an *a priori* estimate for second order derivatives on the boundary we can fix a point $z_0 \in \partial\Omega$ and after a change of coordinates assume that

$$z_0 = 0,$$
$$\rho_{x_i} = \begin{cases} 0 & \text{for } i = 1, \dots, 2n-2, 2n, \\ -1 & \text{for } i = 2n-1, \end{cases}$$
$$\rho_{p\bar{q}}(0) = u_{p\bar{q}}(0) = 0 \quad \text{when } p \ne q,\ p, q = 1, \dots, n-1,$$
$$u_{p\bar{p}}(0) = b_p = -u_n(0)\rho_{p\bar{p}}(0) \quad \text{and} \quad b_1 \le \dots \le b_{n-1},$$

and we can write

$$\rho(z) = -\operatorname{Re} z_n + \sum c_{p\bar{q}} z_p z_{\bar{q}} + O(|z^3|).$$

Then we obtain:

LEMMA 12. *For* $i, j = 1, \dots, n-1$ *we have*

$$|u_{p\bar{q}}(0)| \le C,$$

where $C = C(\Omega, \|f\|_{L^\infty(\Omega)})$.

By standard methods from [C-K-N-S], we obtain

LEMMA 13. *For* $i = 1, \dots n-1$ *we have*

$$|u_{i\bar{n}}(0)| \le C, \tag{9}$$

where $C = C(\Omega, \|f^{1/n}\|_{\mathcal{C}^{0,1}})$.

Proof. Let $L = u^{p\bar{q}}\partial_{p\bar{q}}$. For $k = 1, \dots, 2n-2$ consider the function $w_k = \pm T_k u + (u_{x_k})^2 + (u_{x_n})^2 + A|z|^2 - Bx_{2n-1}$, where $T_k = \rho_{x_{2n-1}}\partial_{x_k} - \rho_{x_k}\partial_{x_{2n-1}}$.

Observe that

$$\begin{aligned} u^{p\bar q}u_{x_{2n-1}\bar q} &= 2u^{p\bar q}u_{n\bar q} + iu^{p\bar q}u_{x_{2n}\bar q} = 2\delta_{pn} + iu^{p\bar q}u_{x_{2n}\bar q},\\ u^{p\bar q}u_{x_{2n-1}p} &= 2u^{p\bar q}u_{p\bar n} - iu^{p\bar q}u_{x_{2n}p} = 2\delta_{qn} - iu^{p\bar q}u_{x_{2n}p},\\ L((u_{x_k})^2) &= 2u_{x_k}(\log f)_{x_k} + u^{p\bar q}u_{x_k\bar q}u_{x_k p},\\ u^{p\bar q}\rho_{x_k p}u_{x_k\bar q} &\le \sqrt{u^{p\bar q}\rho_{x_k p}\rho_{x_k\bar q}}\sqrt{u^{p\bar q}u_{x_k p}u_{x_k\bar q}},\\ \sum u_{p\bar q} &= O\Big(\sum u_{p\bar p}\Big). \end{aligned}$$

Then (using again the inequality between the arithmetic and geometric means) for A enough large, we obtain

$$Lw_k \ge 0.$$

We may choose a neighbourhood U of the origin such that in $S_\varepsilon = \{z \in U \cap \Omega : x_{2n-1} < \varepsilon\}$ we have $x_{2n-1} \ge D|z|^2$ for some constant D. Using Lemma 10 we obtain $(u_{x_k})^2 + (u_{x_n})^2 \le D'|z|^2$ for some other constant D'. Then for B large enough $w_i \le 0$ on ∂S_ε and by the maximum principle we get $w_i \le 0$ on S_ε. Hence, $u_{x_i x_{2n}}(0) = -\rho_{x_i x_{2n}}(0)u_n(0)$ is bounded and we obtain (9). ∎

We can prove a similar lemma for a ball with a constant C not depending on $\|f^{1/n}\|_{\mathcal{C}^{0,1}}$.

Lemma 14. *Let $z_0 = (0,\ldots,0,1)$. If $\Omega = B = \{z \in \mathbb{C}^n : |z_0 - z|^2 < 1\}$ then for $i = 1,\ldots,n-1$ we have*

$$|u_{i\bar n}(0)| \le C,$$

where $C = C(\|f^{1/(n-1)}\|_{\mathcal{C}^{1,1}})$.

Proof. Let $T_k = \bar z_k\frac{\partial}{\partial z_n} - (z_n - 1)\frac{\partial}{z_k}$ for $k = 1,\ldots,n-1$, and let L be as above. Consider the function $w = \pm\operatorname{Re}T_k u + A(|z - z_0|^2 - 1)$. Since T_k on the boundary is tangential, we obtain $|T_k f^{1/(n-1)}| < \tilde C\sqrt{f^{1/(n-1)}}$ where $\tilde C$ depends only on $\|f^{1/n-1}\|_{\mathcal{C}^{1,1}}$. Thus for A large enough we have

$$L(w) = \pm\frac{\operatorname{Re}T_k f^{1/(n-1)}}{(n-1)f^{1/(n-1)}} + A\sum u^{p\bar p} \ge \tilde C f^{-1/2(n-1)} + \frac{A}{2}f^{-1/n} \ge 0.$$

Using the maximum principle we obtain $w \le 0$. So we have

$$|u_{x_{2k-1}x_n}(0)| \le C.$$

Similarly (taking $w = \pm\operatorname{Im}T_k(u-\varphi) + A(|z - z_0|^2 - 1)$), we obtain

$$|u_{x_{2k}x_n}(0)| \le C \quad \text{for } k < n. \ ∎$$

The proof of the next lemma is also the same as in [C-K-N-S].

LEMMA 15. *If* $\|f\|_{L^\infty(\Omega)} > 0$, *then*

$$|u_{n\bar{n}}(0)| \le C,$$

where $C = C(\Omega, \max_{(p,q)\neq(n,n)} |u_{p\bar{q}}(0)|, \|f\|_{L^\infty(\Omega)}, 1/\|f\|_{L^\infty(\Omega)})$.

Proof. There exists $R > 0$ such that $f \ge \|f\|_{L^\infty(\Omega)}/2$ on some ball $B \subset \Omega$ with radius R. Then we have $\Delta u \ge (\|f/2\|_{L^\infty(\Omega)})^{1/n}$. Hence from the Hopf lemma we obtain $-u_{x_{2n-1}}(0) \ge D$ for some constant D. Thus the numbers $1/b_i = -u_n(0)\rho_{i\bar{i}}$ are bounded. As in [G-T-W] we can write

$$f(0) = u_{n\bar{n}}(0)\Big(\prod_{i=1}^{n-1} b_i\Big) - \sum_{j=1}^{n-1} \frac{|u_{j\bar{n}}|^2}{b_j}\Big(\prod_{i=1}^{n-1} b_i\Big).$$

Then

$$0 \le u_{n\bar{n}}(0) = \sum_{j=1}^{n-1} \frac{|u_{j\bar{n}}|^2}{b_j} + \frac{f(0)}{\prod_{i=1}^{n-1} b_i} < C. \quad ∎$$

After a standard regularisation argument, from the above lemmas we obtain Theorems 1 and 6.

REMARK 16. From the proofs we can see that the exponent $\frac{1}{n-1}$ in Theorems 1 and 6 can be replaced by any smaller positive number.

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