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Probability distribution solutions of a general linear equation of infinite order

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Abstract. Let $(\Omega,\mathcal{A},P)$ be a probability space and let $\tau\colon \mathbb{R}\times\Omega\to\mathbb{R}$ be strictly increasing and continuous with respect to the first variable, and $\mathcal{A}$-measurable with respect to the second variable. We obtain a partial characterization and a uniqueness-type result for solutions of the general linear equation

$$F(x)=\int_{\Omega} F(\tau(x,\omega))P(d\omega)$$

in the class of probability distribution functions.

1. Introduction. In this paper we deal with the linear functional equation

$$F(x)=\int_{\Omega} F(\tau(x,\omega))P(d\omega). \tag{1}$$

Several particular cases of (1) appear in various areas of applications. For instance, in the case where $\tau(x,\omega)=x+\omega$ the corresponding equation, called the Integrated Cauchy Functional Equation, is of importance in probability theory (see [27], [28]). G. Choquet and J. Deny were the first to consider that version of (1) (see [3], [9]). The case $\tau(x,\omega)=\alpha x+\omega$ is closely connected with refinement equations (see [8], [15], [26]), which generate wavelets bases (see [4], [7], [20]) and splines (see [6], [19]). They are also fundamental to subdivision schemes (see [5], [10]). Equation (1) also appears in such areas of mathematics as iterated function systems (see [12], [14]), Markov chains (see [11], [21]) and perpetuities (see [13], [16], [29]).

For more information about results concerning equation (1) the reader is referred to the survey paper [1], and to [17], [18] for a complete theory of iterative functional equations.

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In the present paper we deal with the following problem: what can be said about uniqueness and properties of probability distribution (p.d.) solutions of (1) assuming only reasonable conditions on the given mapping τ? We establish a uniqueness-type result which allows us to determine all p.d. solutions, provided we know all continuous p.d. solutions satisfying some special boundary conditions.

2. Preliminaries. Throughout the paper, $(\Omega, \mathcal{A}, P)$ is a probability space and $\tau\colon \mathbb{R} \times \Omega \to \mathbb{R}$ is a mapping such that for every $x \in \mathbb{R}$ the function $\tau(x, \cdot)$ is $\mathcal{A}$-measurable, and for every $\omega \in \Omega$ the function $\tau(\cdot, \omega)$ is strictly increasing and continuous.

We are interested in the following two classes of solutions of (1):

$$\mathcal{I} := \{F\colon \mathbb{R} \to [0,1] \mid F \text{ is a weakly increasing solution of (1) such that } F(-\infty) := \lim_{x\to-\infty} F(x) = 0 \text{ and } F(+\infty) := \lim_{x\to+\infty} F(x) = 1\},$$

$$\mathcal{C} := \{F \in \mathcal{I} : F \text{ is continuous}\}.$$

It will be convenient to consider equation (1) in a more general situation. If $I \subset \mathbb{R}$ is an interval and $\sigma\colon I \times \Omega \to I$ is a mapping which is weakly increasing and continuous with respect to the first variable, and $\mathcal{A}$-measurable with respect to the second variable, then we rewrite (1) as

$$F(x) = \int_{\Omega} F(\sigma(x, \omega))\, P(d\omega). \tag{2}$$

We denote by $\mathcal{C}_\sigma(I)$ the class of all continuous and weakly increasing solutions $F\colon I \to \mathbb{R}$ of (2), and put

$$\mathcal{C}^0_\sigma(I) = \{F \in \mathcal{C}_\sigma(I) : \lim_{x\to\inf I} F(x) = 0 \text{ and } \lim_{x\to\sup I} F(x) = 1\}.$$

We say that a subset S of I is σ-*invariant* if $S \neq \emptyset$ and for every $x \in S$ we have $\sigma(x, \omega) \in S$ for almost all $\omega \in \Omega$.

Given a σ-invariant subinterval J of I define a mapping $\sigma_J\colon J \times \Omega \to J$ by putting $\sigma_J(x, \omega) = \sigma(x, \omega)$ if $\sigma(x, \omega) \in J$, and $\sigma_J(x, \omega) = 0$ otherwise. It is evident that for every function $F\colon I \to [0,1]$ we have $F|_J \in \mathcal{C}^0_{\sigma_J}(J)$ if and only if $F \in \mathcal{C}^0_\sigma(I)$, $\lim_{x\to\inf J} F(x) = 0$ and $\lim_{x\to\sup J} F(x) = 1$. Therefore, for every σ-invariant subinterval J of I we will use the symbol $\mathcal{C}^0_\sigma(J)$ instead of $\mathcal{C}^0_{\sigma_J}(J)$.

Define

$$\mathbf{E}_\sigma = \{x \in I : \sigma(x, \omega) = x \text{ for almost all } \omega \in \Omega\}.$$

Clearly, $\mathbf{E}_\sigma$ is closed. Let $\mathcal{U}_\sigma$ be the family of all open components of $I \setminus \mathbf{E}_\sigma$. Note that each such component is a σ-invariant interval disjoint from $\mathbf{E}_\sigma$.

We now quote the main result from [24] which is the first step in determining the class $\mathcal{I}$ (cf. also [23] where a result of similar type was established in a very particular case of (1)).

THEOREM 1 (see [24, Theorem 2]).

(i) *If* $\mathbf{E}_\tau = \emptyset$, *then* $\mathcal{C} = \mathcal{I}$.

(ii) *If* $\mathbf{E}_\tau \neq \emptyset$, *then* $\mathcal{C} \subsetneq \mathcal{I}$. *Moreover, a function* $F\colon \mathbb{R} \to [0,1]$ *belongs to* $\mathcal{I}$ *if and only if it is weakly increasing,* $F(-\infty) = 0$, $F(+\infty) = 1$ *and on every component* $J \in \mathcal{U}_\tau$, *either* F *is constant or the function given by*

$$F_J(x) = \frac{F(x) - F(\inf J)}{F(\sup J) - F(\inf J)} \tag{3}$$

belongs to $\mathcal{C}_\tau^0(J)$.

We see that p.d. solutions of (1) may be defined arbitrarily on $\mathbf{E}_\tau$ (they just have to meet the requirements in Theorem 1(ii)), whereas their behaviour on every component $J \in \mathcal{U}_\tau$ is determined by functions from $\mathcal{C}_\tau^0(J)$. It turns out that all functions belonging to that class may be described by functions from $\mathcal{C}_\sigma^0(\mathbb{R})$ with a suitable $\sigma\colon \mathbb{R} \times \Omega \to \mathbb{R}$ satisfying $\mathbf{E}_\sigma = \emptyset$. To see this, fix $J \in \mathcal{U}_\tau$, any increasing homeomorphism $\phi_J\colon \mathbb{R} \to J$ and define

$$\sigma(\cdot,\omega) = \phi_J^{-1} \circ \tau(\cdot,\omega) \circ \phi_J. \tag{4}$$

Plainly, σ is strictly increasing and continuous with respect to the first variable, and $\mathcal{A}$-measurable with respect to the second. A simple calculation shows that $F_J \in \mathcal{C}_\tau^0(J)$ if and only if $F_J \circ \phi_J \in \mathcal{C}_\sigma^0(\mathbb{R})$. Moreover, τ-invariant subsets $S \subset J$ are in one-to-one correspondence with σ-invariant sets $\phi_J^{-1}(S)$. In particular, since $J \cap \mathbf{E}_\tau = \emptyset$, we have $\mathbf{E}_\sigma = \emptyset$.

The above argument, jointly with Theorem 1, justifies the assumption $\mathbf{E}_\tau = \emptyset$, which we will adopt from now on.

In Section 3 we prove the main result of this paper. In Section 4 we show how it can be used to describe solutions from the class $\mathcal{I}$ in terms of solutions from a very special subclass (see Corollary 2). We finish the paper with an example, included in Section 5, which demonstrates an application of our results.

3. Uniqueness-type theorem.

Let

$$\mathcal{S}_\sigma = \{S \subset I : S \text{ is a minimal compact } \sigma\text{-invariant interval}\}.$$

The main result of this paper reads as follows.

THEOREM 2. *Assume* $\mathbf{E}_\tau = \emptyset$. *Every* $F \in \mathcal{I}$ *is constant on each interval from* $\mathcal{S}_\tau$. *Moreover, for every* $f : \mathcal{S}_\tau \to [0,1]$ *there is at most one* $F \in \mathcal{I}$ *such that* $F|_I = f(I)$ *for all* $I \in \mathcal{S}_\tau$.

Let us stress that $\mathcal{S}_\tau = \emptyset$ may happen. In such a case (1) has at most one solution in the class of all p.d. functions. Of course, the "monotonicity" of the function f is essential to produce a p.d. solution F.

Proof. For transparency we divide the proof into several parts.

Claim 1. *It is enough to prove the assertion of Theorem 2 under the assumption that F is a continuous p.d. function.*

This follows immediately from assertion (i) of Theorem 1.

In Claims 2–5 we constantly assume the following: $-\infty \le \alpha < \beta \le +\infty$, $I = \mathrm{cl}\,(\alpha, \beta)$ (here and below, cl stands for closure in $\mathbb{R}$), and $\sigma\colon I \times \Omega \to I$ is a mapping which is weakly increasing and continuous with respect to the first variable, $\mathcal{A}$-measurable with respect to the second variable, and such that $\mathbf{E}_\sigma = \emptyset$. We recall that $F(\pm\infty)$ always stands for $\lim_{x\to\pm\infty} F(x)$.

Claim 2. *If there are distinct $F, G \in \mathcal{C}_\sigma(I)$ such that $F(\alpha) = G(\alpha)$ and $F(\beta) = G(\beta)$, then $\mathcal{S}_\sigma$ is non-void.*

Put

$$\begin{aligned} M &= \sup\{|F(x) - G(x)| : x \in I\} > 0,\\ S &= \{x \in I : |F(x) - G(x)| = M\},\\ S_n &= \{x \in I : |F(x) - G(x)| \le M - 1/n\} \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

Evidently, S is a non-void and compact subset of I, and $I \setminus S = \bigcup_{n\in\mathbb{N}} S_n$. Let

$$N = \{x \in I : P(\sigma(x,\omega) \in S) = 1\}.$$

Assume that there exists $x_0 \in I \setminus N$. This means that $P(\sigma(x_0,\omega) \notin S) > 0$, and thus

$$\alpha_0 := P(\sigma(x_0,\omega) \in S_{n_0}) > 0$$

for sufficiently large $n_0 \in \mathbb{N}$. Set

$$\Omega_0 = \{\omega \in \Omega : \sigma(x_0,\omega) \in S_{n_0}\}.$$

Then equation (2) implies

$$\begin{aligned} |F(x_0) - G(x_0)| &\le \int_\Omega |F(\sigma(x_0,\omega)) - G(\sigma(x_0,\omega))|\, P(d\omega)\\ &= \int_{\Omega_0} + \int_{\Omega\setminus\Omega_0} \le \alpha_0\left(M - \frac{1}{n_0}\right) + (1-\alpha_0)M < M, \end{aligned}$$

which shows that $x_0 \notin S$. We infer that $S \subset N$, hence S is σ-invariant.

If $s_1 := \inf S$ and $s_2 := \sup S$, then $\sigma(s_1,\omega) \ge s_1$ and $\sigma(s_2,\omega) \le s_2$ for almost all $\omega \in \Omega$, which, jointly with monotonicity of σ, implies that the interval $[s_1, s_2]$ is σ-invariant.

It remains to apply the Zorn–Kuratowski lemma to the family

$$\{S \subset I : S \text{ is a compact and } \sigma\text{-invariant interval}\}.$$

From now on $\widetilde{I}$ stands for an element of $\mathcal{S}_\sigma$.

Claim 3. *Define* $\phi\colon \widetilde{I} \to \widetilde{I}$ *by*

$$\phi(x) = \sup\{y \in \widetilde{I} : P(\sigma(x,\omega) \geq y) > 0\}.$$

Then:

(i) ϕ *is weakly increasing and left-continuous*;
(ii) *for every* $x \in [\inf \widetilde{I}, \sup \widetilde{I})$ *we have* $x < \phi(x)$.

The fact that ϕ is weakly increasing is an easy consequence of the fact that σ weakly increases as a function of the first variable.

For the left-continuity suppose, on the contrary, that $x_0 \in \widetilde{I}$ and there exists a strictly increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $\widetilde{I}$ such that

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \gamma := \lim_{n\to\infty} \phi(x_n) < \phi(x_0).$$

Choose any numbers ν, ξ such that $\gamma < \nu < \xi < \phi(x_0)$. By the definition of ϕ, the set

$$C := \{\omega \in \Omega : \sigma(x_0,\omega) \geq \xi\}$$

has a positive measure. Let

$$C_n = \{\omega \in C : \sigma(x_n,\omega) \geq \nu\} \quad \text{for } n \in \mathbb{N}.$$

The continuity of σ as a function of the first variable yields

$$\bigcup_{n\in\mathbb{N}} C_n = C.$$

Since $\phi(x_n) < \nu$, we have $P(C_n) = 0$ for $n \in \mathbb{N}$, hence $P(C) = 0$; a contradiction.

Finally, suppose that $\phi(x) \leq x$ for some $x \in [\inf \widetilde{I}, \sup \widetilde{I})$. Then by the definition of ϕ, no $y \in \widetilde{I}$ with $P(\sigma(x,\omega) \geq y) > 0$ exceeds x. Hence

$$P(\sigma(x,\omega) > x) \leq \sum_{n\in\mathbb{N}} P(\sigma(x,\omega) \geq x + 1/n) = 0,$$

which means that $\sigma(x,\omega) \leq x$ for almost all $\omega \in \Omega$. However, the monotonicity of σ with respect to the first variable would then imply that the interval $[\inf \widetilde{I}, x]$ is σ-invariant and $[\inf \widetilde{I}, x] \subsetneq \widetilde{I}$, which contradicts the fact that $\widetilde{I}$ is minimal.

Claim 4. *Define* $\psi\colon \widetilde{I} \to \widetilde{I}$ *by*

$$\psi(x) = \tfrac{1}{2}(x + \phi(x)).$$

Let ψ^n *stand for the nth iterate of* ψ. *Then*:

(i) $\psi^n(\inf \widetilde{I}) < \sup \widetilde{I}$ *for* $n \geq 0$;
(ii) *the sequence* $(\psi^n(\inf \widetilde{I}))_{n\geq 0}$ *is strictly increasing*;
(iii) $\lim_{n\to\infty} \psi^n(\inf \widetilde{I}) = \sup \widetilde{I}$.

Inequality (i) follows directly from the formula of ψ and the fact that the interval $\widetilde{I}$ is non-degenerate (which is a consequence of the assumption $\mathbf{E}_\sigma = \emptyset$).

With the aid of assertion (i) and Claim 3(i) we easily obtain (ii).

For the proof of (iii) set $\gamma = \lim_{n\to\infty} \psi^n(\inf \widetilde{I})$ and suppose that $\gamma < \sup \widetilde{I}$. From the equality

$$\psi^{n+1}(\inf \widetilde{I}) = \tfrac{1}{2}(\psi^n(\inf \widetilde{I}) + \phi(\psi^n(\inf \widetilde{I}))) \quad \text{for } n \in \mathbb{N}$$

we get $\lim_{n\to\infty} \phi(\psi^n(\inf \widetilde{I})) = \gamma$. However, by (ii) and Claim 3(i), the last limit equals $\phi(\gamma)$ and we obtain $\phi(\gamma) = \gamma$, which contradicts Claim 3(ii).

Claim 5. *If $F \in \mathcal{C}_\sigma(I)$, then F is constant on $\widetilde{I}$.*

For $n \geq 0$ let $J_n = [\psi^n(\inf \widetilde{I}), \psi^{n+1}(\inf \widetilde{I})]$. In the light of Claim 4, it suffices to prove that $F|_{J_n}$ is constant for every $n \geq 0$. Put $\xi_n = \psi^n(\inf \widetilde{I})$. Assume inductively that $F(\xi_n) = F(\inf \widetilde{I})$ (which is trivial for $n = 0$) and fix $x \in J_n = [\xi_n, \psi(\xi_n)]$. By Claims 3(ii) and 4(i), we infer that $\psi(\xi_n) < \phi(\xi_n)$, hence the set

$$\Omega_0 := \{\omega \in \Omega : \sigma(\xi_n, \omega) \geq x\}$$

is of a positive probability α_0. Since $F \in \mathcal{C}_\sigma(I)$ and $\sigma(\xi_n, \omega) \geq \inf \widetilde{I}$ for almost all $\omega \in \Omega$, we have

$$\begin{aligned} F(\inf \widetilde{I}) = F(\xi_n) &= \int_\Omega F(\sigma(\xi_n, \omega))\, P(d\omega) = \int_{\Omega_0} + \int_{\Omega\setminus\Omega_0} \\ &\geq \alpha_0 F(x) + (1-\alpha_0) F(\inf \widetilde{I}). \end{aligned}$$

This implies that $F(x) \leq F(\inf \widetilde{I})$, thus $F(x) = F(\inf \widetilde{I})$.

Before we proceed with the proof, let us introduce some notation. If $S \subset \mathbb{R}$ is a τ-invariant interval such that every $F \in \mathcal{C}_\tau(\mathbb{R})$ is constant on S, let $\kappa(S)$ denote a maximal τ-invariant interval such that $S \subset \kappa(S)$ and every $F \in \mathcal{C}_\tau(\mathbb{R})$ is constant on $\kappa(S)$. Obviously, such an interval exists, and the continuity of functions from $\mathcal{C}_\tau(\mathbb{R})$ and of $\tau(\cdot, \omega)$ for $\omega \in \Omega$ implies that it is a closed interval. By Claim 5 (applied for $I = \mathbb{R}$ and $\sigma = \tau$), the symbol $\kappa(S)$ makes sense for every $S \in \mathcal{S}_\tau$. Define

$$\mathcal{M} = \{J \subset \mathbb{R} : J \text{ is a maximal } \tau\text{-invariant interval such that every } F \in \mathcal{C}_\tau(\mathbb{R}) \text{ is constant on } J\}.$$

The families $\mathcal{S}_\tau$, $\kappa(\mathcal{S}_\tau)$, $\mathcal{M}$ each consist of pairwise disjoint non-degenerate closed intervals.

Claim 6. *We have*:

(i) $\kappa(\mathcal{S}_\tau) \subset \mathcal{M}$;

(ii) $\{J \in \mathcal{M} : J$ *is compact*$\} \subset \kappa(\mathcal{S}_\tau)$.

The first assertion is clear. For the second, observe that if $J \in \mathcal{M}$ is compact, then there is $S \in \mathcal{S}_\tau$ with $S \subset J$. Plainly, $\kappa(S) = J$, so $J \in \kappa(\mathcal{S}_\tau)$.

Claim 7. *The set $\bigcup \mathcal{M}$ is closed.*

Suppose that there is $x_0 \in (\mathrm{cl} \bigcup \mathcal{M}) \setminus \bigcup \mathcal{M}$. Then there exists either an increasing sequence of right end-points of intervals from $\mathcal{M}$ which converges to x_0, or a decreasing sequence of left end-points of intervals from $\mathcal{M}$ which converges to x_0. Without loss of generality, assume that the latter case holds true and let $(I_n)_{n\in\mathbb{N}}$ be a sequence of intervals from $\mathcal{M}$ such that $\inf I_{n+1} < \sup I_{n+1} < \inf I_n$ for $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \inf I_n = x_0 = \lim_{n\to\infty} \sup I_n.$$

Since all the intervals I_n are τ-invariant, we infer that

$$P(\tau(\inf I_n, \omega) \geq \inf I_n) = 1 \quad \text{and} \quad P(\tau(\sup I_n, \omega) \leq \sup I_n) = 1,$$

hence $\tau(x_0, \omega) = x_0$ for almost all $\omega \in \Omega$, contrary to the fact that $\mathbf{E}_\tau = \emptyset$.

Claim 8. *For every $g\colon \mathcal{M} \to [0,1]$ there exists at most one $F \in \mathcal{C}$ such that $F|_I = g(I)$ for all $I \in \mathcal{M}$.*

Suppose $F, G \in \mathcal{C}$, $F \neq G$ and $F|_I = g(I) = G|_I$ for all $I \in \mathcal{M}$. By Claim 7, the set $\mathbb{R} \setminus \bigcup \mathcal{M}$ is open. Choose any of its components, (α, β), on which F and G do not coincide.

Let $I = \mathrm{cl}\,(\alpha, \beta)$ and $\widetilde{F} = F|_I$, $\widetilde{G} = G|_I$. It is obvious that $\widetilde{F}$ and $\widetilde{G}$ are continuous, weakly increasing and $\widetilde{F}(\alpha) = \widetilde{G}(\alpha)$, $\widetilde{F}(\beta) = \widetilde{G}(\beta)$. Define a mapping $\sigma\colon I \times \Omega \to I$ as follows. For every $x \in I$ and $\omega \in \Omega$ put

$$\sigma(x,\omega) = \begin{cases} \tau(x,\omega) & \text{if } \tau(x,\omega) \in I, \\ \alpha & \text{if } \tau(x,\omega) < \alpha, \\ \beta & \text{if } \tau(x,\omega) > \beta. \end{cases}$$

It is easily seen that σ is weakly increasing and continuous with respect to the first variable, and $\mathcal{A}$-measurable with respect to the second. Moreover, $\mathbf{E}_\sigma = \emptyset$. Now, we are going to verify that $\widetilde{F}$ and $\widetilde{G}$ satisfy (2).

Fix $x \in I$. Assume that $\beta < +\infty$; then $\beta \in \bigcup \mathcal{M}$, so it is a lower bound of one of the intervals from $\mathcal{M}$, say $I_{t_0} = \mathrm{cl}\,[\beta, \sup I_{t_0})$. This implies that

$$P(\tau(\beta,\omega) \leq \sup I_{t_0}) = 1,$$

and therefore

$$P(\tau(x,\omega) \leq \sup I_{t_0}) = 1. \tag{5}$$

Directly from the definition of σ we infer that

$$\int\limits_{\{\tau(x,\omega)>\beta\}} \widetilde{F}(\sigma(x,\omega))\, P(d\omega) = P(\tau(x,\omega) > \beta) \cdot F(\beta).$$

Condition (5) implies that $\tau(x,\omega) \in I_{t_0}$ for almost all $\omega \in \{\tau(x,\omega) > \beta\}$. Since F is constant on the interval I_{t_0}, we have

$$\int\limits_{\{\tau(x,\omega)>\beta\}} F(\tau(x,\omega))\, P(d\omega) = P(\tau(x,\omega) > \beta) \cdot F(\beta).$$

Hence

$$\int\limits_{\{\tau(x,\omega)>\beta\}} \widetilde{F}(\sigma(x,\omega))\, P(d\omega) = \int\limits_{\{\tau(x,\omega)>\beta\}} F(\tau(x,\omega))\, P(d\omega). \tag{6}$$

In the case where $\beta = +\infty$ the above equality is trivial. Analogously we show that

$$\int\limits_{\{\tau(x,\omega)<\alpha\}} \widetilde{F}(\sigma(x,\omega))\, P(d\omega) = \int\limits_{\{\tau(x,\omega)<\alpha\}} F(\tau(x,\omega))\, P(d\omega). \tag{7}$$

Plainly,

$$\int\limits_{\{\tau(x,\omega)\in I\}} \widetilde{F}(\sigma(x,\omega))\, P(d\omega) = \int\limits_{\{\tau(x,\omega)\in I\}} F(\tau(x,\omega))\, P(d\omega). \tag{8}$$

Summing up equations (6)–(8) we obtain

$$\int\limits_{\Omega} \widetilde{F}(\sigma(x,\omega))\, P(d\omega) = \int\limits_{\Omega} F(\tau(x,\omega))\, P(d\omega),$$

which shows that $\widetilde{F}$ (and $\widetilde{G}$ as well) satisfies (2). Consequently, $\widetilde{F}, \widetilde{G} \in \mathcal{C}_\sigma(I)$.

By Claim 2, there exists $\widetilde{I} \subset I$ such that $\widetilde{I} \in \mathcal{S}_\sigma$. We have just proved that for every $F \in \mathcal{C}$ its restriction $\widetilde{F} = F|_I$ belongs to $\mathcal{C}_\sigma(I)$, thus Claim 5 shows that $\widetilde{F}$, and so F itself, is constant on $\widetilde{I}$. Consequently, the symbol $\kappa(\widetilde{I})$ makes sense.

Fix $x \in \widetilde{I}$. Since $\widetilde{I}$ is σ-invariant, we have

$$\begin{aligned} P(\sigma(x,\omega) \le \beta) &\ge P(\sigma(\sup \widetilde{I},\omega) \le \beta) \ge P(\sigma(\sup \widetilde{I},\omega) \le \sup \widetilde{I}) = 1,\\ P(\sigma(x,\omega) \ge \alpha) &\ge P(\sigma(\inf \widetilde{I},\omega) \ge \alpha) \ge P(\sigma(\inf \widetilde{I},\omega) \ge \inf \widetilde{I}) = 1. \end{aligned}$$

Hence for all $x \in \widetilde{I}$ and almost all $\omega \in \Omega$ we have $\sigma(x,\omega) = \tau(x,\omega)$, which implies that $\widetilde{I}$ is τ-invariant, so $\kappa(\widetilde{I}) \in \mathcal{M}$, a contradiction.

Claim 9. *For every $f\colon \mathcal{S}_\tau \to [0,1]$ there exists at most one $F \in \mathcal{C}$ such that $F|_I = f(I)$ for all $I \in \mathcal{S}_\tau$.*

Suppose that there is $F \in \mathcal{C}$ satisfying $F|_I = f(I)$ for all $I \in \mathcal{S}_\tau$. Define $g\colon \mathcal{M} \to [0,1]$ by

$$g(J) = \begin{cases} f(I) & \text{if } J = \kappa(I) \text{ for some } I \in \mathcal{S}_\tau,\\ 0 & \text{if } J = (-\infty, a] \text{ for some } a \in \mathbb{R},\\ 1 & \text{if } J = [b, +\infty) \text{ for some } b \in \mathbb{R}. \end{cases}$$

In view of Claim 6 and the fact that at least one solution of (1) exists, the definition is correct. Of course, $F|_J = g(J)$ for all $J \in \mathcal{M}$ and Claim 8 implies that F is uniquely determined.

This completes the proof of Theorem 2. ∎

4. Concluding remarks. The following is an immediate consequence of Theorem 2.

COROLLARY 1. *If $\mathbf{E}_\tau = \emptyset$ and there exists a strictly increasing function $F \in \mathcal{I}$, then $\mathcal{I} = \{F\}$.*

Observe that a proof similar to that of Claim 7 shows that $\bigcup \mathcal{S}_\tau$ is closed. Consider any component J of the open set $\mathbb{R} \setminus \bigcup \mathcal{S}_\tau$. Let $\widetilde{F}_J$ stand for a function from $\mathcal{C}$ such that

$$\lim_{x\to\inf J} \widetilde{F}_J(x) = 0 \quad \text{and} \quad \lim_{x\to\sup J} \widetilde{F}_J(x) = 1, \tag{9}$$

provided it exists. By Theorem 2, such a function is then unique. The following corollary is the next step in reducing the investigation of the class $\mathcal{I}$ to some special situations. In fact, now we may focus on solutions $\widetilde{F}_J$ such that $\widetilde{F}_J(x) = 0$ for $x \in (-\infty, \inf J]$ and $\widetilde{F}_J(x) = 1$ for $x \in [\sup J, +\infty)$.

COROLLARY 2. *Assume $\mathbf{E}_\tau = \emptyset$. A function $F\colon \mathbb{R} \to [0,1]$ belongs to $\mathcal{I}$ if and only if it is weakly increasing, continuous, $F(-\infty) = 0$, $F(+\infty) = 1$, $F|_I$ is constant for all $I \in \mathcal{S}_\tau$, and on every component J of $\mathbb{R} \setminus \bigcup \mathcal{S}_\tau$ it is either constant or expressed by* (3), *where $\widetilde{F}_J \in \mathcal{C}$ satisfies* (9) *and is uniquely determined.*

In the case where the component J is bounded one can try to apply known results in order to get the existence of $\widetilde{F}_J$. One of such tools could be Corollary 1 from [22], where $[0,1]$ plays the role of $\operatorname{cl} J$; see also [2].

REMARK 1. *Assume $\mathbf{E}_\tau = \emptyset$. Then $\mathcal{I} \neq \emptyset$ if and only if there exists at least one function $\widetilde{F}_J \in \mathcal{C}$ satisfying* (9) *for some component J of $\mathbb{R} \setminus \bigcup \mathcal{S}_\tau$.*

Proof. Sufficiency is clear. Now suppose that $F \in \mathcal{I}$, but no $\widetilde{F}_J$ exists. Then, since F is continuous, we have

$$(0,1) = F(\mathbb{R}) \setminus \{0,1\} \subset F\Big(\bigcup \mathcal{S}_\tau\Big).$$

However, the last set is countable, a contradiction. ∎

REMARK 2. *Assume $\mathbf{E}_\tau = \emptyset$. If S is a τ-invariant half-line disjoint from $\bigcup \mathcal{S}_\tau$, then every $F \in \mathcal{C}$ is constant on S.*

Proof. If $S = [b, +\infty)$ for some $b \in \mathbb{R}$, one can verify that all arguments in Claims 3–5 work with $\widetilde{I}$ replaced by S. If $S = (-\infty, a]$ for some $a \in \mathbb{R}$, the proof runs analogously. One has to change sup to inf in the formula defining ϕ. ∎

5. Example. We now demonstrate how Corollary 2, jointly with already known results, works in the specific case where

$$\tau_1(x) := \begin{cases} x & \text{if } x \in (-\infty, 0), \\ 3x & \text{if } x \in [0, \frac{1}{3}), \\ \frac{3}{5}x + \frac{4}{5} & \text{if } x \in [\frac{1}{3}, 2), \\ 2x - 2 & \text{if } x \in [2, \infty), \end{cases} \qquad \tau_2(x) := \begin{cases} \frac{3}{5}x - \frac{2}{5} & \text{if } x \in (-\infty, \frac{2}{3}), \\ 3x - 2 & \text{if } x \in [\frac{2}{3}, 1), \\ \frac{2}{3}x + \frac{1}{3} & \text{if } x \in [1, \frac{5}{2}), \\ 2x - 3 & \text{if } x \in [\frac{5}{2}, \infty), \end{cases}$$

and the indices $1, 2$ are chosen with probability $1/2$.

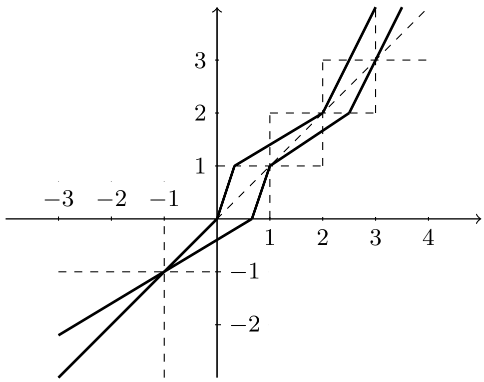

Fig. 1

In this case $\mathbf{E}_\tau = \{-1\}$ (see Figure 1) and, by Theorem 1(ii), $\mathcal{I} \neq \emptyset$ and we have to consider equation (1) separately on $(-\infty, -1)$ and on $(-1, +\infty)$. Fix $F \in \mathcal{I}$. The value $F(-1)$ may be an arbitrary number $a \in [0, 1]$, and -1 is the only possible point of discontinuity, by Theorem 1(ii). The next remark shows that $F|_{(-\infty,-1)} = 0$.

Remark 3. *Assume $\mathbf{E}_\tau = \emptyset$. If either $\tau(x, \omega) \leq x$ for all $x \in \mathbb{R}$ and almost all $\omega \in \Omega$, or $\tau(x, \omega) \geq x$ for all $x \in \mathbb{R}$ and almost all $\omega \in \Omega$, then $\mathcal{I} = \emptyset$.*

Proof. This follows from Remark 2. Indeed, in the first case every half-line $(-\infty, a]$ with $a \in \mathbb{R}$ is τ-invariant, whereas in the second case every half-line $[b, +\infty)$ with $b \in \mathbb{R}$ is τ-invariant. Plainly, $\mathcal{S}_\tau = \emptyset$. ∎

Observe that $\mathcal{S}_\tau = \{[1, 2]\}$, so Corollary 2 implies that $F|_{[1,2]}$ is constant, say c with $a \leq c \leq 1$. Since both $(-1, 0]$ and $[3, +\infty)$ are τ-invariant, Remark 3 yields $F|_{[3,+\infty)} = 1$ and $F|_{(-1,0]} = b$ with $a \leq b \leq c$. Finally, according to [25] we infer that F is the classical Cantor function on $[0, 1]$ and an affine function on $[2, 3]$.

Consequently, any solution $F \in \mathcal{I}$ depends on three parameters $0 \leq a \leq b \leq c \leq 1$ and its graph looks like the one in Figure 2.

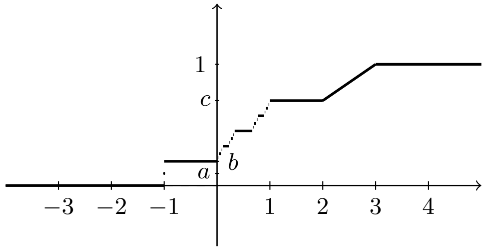

Fig. 2

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