ANNALES
POLONICI MATHEMATICI
98.2 (2010)

L-homology theory of $FSQL$-manifolds and the degree of $FSQL$-mappings

by A. ABBASOV (Biga)

Abstract. A homology theory of Banach manifolds of a special form, called *FSQL*-manifolds, is developed, and also a homological degree of *FSQL*-mappings between *FSQL*-manifolds is introduced.

1. Introduction. In this article the results of the article [3] are generalized to Banach manifolds of a special form, namely to Fredholm Special Quasi Linear (*FSQL*) manifolds. In other words, a homology theory of such manifolds is devised and also the homological degree of *FSQL*-mappings between them is introduced. Every *FSQL*-mapping is an *FQL*-mapping [10] ([1]), and vice versa. However, *FSQL*-mappings are more convenient for the structure of *FSQL*-manifolds.

It is known that the degree of a mapping is a strong tool for proving the existence of solutions of various mathematical problems. For instance, various variants of the nonlinear Hilbert problem ([7], [10], etc.) have been solved with the help of the degree of *FQL*-mappings. Moreover, the homological degree of mappings transforms topological problems into algebraic ones. In this case, the problem of finding the degree of a mapping will be reduced to a combinatorial problem.

2. Definition of $FSQL$-manifolds and $FSQL$-mappings. Let $\xi_p = (X_p, \varphi_p, V_p)$ and $\xi_{p,r} = (X_p, \varphi_{p,r}, V_{p,r})$ be affine bundles with identical total space X_p and with base spaces V_p, $V_{p,r}$ which are p- and r-manifolds ($r \geq p$), respectively.

2010 *Mathematics Subject Classification*: Primary 46-XX; Secondary 58-XX.
Key words and phrases: Fredholm Special Quasi Linear manifold, Fredholm Special Quasi Linear mapping, linear homology.

([1]) See [1] for the proof that every *FQL*-mapping is an *FSQL*-mapping.

DOI: 10.4064/ap98-2-3

Definition 2.1. $\xi_{p,r}$ is called an $(r-p)$-*division* of ξ_p if

$$\forall \alpha' \in V_{p,r}\ \exists \alpha \in V_p, \quad \varphi_{p,r}^{-1}(\alpha') \subset \varphi_p^{-1}(\alpha) \text{ and } \operatorname{codim}(\varphi_{p,r}^{-1}(\alpha')) = r-p \text{ in } \varphi_p^{-1}(\alpha).$$

Obviously, in this case $V_{p,r}$ is an affine bundle with the base space V_p and with fibers of dimension $r-p$.

Let $\eta_m = (Y_m, \psi_m, B_m)$ also be an affine bundle, the base space of which is an m-manifold.

Definition 2.2. A continuous mapping $f_{p,m} : X_p \to Y_m$ is called a *Fredholm Special Linear* (*FSL*) mapping between the affine bundles ξ_p and η_m if for some r there exists an $(r-p)$-division $\xi_{p,r} = (X_p, \varphi_{p,r}, V_{p,r})$ of ξ_p and an $(r-m)$-division $\eta_{m,r} = (Y_m, \psi_{m,r}, B_{m,r})$ of η_m with the same dimension r of the base spaces, such that $f_{p,m}$ induces a bimorphism between $\xi_{p,r}$ and $\eta_{m,r}$.

From this point on, we will denote such $f_{p,m}$ as $f_{p,m,r}$. We will also call the restriction of an *FSL*-mapping to any subset of X_p an *FSL*-mapping.

Obviously, if $f_{p,m,r}$ is a bimorphism between $\xi_{p,r}$ and $\eta_{m,r}$, then it is also a bimorphism between some $(\nu-r)$-divisions $\xi_{p,\nu} = (X_p, \varphi_{p,\nu}, V_{p,\nu})$ and $\eta_{m,\nu} = (Y_m, \psi_{m,\nu}, B_{m,\nu})$ of $\xi_{p,r}$ and $\eta_{m,r}$ for any $\nu > r$.

For simplicity, let us assume that ξ_p and η_m are embedded in Banach spaces E_1 and E_2 with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Let $f_{p,m,r} : X_p \to Y_m$ be a bimorphism between ξ_p and η_m, and Δ_p be a bounded domain in X_p. Let

$$|||f_{p,m,r}|||_{\Delta_p} = \sup \inf\{C \mid \|f_{p,m,r,\alpha'}(u)\|_2 \le C(1+\|u\|_1), \quad \|u\|_1 \le C(1+\|f_{p,m,r,\alpha'}(u)\|_2),\ \forall u \in X_{p,\alpha'}\},$$

where $X_{p,\alpha'}$ is the fiber of $\xi_{p,r}$ over $\alpha' \in V_{p,r}$, $f_{p,m,r,\alpha'}$ is the restriction of $f_{p,m,r}$ onto $X_{p,\alpha'}$, and the supremum is taken over all $X_{p,\alpha'}$ for which $X_{p,\alpha'} \cap \Delta_p \ne \emptyset$.

Definition 2.3. A continuous mapping $f_{p,m} : X_p \to Y_m$ is called an *FSQL-mapping* between the affine bundles ξ_p and η_m if it can be uniformly approximated in each bounded domain Δ_p of X_p by *FSL*-mappings $f_{p,m,r}$ so that

$$|||f_{p,m,r}|||_{\Delta_p} \le C(\Delta_p), \quad \forall r > r(\Delta_p),$$

where $C(\Delta_p)$ is independent of r for $r > r(\Delta_p)$.

Now we shall give definitions of *FSQL*-manifolds and of *FSQL*-mappings between *FSQL*-manifolds. Let $\tilde{X}$ be a Banach manifold and $\{\tilde{X}_p\}$, $\tilde{X}_{p-1} \subset \tilde{X}_p$, $p = 1, 2, \dots$, be a system of open sets covering $\tilde{X}$, i.e. $\tilde{X} = \bigcup \tilde{X}_p$. Let $\xi_p = (X_p, \varphi_p, V_p)$ be an affine bundle, Δ_p be a bounded domain in X_p and $\tilde{\varphi}_p : \tilde{X}_p \to \Delta_p$ be a homeomorphism. In this case, $(\tilde{\varphi}_p, \tilde{X}_p)$ is called a *linear chart* (L-chart) on $\tilde{X}$. We shall say that a *linear structure* (L-structure) is introduced on $\tilde{X}_p$ if the conditions above are satisfied. If an L-structure is defined on $\tilde{X}_{p+1}$, then obviously it is also defined on $\tilde{X}_p$ (as an induced structure). If $\tilde{\varphi}_{p'} : \tilde{X}_{p'} \to \Delta_{p'}$, $\tilde{\varphi}_{p''} : \tilde{X}_{p''} \to \Delta_{p''}$, $p', p'' \geq p$, are two L-structures on $\tilde{X}_p$, then the transition functions $\tilde{\varphi}_{p''} \circ \tilde{\varphi}_{p'}^{-1} : \Delta_{p'} \to \Delta_{p'}$ and $\tilde{\varphi}_{p'} \circ \tilde{\varphi}_{p'}^{-1} : \Delta_{p''} \to \Delta_{p'}$ arise. Let us suppose that they are *FSQL*-mappings between $\xi_{p'} = (X_{p'}, \varphi_{p'}, V_{p'})$ and $\xi_{p''} = (X_{p''}, \varphi_{p''}, V_{p''})$. In that case, we shall say that the two L-structures on $\tilde{X}_p$ are *equivalent.*

Definition 2.4. A class of equivalent L-structures on $\tilde{X}_p$ is called an *FSQL-structure* on $\tilde{X}_p$.

Obviously, an *FSQL*-structure on $\tilde{X}_{p+1}$ induces an *FSQL*-structure on $\tilde{X}_p$. An *FSQL*-structure on $\tilde{X}_p$ is said to be *coordinated* with an *FSQL*-structure on $\tilde{X}_{p+1}$ if it coincides with the induced structure.

Definition 2.5. A collection of *FSQL*-structures on $\tilde{X}_p$, $p = 1, 2, \dots,$ which are coordinated with each other is called an *FSQL-structure* on $\tilde{X}$. A Banach manifold $\tilde{X}$ with an *FSQL*-structure is called an *FSQL-manifold.*

Let $\tilde{X}$, $\tilde{Y}$ be *FSQL*-manifolds,

$$\tilde{X} = \bigcup \tilde{X}_p,\ \tilde{X}_p \subset \tilde{X}_{p+1}\ \forall p, \qquad \tilde{Y} = \bigcup \tilde{Y}_m,\ \tilde{Y}_m \subset \tilde{Y}_{m+1}\ \forall m,$$

$(\tilde{\varphi}_p, \tilde{X}_p)$, $(\tilde{\psi}_m, \tilde{Y}_m)$ be L-charts on $\tilde{X}$, $\tilde{Y}$ and $\tilde{\varphi}_p(\tilde{X}_p) = \Delta_p$, $\tilde{\psi}_m(\tilde{Y}_m) = \Omega_m$ be bounded domains in $\xi_p = (X_p, \varphi_p, V_p)$, $\eta_m = (Y_m, \psi_m, B_m)$, respectively.

Definition 2.6. A continuous mapping $\tilde{f} : \tilde{X} \to \tilde{Y}$ between *FSQL*-manifolds $\tilde{X}$ and $\tilde{Y}$ is called an *FSQL-mapping* if

(a) $\forall p\ \exists m,\ \tilde{f}(\tilde{X}_p) \subset \tilde{Y}_m$,
(b) $f_{p,m} \equiv \tilde{\psi}_m \circ \tilde{f} \circ \tilde{\varphi}_p^{-1} : \Delta_p \to \Omega_m$ is an *FSQL*-mapping between the domains of the affine bundles ξ_p and η_m.

3. ***L*-homology theory of affine bundles. Singular theory.** First, note that the simplicial theory of (n, k)-simplexes is available in [3], it is similar to the finite-dimensional case.

Let H be a real Hilbert space, H^k be a linear subspace of codimension k $(k \geq 0)$ and σ_n be a Euclidean n-simplex. We will name the Cartesian product $\sigma_n \times H^k$ a *Hilbertian simplex* of bi-dimension (n, k) and we will denote it by σ_n^k, that is, $\sigma_n^k = \sigma_n \times H^k$. We will consider σ_n^k to be oriented if σ_n is oriented. In this case, the orientation on σ_n is taken to be the orientation on σ_n^k. From this point on, we will consider σ_n^k to be oriented.

Definition 3.1. A continuous mapping $f_n^k : \sigma_n^k \to X_p$ is called a *singular (n, k)-simplex* in ξ_p if there exists a k-division $(\xi_{p'})$ of ξ_p such that f_n^k induces a bimorphism between $\sigma_n \times H^k$ and $\xi_{p'}$.

It follows from this definition that each singular (n,k)-simplex f_n^k induces some finite-dimensional mapping between the base spaces σ_n and $V_{p'}$ ($p' = p+k$) of these bundles.

DEFINITION 3.2. A finite formal linear combination $\tilde{c}_n^k = \sum_i g_i \cdot f_{n,i}^k$ of singular (n,k)-simplexes in ξ_p with coefficients $g_i \in \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers, is called a *singular* (n,k)*-chain* in ξ_p.

We will denote by $\tilde{C}_n^k(X_p)$ the set of all singular chains in ξ_p of bi-dimension (n,k). Obviously, it is an Abelian group under addition of chains. It is a free group.

DEFINITION 3.3. We define the differential

$$\tilde{\partial}_n^k : \tilde{C}_n^k(X_p) \to \tilde{C}_{n-1}^k(X_p) \quad \forall n \geq 1, \forall k \geq 0$$

as follows:

$$\tilde{\partial}_n^k f_n^k = \sum (-1)^i (f_n^k|_{\sigma_{n-1,i}^k})$$

and we extend it to $\tilde{C}_n^k(X_p)$ by additivity. Moreover,

$$\tilde{\partial}_0^k : \tilde{C}_0^k(X_p) \to 0 \quad \forall k \geq 0.$$

REMARK. Here $\sigma_{n-1,i}^k$ is the $(n-1,k)$-boundary of the simplex σ_n^k, which is located opposite vertex i.

THEOREM 3.4. *The equality*

$$\tilde{\partial}_{n-1}^k \circ \tilde{\partial}_n^k = 0$$

is true for each $n \geq 1$ *and* k.

The proof is similar to the finite-dimensional case.

Analogously to the finite-dimensional case, one can define the groups $\operatorname{Ker} \tilde{\partial}_n^k$, $\operatorname{Im} \tilde{\partial}_{n+1}^k$ and $\tilde{H}_n^k$, i.e. the groups of (n,k)-cycles, (n,k)-boundaries and the (n,k)-homology group (see [3]). However the theory of relative homology of ξ_p, which is introduced in the following section, is more interesting.

4. The relative L-homology of an affine bundle

DEFINITION 4.1. An (n,k)-chain $\tilde{c}_n^k \in \tilde{C}_n^k(X_p)$ is called a *relative cycle* of bi-dimension (n,k) if $\tilde{\partial}_n^k \tilde{c}_n^k \in \tilde{C}_{n-1}^k(X_p \setminus \Delta_p)$.

DEFINITION 4.2. A relative cycle $\tilde{c}_n^k$ is called *homologous to zero* if

$$\exists \tilde{c}_{n+1}^k \in \tilde{C}_{n+1}^k(X_p), \quad \tilde{\partial}_{n+1}^k \tilde{c}_{n+1}^k = \tilde{c}_n^k \oplus \tilde{d}_n^k, \quad \tilde{d}_n^k \in \tilde{C}_n^k(X_p \setminus \Delta_p).$$

It follows from this definition that the sum of relative (n,k)-cycles homologous to zero is also homologous to zero. Therefore the set of relative (n,k)-cycles homologous to zero forms a subgroup of the group of relative (n,k)-cycles.

Now we define the concept of "support" of a singular simplex.

Let f_n^k be a singular simplex in X_p. By definition, it induces a bimorphism between $\sigma_n \times H^k$ and some k-division $\xi_{p'} = (X_p, \varphi_{p'}, V_{p'})$ of ξ_p. Then $(f_n^k)^{-1}(\xi_{p'})$ induces an affine bundle $(\sigma_{n'}^{k'})$, which is a $(k'-k)$-division of σ_n^k: its base space $\sigma_{n'}$ is itself an affine bundle with base space σ_n and fiber $H_{k'-k}$, $n' = n + (k'-k)$, which is the Euclidean $(k'-k)$-space. As σ_n is convex, one can represent $\sigma_{n'}$ in the form of a Cartesian product: $\sigma_{n'} = \sigma_n \times H_{k'-k}$. Therefore the bundle $\sigma_{n'}^{k'}$ is also a Cartesian product, i.e. $\sigma_{n'}^{k'} = \sigma_{n'} \times H^{k'}$, where $H^{k'}$ is a subspace of H of codimension k'. Now we divide $\sigma_{n'}$ into n'-prisms $\sigma_{n',j}$, $j = 0, \pm 1, \pm 2, \ldots$, with bases σ_n (2). Let us choose the orientation of one (n',k')-prism $\sigma_{n',j}^{k'} = \sigma_{n',j} \times H^{k'}$ arbitrarily and coordinate orientations of other (n',k')-prisms with it. Then any two neighboring prisms will induce opposite orientations on the common edge. Obviously, it is possible to divide $\sigma_{n'}$ into n'-prisms so that the restriction of each of the mappings f_n^k to a unique $\sigma_{n',j} \times H^{k'}$ contains the intersection of $f_n^k(\sigma_n^k)$ with Δ_p; this is possible because of the linearity of each f_n^k on H_α^k, the uniform continuity of f_n^k in α, and the boundedness of Δ_p. In this case all the other analogous restrictions will be outside of Δ_p. Thus, we can give the following

Definition 4.3. The restriction of a singular simplex f_n^k to an (n',k')-prism $\sigma_{n'}^{k'}$ is called an (n',k')-*support* of f_n^k if

(a) $n' - n = k' - k$,
(b) $f_n^k(\sigma_{n'}^{k'}) \cap \Delta_p = f_n^k(\sigma_n^k) \cap \Delta_p$.

Let us denote the (n',k')-support of f_n^k by $f_{n'}^{k'}$. From Definition 4.3 it follows that there can be different (n',k')-supports of a singular (n,k)-simplex. But obviously, the difference of two (n',k')-supports of f_n^k is homologous to zero relative to $X_p \setminus \Delta_p$.

Analogously, we shall say that a chain $\tilde{c}_{n'}^{k'} = \sum g_i \cdot f_{n',i}^{k'}$ is an (n',k')-*support* of the chain $\tilde{c}_n^k = \sum g_i \cdot f_{n,i}^k$ if for each i the simplex $f_{n',i}^{k'}$ is an (n',k')-support of $f_{n,i}^k$.

Obviously with the help of the above construction one can construct an (n'',k'')-support of the chain $\tilde{c}_n^k$ for any $n'' > n'$, $k'' > k'$, where $n'' - n = k'' - k$.

(2) For example, in the case of $n = 1$ and $k' - k = 1$ the base space $\sigma_{n'}$, $n' = 2$, can be represented in the form of an infinite band. The line segment which defines the width of this band is σ_n. Having divided this band into segments, we will obtain rectangles-prisms $\sigma_{n',j}$, $j = 0, \pm 1, \pm 2, \ldots$, with bases σ_n.

One can represent each prism $\sigma_{n',j}$, $j = 0, \pm 1, \pm 2, \ldots$, in the form of $\sigma_n \times I_{k'-k}$, where $I_{k'-k}$ is a $(k'-k)$-cube.

Remark. In view of the aforementioned construction, from this point on we will suppose that all simplexes $f^{k'}_{n',i}$ of $\tilde{c}^{k'}_{n'}$ are bimorphisms between $\sigma_n \times H^k$ and $\xi_{p''} = (X_p, \varphi_{p''}, V_{p''})$.

Let $\tilde{c}^k_n$ be a singular cycle relative to $X_p \setminus \Delta_p$ and $\tilde{c}^{k'}_{n'}$ be its (n', k')-support. Let us orient each simplex of $\tilde{c}^{k'}_{n'}$ so that two simplexes which have a common edge induce opposite orientations on this common edge. Then $\tilde{c}^{k'}_{n'}$ is also a singular cycle relative to $X_p \setminus \Delta_p$. Thus the relative cycle $\tilde{c}^{k'}_{n'}$ is oriented (in two possible ways).

Obviously, two supports of a relative cycle $\tilde{c}^k_n$ of the same bi-dimension are homologous to each other relative to $X_p \setminus \Delta_p$.

Lemma 4.4. *If $\tilde{c}^k_n$ is a singular cycle relative to $X_p \setminus \Delta_p$, then for every $l > 0$ its $(n+l, k+l)$-support $\tilde{c}^{k+l}_{n+l}$ is also a singular cycle relative to $X_p \setminus \Delta_p$, and if an $(n+l, k+l)$-support $\tilde{c}^{k+l}_{n+l}$ of $\tilde{c}^k_n$ is a singular cycle relative to $X_p \setminus \Delta_p$ for some $l > 0$, then $\tilde{c}^k_n$ is also a singular cycle relative to $X_p \setminus \Delta_p$ (3).*

Indeed, as $\tilde{c}^k_n$ is a singular cycle relative to $X_p \setminus \Delta_p$, $\tilde{\partial}^k_n \tilde{c}^k_n \in \tilde{C}^k_{n-1}(X_p \setminus \Delta_p)$. Because of the definition of a support of a chain and the construction of the prism, the boundary of the $(n+l, k+l)$-support $\tilde{c}^{k+l}_{n+l}$ also belongs to $X_p \setminus \Delta_p$ for every $l > 0$. For the proof of the second statement of this lemma, it is enough to apply the construction from the definition of the support of a function in reverse order.

Lemma 4.5. *If $\tilde{c}^k_n \sim 0$ $(X_p, X_p \setminus \Delta_p)$, then $\tilde{c}^{k+l}_{n+l} \sim 0$ $(X_p, X_p \setminus \Delta_p)$ for all $l > 0$, and if $\tilde{c}^{k+l}_{n+l} \sim 0$ $(X_p, X_p \setminus \Delta_p)$ for some $l > 0$, then $\tilde{c}^k_n \sim 0$ $(X_p, X_p \setminus \Delta_p)$ (4).*

Indeed, if $\tilde{c}^k_n \sim 0$ $(X_p, X_p \setminus \Delta_p)$, then

$$\exists \tilde{c}^k_{n+1} \in \tilde{C}^k_{n+1}(X_p), \quad \tilde{\partial}^k_{n+1} \tilde{c}^k_{n+1} = \tilde{c}^k_n \oplus \tilde{d}^k_n, \quad \tilde{d}^k_n \in \tilde{C}^k_n(X_p \setminus \Delta_p).$$

In this case one can construct an $(n+l+1, k+l)$-support $\tilde{c}^{k+l}_{n+l+1}$ of $\tilde{c}^k_{n+1}$ such that

$$\tilde{\partial}^{k+l}_{n+l+1} \tilde{c}^{k+l}_{n+l+1} = \tilde{c}^{k+l}_{n+l} \oplus \tilde{d}^{k+l}_{n+l}, \quad \tilde{d}^{k+l}_{n+l} \in \tilde{C}^{k+l}_{n+l}(X_p \setminus \Delta_p),$$

where $\tilde{c}^{k+l}_{n+l}$ and $\tilde{d}^{k+l}_{n+l}$ are $(n+l, k+l)$-supports of $\tilde{c}^k_n$ and $\tilde{d}^k_n$, respectively. For the proof of the second statement of this lemma it is enough to apply the construction from the definition of support of a function in reverse order.

In view of Lemmas 4.4 and 4.5 we can give a new definition of homology to zero, which is equivalent to the previous one.

(3) Here and in the following, $k' = k + l$, $n' = n + l$.

(4) $\tilde{c}^k_n \sim 0$ $(X_p, X_p \setminus \Delta_p)$ means that $\tilde{c}^k_n \sim 0$ relative to $X_p \setminus \Delta_p$.

DEFINITION 4.6 (equivalent to Definition 4.2). A relative cycle $\tilde{c}_n^k$ is called *homologous to zero* if for some $l > 0$ its $(n+l, k+l)$-support $\tilde{c}_{n+l}^{k+l}$ is homologous to zero (in the sense of Definition 4.2).

5. Calculation of relative L-homology of an affine bundle. In this section we will assume that the base space V_{p_0} of the affine bundle $\xi_{p_0} = (X_{p_0}, \varphi_{p_0}, V_{p_0})$ does not have boundary, and the bounded domain Δ_{p_0} is of the form $X_{p_0} \cap B_1(R)$, where $B_1(R)$ is the open ball in E_1 of radius R with center at zero ([5]).

THEOREM 5.1. *For any p_0 and $k \geq 0$,*

$$\tilde{H}_n^k(X_{p_0}, X_{p_0} \setminus \Delta_{p_0}) \cong \begin{cases} 0, & n \neq p_0 + k, \\ \mathbb{Z}, & n = p_0 + k. \end{cases}$$

The proof reduces to calculating $\tilde{H}_n(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k})$ where $W_{p_0,p_0+k} = \varphi_{p_0,p_0+k}(\Delta_{p_0})$, φ_{p_0,p_0+k} is the projection of the k-division $(X_{p_0}, \varphi_{p_0,p_0+k}, V_{p_0,p_0+k})$ of $(X_{p_0}, \varphi_{p_0}, V_{p_0})$.

Before proving the theorem we state two relevant lemmas.

Let $\tilde{c}_n^k = \sum g_i \cdot f_{n,i}^k$ be an (n,k)-chain in $\tilde{C}_n^k(X_{p_0})$, $\sigma_n^k = \sigma_n \times H^k$ be a Hilbertian (n,k)-simplex and $s : \sigma_n \to \sigma_n^k$ be a continuous section of $\sigma_n \times H^k$. Let us consider the n-chain $\tilde{c}_n = \sum g_i \cdot f_{n,i}$ in V_{p_0,p_0+k}, where

$$f_{n,i} = \varphi_{p_0,p_0+k} \circ f_{n,i}^k \circ s : \sigma_n \to V_{p_0,p_0+k}.$$

In other words, $\tilde{c}_n$ is the projection (by means of φ_{p_0,p_0+k}) of the chain $\tilde{c}_n^k$ onto V_{p_0,p_0+k}.

LEMMA 5.2. *$\tilde{c}_n^k$ is a cycle relative to $X_{p_0} \setminus \Delta_{p_0}$ if and only if $\tilde{c}_n$ is a cycle relative to $V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}$.*

Indeed, if $\tilde{c}_n^k$ is a cycle relative to $X_{p_0} \setminus \Delta_{p_0}$, then $\tilde{\partial}_n^k \tilde{c}_n^k \in \tilde{C}_{n-1}^k(X_{p_0} \setminus \Delta_{p_0})$. As $\tilde{c}_n$ is the projection (by means of φ_{p_0,p_0+k}) of $\tilde{c}_n^k$ onto V_{p_0,p_0+k}, then $\tilde{\partial}_n \tilde{c}_n \in \tilde{C}_{n-1}(V_{p_0,p_0+k} \setminus W_{p_0,p_0+k})$. The converse implication is self-evident.

LEMMA 5.3. *$\tilde{c}_n^k \sim 0$ $(X_{p_0}, X_{p_0} \setminus \Delta_{p_0})$ if and only if*

$$\tilde{c}_n \sim 0 \ (V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}).$$

Indeed, if $\tilde{c}_n^k \sim 0$ $(X_{p_0}, X_{p_0} \setminus \Delta_{p_0})$, it follows that

$$\exists \tilde{c}_{n+1}^k \in \tilde{C}_{n+1}^k(X_{p_0}), \quad \tilde{\partial}_{n+1}^k \tilde{c}_{n+1}^k = \tilde{c}_n^k \oplus \tilde{d}_n^k, \quad \tilde{d}_n^k \in \tilde{C}_n^k(X_{p_0} \setminus \Delta_{p_0}).$$

([5]) Recall that the affine bundle ξ_{p_0} is embedded in a Banach space E_1.

Therefore

$$\tilde{\partial}_{n+1}\tilde{c}_{n+1} = \tilde{c}_n \oplus \tilde{d}_n, \qquad \tilde{d}_n \in \tilde{C}_n(V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}),$$

where $\tilde{c}_{n+1}$, $\tilde{c}_n$ and $\tilde{d}_n$ are the projections (by means of φ_{p_0,p_0+k}) of the chains $\tilde{c}^k_{n+1}$, $\tilde{c}^k_n$ and $\tilde{d}^k_n$ onto V_{p_0,p_0+k}, respectively. The converse implication is self-evident.

Proof of Theorem 5.1. Let $\tilde{c}^k_n \in [\tilde{c}^k_n] \in \tilde{H}^k_n(X_{p_0}, X_{p_0} \setminus \varDelta_{p_0})$, and $\tilde{c}_n$ be the projection of $\tilde{c}^k_n$ onto V_{p_0,p_0+k}. By Lemma 5.2, $\tilde{c}_n$ is a cycle relative to $V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}$.

1) Let $n \neq p_0+k$. Then, as is known from the theory of finite-dimensional homology,

$$\tilde{c}_n \sim 0 \ (V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}),$$

i.e. the n-dimensional singular cycle $\tilde{c}_n$ in V_{p_0,p_0+k} is homologous to zero relative to $V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}$. By Lemma 5.3,

$$\tilde{c}^k_n \sim 0 \ (X_{p_0},\ X_{p_0} \setminus \varDelta_{p_0}).$$

Hence,

$$\tilde{H}^k_n(X_{p_0}, X_{p_0} \setminus \varDelta_{p_0}) \cong 0 \text{ for } n \neq p_0 + k.$$

2) Let $n = p_0 + k$. If $\tilde{c}_{p_0+k}$ is a cycle relative to $V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}$, then

$$[\tilde{c}_{p_0+k}] \in \tilde{H}_{p_0+k}(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}).$$

Therefore

$$\exists d \in \mathbb{Z}, \qquad [\tilde{c}_{p_0+k}] = d \cdot [\tilde{1}_{p_0+k}],$$

where $[\tilde{1}_{p_0+k}]$ is the unit element of $\tilde{H}_{p_0+k}(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k})$. By Lemma 5.3,

$$[\tilde{c}^k_{p_0+k}] = d \cdot [\tilde{1}^k_{p_0+k}],$$

where $[\tilde{1}^k_{p_0+k}]$ is the unit element of $\tilde{H}^k_{p_0+k}(X_{p_0}, X_{p_0} \setminus \varDelta_{p_0})$. By the above-mentioned construction, the mapping

$$[\tilde{c}^k_{p_0+k}] \mapsto d \in \mathbb{Z}$$

is an isomorphism. Thus,

$$\tilde{H}^k_{p_0+k}(X_{p_0}, X_{p_0} \setminus \varDelta_{p_0}) \cong \mathbb{Z}. \ \blacksquare$$

Remark. Actually we proved that

$$\tilde{H}^k_n(X_{p_0}, X_{p_0} \setminus \varDelta_{p_0}) \cong \tilde{H}_n(V_{p_0,p_0+k}, V_{p_0,p_0+k} \setminus W_{p_0,p_0+k}) \cong \begin{cases} 0, & n \neq p_0 + k, \\ \mathbb{Z}, & n = p_0 + k. \end{cases}$$

As $\tilde{\varphi}_{p_0}(\tilde{X}_{p_0}) = \Delta_{p_0}$ and $\Delta_{p_0} \subset X_{p_0}$, the spaces $(\tilde{X}, \tilde{X} \setminus \tilde{X}_{p_0})$ and $(X_{p_0}, X_{p_0} \setminus \Delta_{p_0})$ are homeomorphic to each other. Therefore

$$\tilde{H}_n^k(\tilde{X}, \tilde{X} \setminus \tilde{X}_{p_0}) \cong \begin{cases} 0, & n \neq p_0 + k, \\ \mathbb{Z}, & n = p_0 + k. \end{cases}$$

for every integer $k \geq 0$ ([6]).

6. L-homological degree of an $FSQL$-mapping between $FSQL$-manifolds. We shall consider a simpler case for the definition of L-homological degree of $FSQL$-mappings between $FSQL$-manifolds.

We will suppose that

1) The $FSQL$-manifolds $\tilde{X}$, $\tilde{Y}$ are embedded in the Banach spaces E_x, E_y with the norms $\|\cdot\|_x$, $\|\cdot\|_y$, respectively.

2) The mappings $\tilde{\varphi}_p$, $\tilde{\varphi}_p^{-1}$, $\tilde{\psi}_m$, $\tilde{\psi}_m^{-1}$ are uniformly continuous.

3) $\tilde{f} : \tilde{X} \to \tilde{Y}$ is an $FSQL$-mapping which satisfies an a priori estimate

$$\|x\|_x \leq \Phi(\|\tilde{f}(x)\|_y), \tag{6.1}$$

where Φ is some positive monotone function.

For simplicity, suppose that Φ is the identity mapping. Let us consider the equation

$$\tilde{f}(x) = y_0, \quad y_0 \in \tilde{Y}. \tag{6.2}$$

Under condition (6.1), all the solutions of (6.2) belong to $\tilde{X}_{R_0} = \tilde{X} \cap B_x(R_0)$, where $B_x(R_0)$ is the open ball in E_x of radius $R_0 = \|y_0\|_y$ with center at zero. According to the definition of an $FSQL$-manifold,

$$\exists p_0, \forall p \geq p_0 : \quad \tilde{X}_{R_0} \tilde{X}_p,$$

and according to the definition of $FSQL$-mappings between $FSQL$-manifolds,

$$\exists m_0, \forall m \geq m_0 : \quad \tilde{f}(\tilde{X}_p) \subset \tilde{Y}_m.$$

Let p and m be numbers for which all the above mentioned conditions are satisfied. Then to define the degree of $\tilde{f}$ at the point $y_0 \in \tilde{Y}$ we can consider the restriction of $\tilde{f}$ to $\tilde{X}_p$. As $\tilde{\varphi}_p$ and $\tilde{\psi}_m$ are homeomorphisms, equation (6.2) holds in $\tilde{X}_{R_0}$ if and only if the equation

$$f_{p,m}(u) = w_0, \qquad w_0 = \tilde{\psi}_m(y_0)$$

holds in $\tilde{\varphi}_p(\tilde{X}_{R_0})$, where $f_{p,m} \equiv \tilde{\psi}_m \circ \tilde{f} \circ \tilde{\varphi}_p^{-1} : \Delta_p \to \Omega_m$, $\tilde{\varphi}_p(\tilde{X}_{R_0}) \subset \Delta_p$.

According to the definition of $FSQL$-manifolds, $f_{p,m}$ is an $FSQL$-mapping between the affine bundles ξ_p and η_m. Let $\{f_{p,m,r}\}$ be a sequence of FSL-mappings which is uniformly convergent to $f_{p,m}$ on Δ_p. Let us consider the

([6]) Recall that p_0 is the dimension of the base space V_{p_0} of the affine bundle ξ_{p_0}.

equation

$$f_{p,m,r}(u) = w_0. \tag{6.3}$$

We will search for its solutions in $\tilde{\varphi}_p(\tilde{X}_{R'_0})$, where $\tilde{X}_{R'_0} = \tilde{X} \cap B_x(R'_0)$, $R'_0 = \|y_0\|_y + 2\delta$, $\delta > 0$.

Remark. $\tilde{X}_{R'_0} \subset \tilde{X}_p$ for large enough p, therefore $\tilde{\varphi}_p(\tilde{X}_{R'_0}) \subset \Delta_p$.

Obviously, $\tilde{f}(x) \in \tilde{Y} \setminus B_y(R_0)$ at $x \in \tilde{X} \setminus B_x(R_0)$, where $B_y(R_0)$ is the open ball in E_y of radius R_0 with center at zero. Therefore $\tilde{f}$ is a mapping of pairs $(\tilde{X}, \tilde{X} \setminus B_x(R_0))$ and $(\tilde{Y}, \tilde{Y} \setminus B_y(R_0))$, and $f_{p,m}$ is a mapping of pairs $(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R_0})t)$ and $(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0}))$ (7).

By the definition of $FSQL$-mapping,

$$\forall u \in \Delta_p : \quad \|f_{p,m}(u) - f_{p,m,r}(u)\|_2 < \delta_1, \quad \delta_1 > 0.$$

for sufficiently large r. As the L-charts $\tilde{\varphi}_p$, $\tilde{\varphi}_p^{-1}$, $\tilde{\psi}_m$, $\tilde{\psi}_m^{-1}$ are uniformly continuous,

$$\forall x \in \tilde{X}_p : \quad \|\tilde{f}(x) - \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x)\|_y < \delta, \quad \delta > 0.$$

for a proper choice of δ_1. Therefore $f_{p,m,r}$ will be a mapping of pairs $(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0}))$ and $(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta}))$ for sufficiently large r, where $\tilde{Y}_{R_0-\delta} = \tilde{Y} \cap B_y(R_0 - \delta)$, $B_y(R_0 - \delta)$ is the open ball in E_y of radius $R_0 - \delta$ with center at zero.

Let $[\tilde{\omega}^k_{r+k}] \in \tilde{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0}))$, $\tilde{\omega}^k_{r+k} \in [\tilde{\omega}^k_{r+k}]$, $\tilde{\omega}^k_{r+k} = \sum g_i \cdot f^k_{r+k,i}$ and for any i, $f^k_{r+k,i} : \sigma_{r+k} \times H^k \to \xi_{p''}$ where $\xi_{p''} = (X_p, \varphi_{p''}, V_{p''})$, and $f_{p,m,r} : \Delta_p \to \Omega_m$ is an FSL-mapping which satisfies the above mentioned conditions. One can construct an affine bundle $\xi_{p,\nu}$, $\nu \geq r$, which is a common division of $\xi_{p,r}$ and $\xi_{p''}$. Let us take an $(r+\nu, \nu)$-support $\tilde{\omega}^\nu_{r+\nu} = \sum g_i \cdot f^\nu_{r+\nu,i}$ of $\tilde{\omega}^k_{r+k}$. Then there exists a singular chain $\tilde{c}^\nu_{r+\nu} = \sum g_i \cdot (f_{p,m,r} \circ f^\nu_{r+\nu,i})$. By Lemma 4.4, $\tilde{\omega}^\nu_{r+\nu}$ is a relative cycle. As $f_{p,m,r}$ is a mapping of the above-mentioned pairs, $\tilde{c}^\nu_{r+\nu}$ is also a relative cycle, i.e. $\lfloor \tilde{c}^\nu_{r+\nu} \rfloor \in \tilde{H}^\nu_{r+\nu}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta}))$. Obviously, the class $[\tilde{\omega}^k_{r+k}]$ corresponds to $\lfloor \tilde{\omega}^\nu_{r+\nu} \rfloor$ under the natural isomorphism $\tilde{H}^\nu_{r+\nu}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0})) \to \tilde{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'0}))$, and the class $[\tilde{c}^k_{r+k}]$ corresponds to $\lfloor \tilde{c}^\nu_{r+\nu} \rfloor$ under the natural isomorphism

$$\tilde{H}^\nu_{r+\nu}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \to \tilde{H}^k_{r+k}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})).$$

Therefore $f_{p,m,r}$ induces a homomorphism

$$f_{p,m,r,*} : \tilde{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0})) \to \tilde{H}^k_{r+k}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})).$$

---

(7) Recall that $\tilde{\psi}_m(\tilde{Y}_m) = \Omega_m$, where $(\tilde{\psi}_m, \tilde{Y}_m t)$ is the L-chart on $\tilde{Y}$.

Let $[\tilde{1}^k_{r+k}]$ be the generator of the group $\tilde{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_1}))$ and $[\tilde{c}^k_{r+k}] = f_{p,m,r,*}[\tilde{1}^k_{r+k}]$. As $\tilde{H}^k_{r+k}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \cong \mathbb{Z}$, some number in $\mathbb{Z}$ corresponds to the element $[\tilde{c}^k_{r+k}]$. Let us denote that number by $\deg_H(f_{p,m,r})$.

Definition 6.1. The number $\deg_H(f_{p,m,r})$ is called an *L-homological degree* of the *FSL*-mapping $f_{p,m,r}$.

The sign of $\deg_H(f_{p,m,r})$ depends on the choice of the generators of the groups $\tilde{H}^k_{r+k}(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0}))$ and $\tilde{H}^k_{r+k}(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta}))$, but its absolute value is invariable. The latter fact is not important for the proof of the existence of a solution of equation (6.2) (see Theorem 6.6). One can prove that the degree of $f_{p,m,r}$ is well defined by Definition 6.1.

One can prove that $\{|\deg_H(f_{p,m,r})|\}$ stabilizes for sufficiently large r ([8]). Therefore we can give the following

Definition 6.2. $\deg_H(f_{p,m}) = \lim_{r\to\infty} |\deg_H(f_{p,m,r})|$.

Definition 6.3. $\deg_H(\tilde{f}) = \deg_H(f_{p,m})$.

As $f_{p,m} \equiv \tilde{\psi}_m \circ \tilde{f} \circ \tilde{\varphi}_p^{-1}\tilde{\psi}_m$, and $\tilde{\varphi}_p$ are homeomorphisms, the degree of $\tilde{f}$ is well defined by Definition 6.3.

Lemma 6.4. *Let* $\deg_H(f_{p,m,r}) \neq 0$. *Then the equation* (6.3) *has a solution in* $\tilde{\varphi}_p(\tilde{X}_{R'_0})$.

Proof. As $f_{p,m,r}$ is a bimorphism, it induces some finite-dimensional continuous mapping $g_{p,m,r} : V_{p,r} \to B_{m,r}$. The commutativity of the diagram

$$\begin{array}{ccc} (\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0})) & \xrightarrow{f_{p,m,r}} & (\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \\ {\scriptstyle \varphi_{p,r}}\downarrow & & \downarrow{\scriptstyle \psi_{m,r}} \\ (V_{p,r}, V_{p,r} \setminus \varphi_{p,r}(\tilde{\varphi}_p(\tilde{X}_{R'_0}))) & \xrightarrow{g_{p,m,r}} & (B_{m,r}, B_{m,r} \setminus \psi_{m,r}(\tilde{\psi}_m(\tilde{Y}_{R_0-\delta}))) \end{array}$$

yields the commutativity of

$$\begin{array}{ccc} \tilde{H}^0_r(\Delta_p, \Delta_p \setminus \tilde{\varphi}_p(\tilde{X}_{R'_0})) & \xrightarrow{f_{p,m,r,*}} & \tilde{H}^0_r(\Omega_m, \Omega_m \setminus \tilde{\psi}_m(\tilde{Y}_{R_0-\delta})) \\ {\scriptstyle \varphi_{p,r,*}}\downarrow & & \downarrow{\scriptstyle \psi_{m,r,*}} \\ \tilde{H}_r(V_{p,r}, V_{p,r} \setminus \varphi_{p,r}(\tilde{\varphi}_p(\tilde{X}_{R'_0}))) & \xrightarrow{g_{p,m,r,*}} & \tilde{H}_r(B_{m,r}, B_{m,r} \setminus \psi_{m,r}(\tilde{\psi}_m(\tilde{Y}_{R_0-\delta}))) \end{array}$$

As $\varphi_{p,r,*}$ and $\psi_{m,r,*}$ are isomorphisms (see Theorem 5.1),

$$\deg_H(f_{p,m,r}) = \deg_H(g_{p,m,r,*}).$$

([8]) Because of its length, the proof of this statement is given in the appendix.

Here $\deg_H(g_{p,m,r})$ is the homological degree of $g_{p,m,r}$. Thus, $\deg_H(g_{p,m,r}) \neq 0$ as $\deg_H(f_{p,m,r}) \neq 0$. Then, as is known from finite-dimensional analysis,

$$\exists \alpha_0' \in V_{p,r}, \quad g_{p,m,r}(\alpha_0') = \beta_0', \quad \beta_0' = \psi_{m,r}(w_0).$$

As $f_{p,m,r,\alpha_0'}$ is an isomorphism between the fibers $X_{p,\alpha_0'}$ $(X_{p,\alpha_0'} = \varphi_{p,r}^{-1}(\alpha_0'))$ and $Y_{m,\beta_0'}$ $(Y_{m,\beta_0'} = \psi_{m,r}^{-1}(\beta_0'))$ of the affine bundles $\xi_{p,r}$ and $\eta_{m,r}$, there exists a unique point $u_0 \in \varphi_{p,r}^{-1}(\alpha_0')$ such that

$$f_{p,m,r}(u_0) = w_0. \tag{6.4}$$

However, in this case, it could happen that $u_0 \notin \tilde{\varphi}_p(\tilde{X}_{R_0'})$. Let us show that this is not the case. Obviously,

$$\forall u \in \varDelta_p : \quad \|f_{p,m}(u) - f_{p,m,r}(u)\|_2 < \delta_1, \quad \delta_1 > 0,$$

for sufficiently large r. As the L-charts $\tilde{\varphi}_p$, $\tilde{\varphi}_p^{-1}$, $\tilde{\psi}_m$, $\tilde{\psi}_m^{-1}$ are uniformly continuous,

$$\forall x \in \tilde{X}_p : \quad \|\tilde{f}(x) - \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x)\|_y < \delta, \quad \delta > 0.$$

If $u_0 \notin \tilde{\varphi}_p(\tilde{X}_{R_0'})$, then $x_0 = \tilde{\varphi}_p^{-1}(u_0) \notin X_{R_0'}$, i.e. $\|x_0\|_x > R_0'$. Then it follows from the estimate (6.1) that $\|\tilde{f}(x_0)\|_y > R_0'$. As $R_0' = R_0 + 2\delta$, $R_0 = \|y_0\|_y$, we have

$$\begin{aligned}\|\tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x_0)\|_y &\geq \|\tilde{f}(x_0)\|_y - \|\tilde{f}(x_0) - \tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x_0)\|_y \\ &\geq (\|y_0\|_y + 2\delta) - \delta > \|y_0\|_y,\end{aligned}$$

i.e. $\tilde{\psi}_m^{-1} \circ f_{p,m,r} \circ \tilde{\varphi}_p(x_0) \neq y_0$, hence $f_{p,m,r}(u_0) \neq w_0$, which contradicts the equality (6.4). Thus $u_0 \in \tilde{\varphi}_p(\tilde{X}_{R_0'})$. ■

Using the local stability of $|\deg_H(f_{p,m,r})|$ it is not difficult to prove the following:

THEOREM 6.5. *Let $\{\tilde{f}_t\}$ be a family of FSQL-mappings between $\tilde{X}$ and $\tilde{Y}$, which continuously depends on $t \in [0,1]$ (uniformly in each ball) and for each $t \in [0,1]$ an a priori estimate* (6.1) *is satisfied, where the function Φ does not depend on t. Then*

$$\deg_H(\tilde{f}_1) = \deg_H(\tilde{f}_0).$$

THEOREM 6.6 ([9]). *Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be an FSQL-mapping which satisfies an a priori estimate* (6.1) *and $\deg_H(\tilde{f}) \neq 0$. Then equation* (6.2) *has a solution for each $y_0 \in \tilde{Y}$.*

Proof. Because of Definition 6.3,

$$\deg_H(f_{p,m}) \neq 0,$$

([9]) A similar theorem, for a simple case, is proved in [10].

and because of Definition 6.2,

$$\deg_H(f_{p,m,r}) \neq 0$$

for sufficiently large r. By Lemma 6.4, in this case equation (6.3) has a solution in $\tilde{\varphi}_p(\tilde{X}_{R'_0})$. Let

$$N_r = \{u \in \tilde{\varphi}_p(\tilde{X}_{R'_0}) \mid f_{p,m,r}(u) = w_0\}, \qquad N = \overline{\bigcup_{r\geq r_0} N_r}.$$

Let us prove that N is compact. First, we shall prove that N_r is compact. For this purpose we will construct its finite ε-covering. Let $u_0 \in N_r$ and $B_1(u_0,\varepsilon)$ the ball in E_1 of radius ε with center at u_0. Let us consider the function

$$P_{u_0}(\alpha') = \inf_u\{\|f_{p,m,r,\alpha'}(u) - w_0\|_2 \mid u \in X_{p,\alpha'} \setminus B_1(u_0,\varepsilon)\},$$

where $X_{p,\alpha'}$ is the fiber of the subbundle $\xi_{p,r} = (X_p, \varphi_{p,r}, V_{p,r})$ above $\alpha' \in V_{p,r}$ and $f_{p,m,r,\alpha'}$ is the restriction of $f_{p,m,r}$ to $X_{p,\alpha'}$. It is continuous in $\varphi_{p,r}(\tilde{\varphi}_p(\tilde{X}_{R'_0}))$. Let C be the constant from Definition 2.3. Then for $u \in X_{p,\alpha'_0} \setminus B_1(u_0,\varepsilon)$,

$$\begin{aligned}(6.5)\qquad \|f_{p,m,r,\alpha'_0}(u) - w_0\|_2 &= \|f_{p,m,r,\alpha'_0}(u) - f_{p,m,r,\alpha'_0}(u_0)\|_2 \\ &= \|f_{p,m,r,\alpha'_0}(u-u_0)\|_2 \geq \frac{1}{C}\cdot\|u-u_l\|_1 > \frac{\varepsilon}{C}.\end{aligned}$$

As $\|u-u_0\|_1 > \varepsilon$ we have $P_{u_0}(\alpha'_0) > \varepsilon/C$. Then there exists a neighborhood $U(\alpha'_0)$ in which

$$P_{u_0}(\alpha') > \frac{\varepsilon}{2C}.$$

Let $u \in X_{p,\alpha'} \setminus B_1(u_0,\varepsilon)$. Then

$$\begin{aligned}&\|f_{p,m,r,\alpha'}(u) - w_0\|_2 \\ &\quad= \|f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'_0}(u_0)\|_2 \\ &\quad= \|(f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'}(u_0)) + (f_{p,m,r,\alpha'}(u_0) - f_{p,m,r,\alpha'_0}(u_0))\|_2 \\ &\quad\geq \|f_{p,m,r,\alpha'}(u) - f_{p,m,r,\alpha'}(u_0)\|_2 - \|f_{p,m,r,\alpha'}(u_0) - f_{p,m,r,\alpha'_0}(u_0)\|_2 \\ &\quad\geq \|f_{p,m,r,\alpha'}(u-u_0)\|_2 - \|(f_{p,m,r,\alpha'} - f_{p,m,r,\alpha'_0})(u_0)\|_2 \\ &\quad\geq \frac{\varepsilon}{C} - \|f_{p,m,r,\alpha'} - f_{p,m,r,\alpha'_0}\|\cdot\|u_0\|_1.\end{aligned}$$

Let us denote the last difference by A. As the family $\{f_{p,m,r,\alpha'}\}$ of affine mappings is uniformly continuous in α',

$$\exists\lambda > 0, \quad \|f_{p,m,r,\alpha'} - f_{p,m,r,\alpha'_0}\| < \frac{\varepsilon}{2C\cdot\max\{\|u_\tau\|_1\}} \quad \text{if } \rho_r(\alpha',\alpha'_0) < \lambda,$$

where $u_\tau \in N_r$, and $\rho_r(\alpha', \alpha'_0)$ is a metric on $V_{p,r}$. Then

$$A > \frac{\varepsilon}{C} - \frac{\varepsilon}{2C \cdot \max\{\|u_\tau\|_1\}} \cdot \max\{\|u_\tau\|_1\} = \frac{\varepsilon}{2C} \ ({}^{10}).$$

So, the neighborhood $U(\alpha'_0)$ contains a ball $W(\alpha'_l) = \{\alpha' \mid \rho_r(\alpha', \alpha'_l) < \lambda\}$ of some radius λ, where λ depends only on ε. Therefore there exists a finite covering of the bounded finite-dimensional set $\varphi_{p,r}(N_r)$ by balls $W(\alpha'_l)$: $\varphi_{p,r}(N_r) \subset \bigcup W(\alpha'_l)$. Then the balls $B_1(u_l, \varepsilon)$ form an ε-covering of the set N_r, as for $u \notin \bigcup B_1(u_l, \varepsilon)$, $u \notin N_r$ because of (6.5).

Now we will prove that N is compact. Let

$$N_r^\varepsilon = \{u \in \tilde{\varphi}_p(\tilde{X}_{R'_0}) \mid \|f_{p,m,r}(u) - w_0\|_2 < \varepsilon\}.$$

By the definition of *FSQL*-mapping for each $\varepsilon > 0$ there exists μ such that

$$\|f_{p,m,r}(u) - f_{p,m}(u)\|_2 < \frac{\varepsilon}{8C} \quad \text{for } r \geq \mu \text{ and } u \in \tilde{\varphi}_p(\tilde{X}_{R'_0}). \tag{6.6}$$

Let $u \in N_r$, i.e. $f_{p,m,r}(u) = w_0$, and $r \geq \mu$. Then taking into account (6.6) we have

$$\begin{aligned}\|f_{p,m,\mu}(u) - w_0\|_2 &\leq \|f_{p,m,\mu}(u) - f_{p,m}(u)\|_2 + \|f_{p,m}(u) - f_{p,m,r}(u)\|_2 \\ &\quad + \|f_{p,m,r}(u) - w_0\|_2 \leq \frac{\varepsilon}{4C}.\end{aligned}$$

Hence $N_r \subset N_\mu^{\varepsilon/4C}$ at $r \geq \mu$. Therefore $N \subset N_{r_0} \cup \cdots \cup N_{\mu-1} \cup N_\mu^{\varepsilon/4C}$. Now we shall construct a finite ε-covering for N. It is already constructed for each $N_{r_0}, \ldots, N_{\mu-1}$; therefore it is sufficient to construct a finite covering only for $N_\mu^{\varepsilon/4C}$. Let $\varphi_{p,\mu}$ be the projection of $\xi_{p,\mu} = (X_p, \varphi_{p,\mu}, V_{p,\mu})$, on which $f_{p,m,\mu}$ is defined. Let us consider a ball $B_1(u_0, \varepsilon)$, where $u_0 \in N_\mu^{\varepsilon/4C}$. The intersection of $N_\mu^{\varepsilon/4C}$ with the plane X_{p,α''_0}, where $\alpha''_0 = \varphi_{p,\mu}(u_0)$, is contained in $B_1(u_0, \varepsilon/2)$. Indeed, if $u \notin B_1(u_0, \varepsilon/2)$, then $\|u - u_0\|_1 > \varepsilon/2$, hence

$$\begin{aligned}&\|f_{p,m,\mu,\alpha''_0}(u) - w_0\|_2 \\ &\qquad \geq \|f_{p,m,\mu,\alpha''_0}(u) - f_{p,m,\mu,\alpha''_0}(u_0)\|_2 - \|f_{p,m,\mu,\alpha''_0}(u_0) - w_0\|_2 \\ &\qquad \geq \|(f_{p,m,\mu,\alpha''_0})^{-1}\| \cdot \|u - u_0\|_1 - \|f_{p,m,\mu,\alpha''_0}(u_0) - w_0\|_2 \\ &\qquad \geq \frac{1}{C} \cdot \|u - u_0\|_1 - \frac{\varepsilon}{4C} \geq \frac{1}{C} \cdot \frac{\varepsilon}{2} - \frac{\varepsilon}{4C} = \frac{\varepsilon}{4C},\end{aligned}$$

i.e. $u \notin N_\mu^{\varepsilon/4C}$. This contradicts the assumption. From this it follows that for the continuous function

$$P'_{u_0}(\alpha'') = \inf_u \{\|f_{p,m,\mu,\alpha''}(u) - w_0\|_2 \mid u \in X^\mu_{p,\alpha''} \setminus B_1(u_0, \varepsilon)\},$$

---

(10) The set N_r is bounded, therefore $\max\{\|u_\tau\|_1\} < \infty$.

we have

$$P'_{u_0}(\alpha''_0) > \varepsilon/4C.$$

Hence, as above, from the covering $N_\mu^{\varepsilon/4C}$ by balls $B_1(u,\varepsilon)$, one can select a finite subcovering. As ε is arbitrary, it is proved that N is compact.

Now let $\{u_r\} \subset \tilde{\varphi}_p(\tilde{X}_{R'_0})$ be some sequence of solutions of (6.3). As $\{u_r\} \subset N$, there exists a subsequence converging to some $u_0 \in N$. As $\{f_{p,m,r}\}$ uniformly converges to $f_{p,m}$ in $\tilde{\varphi}_p(\tilde{X}_{R'_0})$, $f_{p,m}(u_0) = w_0$. Therefore, $\tilde{f}(x_0) = y_0$, where $x_0 = \tilde{\varphi}_p^{-1}(u_0)$, i.e. x_0 is a solution of equation (6.2). ∎

7. Appendix. The proof of stabilization of $\{|\deg_H(f_{p,m,r})|\}$**.** First we recall that η_m is embedded in the Banach space E_2. Let $f_{p,m,r'} : \xi_{p,r'} \to \eta_{m,r'}$ and $f_{p,m,r''} : \xi_{p,r''} \to \eta_{m,r''}$ be two *FSL*-mappings which are close enough to each other in $\Delta_p \subset X_p$. Without restriction of generality one can suppose that $f_{p,m,r'} : \xi_{p,\nu} \to \eta_{m,\nu,1}$ and $f_{p,m,r''} : \xi_{p,\nu} \to \eta_{m,\nu,2}$ are bimorphisms between the aforesaid bundles, where $\xi_{p,\nu} = (X_p, \varphi_{p,\nu}, V_{p,\nu})$, $\nu \geq r', r''$, is a common division of $\xi_{p,r'}$, $\xi_{p,r''}$ and $\eta_{m,\nu,1}$, $\eta_{m,\nu,2}$ are divisions of η_m of the same codimension ν. Let us introduce the following notations:

We denote the mappings $f_{p,m,r'}$ and $f_{p,m,r''}$ by $f_{p,m,\nu,1}$ and $f_{p,m,\nu,2}$, respectively. We denote the fibers of $\xi_{p,\nu} = (X_p, \varphi_{p,\nu}, V_{p,\nu})$ by $X_{p,\alpha}$:

$$X_{p,\alpha} = \varphi_{p,\nu}^{-1}(\alpha), \quad \alpha \in V_{p,\nu}.$$

We denote the fibers of $\eta_m = (Y_m, \psi_m, B_m)$ by $Y_{m,\beta}$:

$$Y_{m,\beta} = \psi_m^{-1}(\beta), \quad \beta \in B_m.$$

We denote the fibers of $\eta_{m,\nu,1} = (Y_m, \psi_{m,\nu,1}, B_{m,\nu,1})$ by $Y_{m,\nu,\beta_1,1}$:

$$Y_{m,\nu,\beta_1,1} = \psi_{m,\nu,1}^{-1}(\beta_1), \quad \beta_1 \in B_{m,\nu,1}.$$

We denote the fibers of $\eta_{m,\nu,2} = (Y_m, \psi_{m,\nu,2}, B_{m,\nu,2})$ by $Y_{m,\nu,\beta_2,2}$:

$$Y_{m,\nu,\beta_2,2} = \psi_{m,\nu,2}^{-1}(\beta_2), \quad \beta_2 \in B_{m,\nu,2}.$$

Finally

$$f_{p,m,\nu,1}(X_{p,\alpha}) = Y_{m,\nu,\beta_1(\alpha),1}, \quad f_{p,m,\nu,2}(X_{p,\alpha}) = Y_{m,\nu,\beta_2(\alpha),2}.$$

As $f_{p,m,\nu,1}$ and $f_{p,m,\nu,2}$ are close to each other in Δ_p, the fibers $Y_{m,\nu,\beta_1(\alpha),1}$ and $Y_{m,\nu,\beta_2(\alpha),2}$ are also close to each other for any $\alpha \in V_{p,\nu}$, i.e.

$$\begin{aligned}&\operatorname{dist}(Y_{m,\nu,\beta_1(\alpha),1}, Y_{m,\nu,\beta_2(\alpha),2})\\ &\quad = \sup\{\rho(w, Y'_{m,\nu,\beta_1(\alpha),1}) \mid w \in Y'_{m,\nu,\beta_2(\alpha),2} \cap B_2(1))\} < \varepsilon, \quad \varepsilon > 0 \ (^{11}).\end{aligned}$$

(11) Here $Y'_{m,\nu,\beta_1(\alpha),1}$, $Y'_{m,\nu,\beta_2(\alpha),2}$ are the subspaces of E_2 which are parallel translates of $Y_{m,\nu,\beta_1(\alpha),1}$, $Y_{m,\nu,\beta_2(\alpha),2}$ respectively through the origin of E_2, $B_2(1)$ is the ball of radius one in E_2 with center at zero, and $\rho(w, Y'_{m,\nu,\beta_1(\alpha),1})$ is the distance between w and $Y'_{m,\nu,\beta_1(\alpha),1}$.

Therefore $Y_{m,\nu,\beta_2(\alpha),2}$ is close to $Y_{m,\beta(\alpha)}$, which contains $Y_{m,\nu,\beta_1(\alpha),1}$. Then it is possible to take the orthogonal projection of each fiber $Y_{m,\nu,\beta_2(\alpha),2}$ onto $Y_{m,\beta(\alpha)}$. Let us denote this projection by $\pi_{\beta(\alpha)}$, $\alpha \in V_{p,\nu}$. By construction:

1) $\pi_{\beta(\alpha)}$ is an affine isomorphism between $Y_{m,\nu,\beta_2(\alpha),2}$ and its image.
2) $\pi = \{\pi_{\beta(\alpha)} \mid \alpha \in V_{p,\nu}\}$ is an isomorphism between $\{Y_{m,\nu,\beta_2(\alpha),2}\}$ and its image.
3) $f_{p,m,\nu,3} = \pi \circ f_{p,m,\nu,2}$ is an *FSL*-mapping.
4) The mappings $f_{p,m,\nu,3}$ and $f_{p,m,\nu,2}$ are close to each other in Δ_p, hence $f_{p,m,\nu,3}$ is close to $f_{p,m,\nu,1}$ in Δ_p.

Remark. The difference between the mappings $f_{p,m,\nu,3}$ and $f_{p,m,\nu,2}$ is that $f_{p,m,\nu,1}(X_{p,\alpha})$ and $f_{p,m,\nu,3}(X_{p,\alpha})$ are contained in the same $Y_{m,\beta(\alpha)}$ for each $\alpha \in V_{p,\nu}$.

As all the mappings obeying $f_{p,m,\nu,3} = \pi \circ f_{p,m,\nu,2}$ are *FSL*-mappings and π is an isomorphism,

$$\begin{aligned}\deg_H f_{p,m,\nu,3} &= \deg_H(\pi \circ f_{p,m,\nu,2}) \\ &= (\deg_H \pi)\cdot(\deg_H f_{p,m,\nu,2}) = \deg_H f_{p,m,\nu,2}.\end{aligned}$$

Now let us prove that

$$\deg_H f_{p,m,\nu,3} = \deg_H f_{p,m,\nu,1}.$$

For this purpose we take an $(\nu+1,k)$-prism $\sigma^k_{\nu+1} = \sigma^k_\nu \times I_1$, where I_1 is a 1-cube, that is, a line segment. We will consider the singular $(\nu+1,k)$-chain

$$\tilde\sigma^k_{\nu+1} = \sum g_i \cdot [t \cdot f_{p,m,\nu,1} \circ f^k_{\nu,i}(u) + (1-t) \circ f_{p,m,\nu,3} \circ f^k_{\nu,i}(u)] \ (^{12}).$$

One can show that $\tilde\partial^k_{\nu+1}\tilde\sigma^k_{\nu+1} \in \tilde C^k_\nu(X_p \setminus \Delta_p)$, i.e. the relative cycles $\sigma^k_{\nu,1} = \sum g_i \cdot f_{p,m,\nu,1} \circ f^k_{\nu,i}$ and $\tilde\sigma^k_{\nu,3} = \sum g_i \cdot f_{p,m,\nu,3} \circ f^k_{\nu,i}$ are homologous to each other relative to $X_p \setminus \Delta_p$. Hence

$$\deg_H f_{p,m,\nu,3} = \deg_H f_{p,m,\nu,1}.$$

Therefore

$$\deg_H f_{p,m,\nu,2} = \deg_H f_{p,m,\nu,1}.$$

Thus,

$$\deg_H f_{p,m,r'} = \deg_H f_{p,m,r''}$$

for sufficiently large r' and r''.

Acknowledgments. I would like to thank Dr. Ibrahim Turkyilmaz, Nazmi Yaganoglu and Yavuz Yildirim for their assistance in preparing this manuscript. I am also grateful to an anonymous referee for extremely helpful comments and suggestions.

(^{12}) Because of the note mentioned above, $\tilde\sigma^k_{\nu+1}$ is a chain in X_p.

References

[1] A. Abbasov, *A special quasi-linear mapping and its degree*, Turkish J. Math. 24 (2000), 1–14.

[2] —, *Quasi-linear manifolds and quasi-linear mapping between them*, ibid. 28 (2004), 1–11.

[3] —, *The homological theory of degree of FQL-mappings*, ibid. 30 (2006), 129–138.

[4] Yu. G. Borisovich, V. G. Zvyagin and Yu. I. Sapronov, *Nonlinear Fredholm mappings and Leray–Schauder theory*, Uspekhi Mat. Nauk 32 (1977), no. 4, 3–54 (in Russian).

[5] D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. 92 (1970), 102–163.

[6] J. Eells, *Fredholm structures*, in: Proc. Sympos. Pure Math. 18, Amer. Math. Soc. Providence, RI, 1970, 62–85.

[7] M. A. Efendiev, *The degree of a Fredholm quasilinear mapping of quasicylindrical domains and the nonlinear Hilbert problem in an annulus*, Izv. Akad. Nauk Azerbaĭdzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk 1979, no. 5, 18–23 (in Russian).

[8] Yu. E. Gliklikh, *Analysis on Riemannian Manifolds and Problems of Mathematical Physics*, Univ. of Voronezh, 1989 (in Russian).

[9] M. W. Hirsch, *Differential Topology*, Springer, 1976.

[10] A. I. Shnirelman, *The degree of quasi-linear mapping and the nonlinear Hilbert problem*, Mat. Sb. 89 (1972), 336–389 (in Russian).

A. Abbasov
Faculty of Economics and Administrative Sciences
Çanakkale Onsekiz Mart University
Biga, Çanakkale, Turkey
E-mail: abakif@hotmail.com

Received 8.6.2009
and in final form 25.3.2010 (2026)