ANNALES
POLONICI MATHEMATICI
95.2 (2009)

Uniqueness results for operators in the variational sequence

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Abstract. We prove that the most interesting operators in the Euler–Lagrange complex from the variational bicomplex in infinite order jet spaces are determined up to multiplicative constant by the naturality requirement, provided the fibres of fibred manifolds have sufficiently large dimension. This result clarifies several important phenomena of the variational calculus on fibred manifolds.

0. Introduction. In the present note we study the Euler–Lagrange variational complex (called simply the *variational sequence*) in infinite order jet spaces from the natural operator point of view. The notion (used in the present note) of the variational sequence can be found in the handbook [8] by R. Vitolo. The concept of natural operators can be found in the fundamental monograph [3] by I. Kolář, P. Michor and J. Slovák. Variational sequences have many applications (e.g. in the variational calculus and theoretical mechanics). That is why they have been studied by many authors, e.g. D. Krupka [5], D. Krupka and J. Musilová [6], I. M. Anderson [1], etc. In [8], one can find over a hundred references concerning variational sequences and their applications.

Let $\pi : E \to M$ be a fibred manifold with n-dimensional bases and m-dimensional fibres. In [8], the variational sequence (of interest to us) has been introduced basically in two ways. The first is through the variational bicomplex associated to $\pi : E \to M$; this approach can be found e.g. in Section 3 of [8]. The variational bicomplex is presented in the diagram (3.5) in [8] (see Definition 3.1 there). The second way is through a spectral sequence; this approach can be found e.g. in Section 4 of [8]. In the present note we use the first approach only. Then the variational sequence (also called the Euler–Lagrange complex)

$$0 \to \mathbb{R} \to E_0^{0,0} \to E_0^{0,1} \to \cdots \to E_0^{0,n-1} \to E_0^{0,n}$$
$$\to E_1^{1,n} \to E_1^{2,n} \to \cdots \to E_1^{p,n} \to E_1^{p+1,n} \to \cdots$$

2000 *Mathematics Subject Classification*: 49Q99, 49S05, 58A20, 58A32.
Key words and phrases: natural operator, variational bicomplex, Euler–Lagrange complex, interior Euler operator, Helmholtz operator.

DOI: 10.4064/ap95-2-3

is the most important part of the variational bicomplex associated to $\pi : E \to M$ (see Definition 3.2 in [8]).

Let us briefly present the main results of the present note. Given an integer $p \geq 1$ we have the interior Euler operator $I = I_p : E_0^{p,n} \to E_0^{p,n}$ (see Section 3.1 in [8]). By [8], $E_1^{p,n} = E_0^{p,n}/\ker(I_p)$ and we have the quotient monomorphism (representation) $E_1^{p,n} \to \mathcal{V}^p \subset E_0^{p,n}$ onto $\mathcal{V}^p = \mathrm{im}(I_p)$. Then the operator $E_1^{p,n} \to E_1^{p+1,n}$ from the variational sequence is the composition of the so called Helmholtz operator $H = I_{p+1} \circ d_{V\,|\,\mathrm{im}(I_p)} : \mathrm{im}(I) = \mathcal{V}^p \to E_0^{p+1,n}$ with the quotient projection $E_0^{p+1,n} \to E_1^{p+1,n}$, where $d_V : E^{p,n} \to E_0^{p+1,n}$ is the vertical differentiation (see [8]). The main result of the present note can be formulated in the following two theorems.

Theorem 1. *Let $\pi : E \to M$ be a fibred manifold with n-dimensional base and m-dimensional fibres. Let $p \geq 1$ be an integer. If $m \geq p$, then any operator $D : E_0^{p,n} \to E_0^{p,n}$ of the interior Euler operator type* (*in the sense of Definition* 1 *from Section* 1 *below*) *is of the form $D = cI$, $c \in \mathbb{R}$.*

Theorem 2. *Let $\pi : E \to M$ be a fibred manifold with n-dimensional base and m-dimensional fibres. Let $p \geq 1$ be an integer. If $m \geq p+1$ then any operator $F : \mathrm{Im}(I) = \mathcal{V}^p \to E_0^{p+1,n}$ of the Helmholtz operator type* (*in the sense of Definition* 2 *from Section* 3 *below*) *is of the form $F = cH$, $c \in \mathbb{R}$.*

Roughly speaking, the above theorems give uniqueness results for the arrows $E_1^{p,n} \to E_1^{p+1,n}$ in the Euler–Lagrange complex, provided the fibres of $\pi : E \to M$ are of sufficiently high dimension.

We recall that the uniqueness result for the arrow $E_0^{0,n} \to E_1^{1,n}$ was proved by I. Kolář in [2].

We observe that Theorem 1 with no assumption on D is false (the identity map $\mathrm{id} : E_0^{p,n} \to E_0^{p,n}$ is a counterexample).

The proofs of Theorems 1 and 2 will occupy the rest of this note. We first reformulate the theorems into finite jet versions. Then using a generalization of the technique from [7] we prove the latter.

1. A reformulation of Theorem 1. Using the definition of $E_0^{p,n}$ from [8], one can easily see that $E_0^{p,n}$ is the injective limit of the sequence of inclusions

$$(1)\qquad \cdots \subset \mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^*J^rE \otimes \bigwedge^n T^*M) \subset \mathcal{C}^\infty_{J^{r+1}E}(J^{r+1}E, \bigwedge^p V^*J^{r+1}E \otimes \bigwedge^n T^*M) \subset \cdots$$

given by the pull-back with respect to the jet projections $\pi_r^{r+1} : J^{r+1}E \to J^rE$ for all natural numbers r, where given two fibred manifolds $Z_1 \to N$ and $Z_2 \to N$ over the same base N we denote the space of all base preserving fibred manifold morphisms of Z_1 into Z_2 by $\mathcal{C}^\infty_N(Z_1, Z_2)$. (Indeed, if we set $K^r = \mathcal{C}^\infty_{J^rE}(J^rE, \bigwedge^p V^*J^rE \otimes \bigwedge^n T^*M)$ and use the notations of [8] and local coordinate arguments, we have the inclusions $K^r \subset \mathcal{C}^p\Omega^p_{r+1} \wedge \bar{\Omega}^{(0,n)}_{r+1} \subset K^{r+1}$.) Taking into account the definition of I one can easily see that:

1. For any natural number r we have

$$(2)\qquad (\mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^*J^rE \otimes \bigwedge^n T^*M)) \subset \mathcal{C}^\infty_{J^qE}(J^sE, \textstyle\bigwedge^{p-1} V^*J^qE \wedge V^*E \otimes \bigwedge^n T^*M)$$

for some integers s, q with $s \ge q \ge r$.

2. For any natural number r the restriction

$$(3)\qquad I^r : \mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^*J^rE \otimes \bigwedge^n T^*M) \to \mathcal{C}^\infty_{J^qE}(J^sE, \textstyle\bigwedge^{p-1} V^*J^qE \wedge V^*E \otimes \bigwedge^n T^*M)$$

of I is a regular π^s_r-local $\mathcal{FM}_{n,m}$-natural operator (see below for the definitions).

Definition 1. We say that an operator $D : E^{p,n}_0 \to E^{p,n}_0$ is of the *interior Euler operator type* if the above properties 1 and 2 hold for D playing the role of I.

Clearly, Theorem 1 is an immediate consequence of the following proposition.

Proposition 1. *Let m, n, r, s, q, p be natural numbers with $m \ge p$ and $s \ge q \ge r$. Then the vector space of all π^s_r-local and $\mathcal{FM}_{n,m}$-natural* (*regular*) *operators*

$$D : \mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^*J^rE \otimes \bigwedge^n T^*M) \to \mathcal{C}^\infty_{J^qE}(J^sE, \textstyle\bigwedge^{p-1} V^*J^qE \wedge V^*E \otimes \bigwedge^n T^*M)$$

is of dimension ≤ 1.

A general notion of natural operators can be found in [3]. In particular, $\mathcal{FM}_{n,m}$ denotes the category of all fibred manifolds $\pi : E \to M$ with n-dimensional bases and m-dimensional fibres and their fibred embeddings. The naturality of an operator D (as in Proposition 1) means that D transforms pairs of f-related morphisms into pairs of f-related morphisms for any $\mathcal{FM}_{n,m}$-map f between arbitrary $\mathcal{FM}_{n,m}$-objects. The regularity means that D transforms smoothly parametrized families into smoothly parametrized families. The locality means that $D(\lambda)_u$ depends only on the germ $\mathrm{germ}_{\pi^s_r(u)}(\lambda)$ for any $u \in J^sE$.

2. Proof of Proposition 1. From now on, $\mathbb{R}^{n,m}$ denotes the trivial bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $x^1, \dots, x^n$, $y^1, \dots, y^m$ are the usual coordinates on $\mathbb{R}^{n,m}$. Let D be an operator as in the statement. Since an $\mathcal{FM}_{n,m}$-map $(x, y - \sigma(x))$ sends $j^s_0(\sigma)$ into $\Theta = j^s_0(0) \in J^s_0(\mathbb{R}^n, \mathbb{R}^m) = J^s_0(\mathbb{R}^{n,m})$, $J^s(\mathbb{R}^{n,m})$

is the $\mathcal{FM}_{n,m}$-orbit of Θ. Then D is uniquely determined by the evaluations

$$D(\lambda)_\Theta(w_1,\dots,w_{p-1},v)\in \bigwedge^n T_0^*\mathbb{R}^n$$

for all $\lambda\in \mathcal{C}^\infty_{J^r\mathbb{R}^{n,m}}(J^r(\mathbb{R}^{n,m}),\bigwedge^p V^*J^r(\mathbb{R}^{n,m})\otimes\bigwedge^n T^*\mathbb{R}^n)$, all $v\in T_0\mathbb{R}^m = V_{(0,0)}\mathbb{R}^{n,m}$ and all $w_1,\dots,w_{p-1}\in V_{\pi^s_q(\Theta)}J^q(\mathbb{R}^{n,m})$.

Then by the multi-linearity of the above evaluations with respect to $w_1,\dots,w_{p-1},v$, we see that D is uniquely determined by the evaluations

$$D(\lambda)_\Theta\bigg(\frac{d}{dt}\bigg|_0(tj^q_0(f_1(x)u_1)),\dots,\frac{d}{dt}\bigg|_0(tj^q_0(f_{p-1}(x)u_{p-1})),w\bigg)\in \bigwedge^n T_0^*\mathbb{R}^n$$

for all $\lambda\in \mathcal{C}^\infty_{J^r\mathbb{R}^{n,m}}(J^r(\mathbb{R}^{n,m}),\bigwedge^p V^*J^r(\mathbb{R}^{n,m})\otimes\bigwedge^n T^*\mathbb{R}^n)$ and all maps $f_1,\dots,f_{p-1}:\mathbb{R}^n\to\mathbb{R}$ and all $u_1,\dots,u_{p-1},w\in\mathbb{R}^m$. Using a density argument (since $m\ge p$) one can assume that $u_1,\dots,u_{p-1},w$ are linearly independent and that $f_1(0),\dots,f_{p-1}(0)$ are not equal to zero. Then using the invariance of D with respect to $\mathcal{FM}_{n,m}$-maps of the form $\mathrm{id}_{\mathbb{R}^n}\times\psi$ with linear ψ we can assume $u_1=e_1,\dots,u_{p-1}=e_{p-1}, w=e_p$, where $e_1,\dots,e_n$ is the usual basis in $\mathbb{R}^m$. Then using the invariance of D with respect to $\mathcal{FM}_{n,m}$-maps

$$\bigg(x^1,\dots,x^n,\frac{1}{f_1(x)}\,y^1,\dots,\frac{1}{f_{p-1}(x)}\,y^{p-1},y^p,\dots,y^m\bigg)$$

preserving Θ, we see that D is uniquely determined by the evaluations

$$D(\lambda)_\Theta\bigg(\frac{d}{dt}\bigg|_0(tj^q_0(e_1)),\dots,\frac{d}{dt}\bigg|_0(tj^q_0(e_{p-1})),\frac{\partial}{\partial y^p}\bigg|_0\bigg)\in \bigwedge^n T_0^*\mathbb{R}^n \tag{4}$$

for all $\lambda\in \mathcal{C}^\infty_{J^r\mathbb{R}^{n,m}}(J^r(\mathbb{R}^{n,m}),\bigwedge^p V^*J^r(\mathbb{R}^{n,m})\otimes\bigwedge^n T^*\mathbb{R}^n)$.

Fix such a λ. Using the invariance of D with respect to $\mathcal{FM}_{n,m}$-maps

$$\psi_\tau=\bigg(x^1,\dots,x^n,\frac{1}{\tau^1}\,y^1,\dots,\frac{1}{\tau^m}\,y^n\bigg)$$

for $\tau^j\neq 0$ we get the homogeneity condition

$$\begin{aligned}D((\psi_\tau)_*\lambda)_\Theta&\bigg(\frac{d}{dt}\bigg|_0(tj^q_0(e_1)),\dots,\frac{d}{dt}\bigg|_0(tj^q_0(e_{p-1})),\frac{\partial}{\partial y^p}\bigg|_0\bigg)\\&=\tau^1\cdots\tau^p D(\lambda)_\Theta\bigg(\frac{d}{dt}\bigg|_0(tj^q_0(e_1)),\dots,\frac{d}{dt}\bigg|_0(tj^q_0(e_{p-1})),\frac{\partial}{\partial y^p}\bigg|_0\bigg)\end{aligned}$$

for $\tau=(\tau^1,\dots,\tau^m)$. By Corollary 19.8 in [3] of the non-linear Peetre theorem we can assume that λ is a polynomial (of arbitrary degree). The regularity of D implies that

$$D(\lambda)_\Theta\bigg(\frac{d}{dt}\bigg|_0(tj^q_0(e_1)),\dots,\frac{d}{dt}\bigg|_0(tj^q_0(e_{p-1})),\frac{\partial}{\partial y^p}\bigg|_0\bigg)$$

is smooth with respect to the coefficients of λ. Then by the homogeneous function theorem (and the above type of homogeneity) we deduce that the evaluations (4) are fully determined by the evaluations (4) for

$$\lambda = x^\beta dy^1_{\alpha_1} \wedge \cdots \wedge dy^p_{\alpha_p} \otimes dx^\mu \tag{5}$$

for all multi-indices $\beta, \alpha_1, \dots, \alpha_p \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha_1| \le r, \dots, |\alpha_p| \le r$, where (x^i, y^j_α) is the induced coordinate system on $J^s(\mathbb{R}^{n,m})$ and $dx^\mu = dx^1 \wedge \cdots \wedge dx^n$. Moreover, if we denote by W the vector space spanned by all λ of the form (5), then (4) for $\lambda \in W$ depends linearly on λ.

Then by the invariance of D^r with respect to the $\mathcal{FM}_{m,n}$-maps

$$(\tau^1 x^1, \dots, \tau^n x^n, y^1, \dots, y^m)$$

for $\tau^i \neq 0$ we get

$$\begin{aligned} D(x^\beta dy^1_{\alpha_1} \wedge \cdots \wedge dy^p_{\alpha_p} \otimes dx^\mu)_\Theta \bigg(\frac{d}{dt}\bigg|_0 (tj^q_0(e_1)), \dots, \\ \frac{d}{dt}\bigg|_0 (tj^q_0(e_{p-1})), \frac{\partial}{\partial y^p}\bigg|_0 \bigg) = 0 \end{aligned} \tag{6}$$

if only $\beta \neq \alpha_1 + \cdots + \alpha_p$ for all $\beta, \alpha_1, \dots, \alpha_p$ as above.

Define $\alpha = \alpha_1 + \cdots + \alpha_p = (\alpha^1, \dots, \alpha^n)$. Suppose that $\alpha^i \neq 0$ for some $i = 1, \dots, n$. The map

$$\psi = (x^1, \dots, x^n, y^1, \dots, y^p + x^i y^p, \dots, y^m)^{-1}$$

is an $\mathcal{FM}_{m,n}$-map near 0. It preserves $x^1, \dots, x^n$, Θ, $\frac{d}{dt}\big|_0 (tj^q_0(e_1)), \dots, \frac{d}{dt}\big|_0 (tj^q_0(e_{p-1}))$ and $\frac{\partial}{\partial y^p}\big|_0$. It sends y^p_ϱ (for $|\varrho| \le r$) into

$$y^p_\varrho + x^i y^p_\varrho + y^p_{\varrho - 1_i}$$

where the third summand does not occur if $\varrho^i = 0$. (Indeed, we have

$$\begin{aligned} y^p_\varrho \circ J^r\psi^{-1}(j^r_{x_0}\eta) &= \partial_\varrho(\eta^p + x^i\eta^p)(x_0) \\ &= \partial_\varrho\eta^p(x_0) + x^i_o\partial_\varrho\eta^p(x_0) + \partial_{\varrho-1_i}\eta^p(x_0) \\ &= (y^p_\varrho + x^i y^p_\varrho + y^p_{\varrho-1_i})(j^r_{x_0}\eta) \end{aligned}$$

for $j^r_{x_0}\eta \in J^r\mathbb{R}^{n,m}$ where ∂_ϱ is the iterated partial derivative as indicated multiplied by $1/\varrho!$). It preserves the other y^j_σ. Thus it sends dy^p_ϱ into $dy^p_\varrho + x^i dy^p_\varrho + dy^p_{\varrho-1_i}$ as $dx^i = 0$ on $V\mathbb{R}^{n,m}$. Then using the invariance of D with respect to ψ, from (6) for $\beta = \alpha - 1_i$ we deduce that the evaluation (4) for λ as in (5) and $\beta = \alpha$ is a linear combination of evaluations (4) for λ as in (5) and $\beta = \alpha - 1_i$ and $\alpha - 1_i$ playing the role of α.

Continuing this process we deduce that the evaluations (4) for λ as in (5) for $\beta = \alpha$ are determined by the evaluation (4) for λ as in (5) with $\beta = \alpha = (0)$.

That is why the vector space of all D in question is 1-dimensional. ∎

3. A reformulation of Theorem 2. Using quite similar arguments to the ones from Section 1 we see that:

1. For any natural number r we have

$$H(\mathrm{Im}(I^r)) \subset \mathcal{C}^\infty_{J^{\bar q}E}(J^{\bar s}E, \textstyle\bigwedge^p V^* J^{\bar q}E \wedge V^*E \otimes \bigwedge^n T^*M)$$

for some integers $\bar s, \bar q$ with $\bar s \geq \bar q \geq s$, where

$$I^r : \mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^* J^rE \otimes \bigwedge^n T^*M) \to \mathcal{C}^\infty_{J^qE}(J^sE \bigwedge^{p-1} V^* J^qE \wedge V^*E \otimes \bigwedge^n T^*M) \tag{7}$$

is the restriction of the interior Euler operator (as in Section 1).

2. For any natural number r, the restriction

$$H^r : \mathrm{Im}(I^r) \to \mathcal{C}^\infty_{J^{\bar q}E}(J^{\bar s}E, \textstyle\bigwedge^p V^* J^{\bar q}E \wedge V^*E \otimes \bigwedge^n T^*M) \tag{8}$$

of H is a regular $\pi^{\bar s}_s$-local $\mathcal{FM}_{m,n}$-natural operator.

Definition 2. We say that an operator $F : \mathrm{Im}(I) \to E^{p+1,n}_0$ is of the *Helmholtz operator type* if the above properties 1 and 2 hold for F playing the role of H.

It is clear that Theorem 2 is an immediate consequence of the following proposition.

Proposition 2. *Let n, m, p, r, s, q, $\bar s$, $\bar q$ be natural numbers as above with $m \geq p+1$. Then any $\pi^{\bar s}_s$-local and $\mathcal{FM}_{n,m}$-natural (regular) operator*

$$F : \mathrm{Im}(I^r) \to \mathcal{C}^\infty_{J^{\bar q}E}(J^{\bar s}E, \textstyle\bigwedge^p V^* J^{\bar q}E \wedge V^*E \otimes \bigwedge^n T^*M)$$

is of the form $F = cH^r$, $c \in \mathbb{R}$.

Remark 1. Proposition 2 for $p = 1$ and $r = 1, 2$ was proved in [4], and for $p = 1$ and all r in [7].

Let F be an operator as above. Considering the composition

$$F \circ I^r : \mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^* J^rE \otimes \bigwedge^n T^*M) \to \mathcal{C}^\infty_{J^{\bar q}E}(J^{\bar s}E, \bigwedge^p V^* J^{\bar q}E \wedge V^*E \otimes \bigwedge^n T^*M)$$

we see that Proposition 2 is an immediate consequence of the following one.

Proposition 3. *Let n, m, p, r, s, q be arbitrary natural numbers with $s \geq q \geq r$ and $m \geq p+1$. Then the vector space of all π^s_r-local and $\mathcal{FM}_{n,m}$-natural (regular) operators*

$$D : \mathcal{C}^\infty_{J^rE}(J^rE, \textstyle\bigwedge^p V^* J^rE \otimes \bigwedge^n T^*M) \to \mathcal{C}^\infty_{J^qE}(J^sE, \bigwedge^p V^* J^qE \wedge V^*E \otimes \bigwedge^n T^*M)$$

is of dimension ≤ 1.

4. Proof of Proposition 3. Let D be an operator as in the statement. Then (by a similar argument to the proof of Proposition 1) D is uniquely determined by the evaluations

$$(9)\qquad D(\lambda)_{\Theta}\left(\frac{d}{dt}\bigg|_0 (tj^q_0(e_1)),\ldots,\frac{d}{dt}\bigg|_0 (tj^q_0(e_p)),\frac{\partial}{\partial y^{p+1}}\bigg|_0\right)\in \bigwedge^n T^*_0\mathbb{R}^n$$

for all $\lambda \in \mathcal{C}^\infty_{J^r\mathbb{R}^n}(J^r(\mathbb{R}^{n,m}),\bigwedge^p V^*J^r(\mathbb{R}^{n,m})\otimes \bigwedge^n T^*\mathbb{R}^n)$, where $\Theta = j^s_0(0) \in J^s\mathbb{R}^{n,m}$.

Fix λ as above. Using the invariance of D with respect to the $\mathcal{FM}_{m,n}$-maps

$$\psi_\tau = \left(x^1,\ldots,x^n,\frac{1}{\tau^1}\,y^1,\ldots,\frac{1}{\tau^m}\,y^m\right)$$

for $\tau^j \neq 0$ we get the homogeneity condition

$$\begin{aligned}
&D((\psi_\tau)_*\lambda)_{\Theta}\left(\frac{d}{dt}\bigg|_0 (tj^q_0(e_1)),\ldots,\frac{d}{dt}\bigg|_0 (tj^q_0(e_p)),\frac{\partial}{\partial y^{p+1}}\bigg|_0\right)\\
&\qquad=\tau^1\cdots\tau^{p+1}D(\lambda)_{\Theta}\left(\frac{d}{dt}\bigg|_0 (tj^s_0(e_1)),\ldots,\frac{d}{dt}\bigg|_0 (tj^q_0(e_p)),\frac{\partial}{\partial y^{p+1}}\bigg|_0\right)
\end{aligned}$$

for $\tau = (\tau^1,\ldots,\tau^m)$. By Corollary 19.8 in [3] of the non-linear Peetre theorem we can assume that λ is a polynomial (of arbitrary degree). The regularity of D implies that

$$D(\lambda)_{\Theta}\left(\frac{d}{dt}\bigg|_0 (tj^q_0(e_1)),\ldots,\frac{d}{dt}\bigg|_0 (tj^q_0(e_p)),\frac{\partial}{\partial y^{p+1}}\bigg|_0\right)$$

is smooth with respect to the coefficients of λ. Then by the homogeneous function theorem (and the above type of homogeneity) we deduce that the evaluations (9) are fully determined by the evaluations (9) for

$$(10)\qquad \lambda = x^\beta y^{\sigma(p+1)}_{\alpha_{p+1}}dy^{\sigma(1)}_{\alpha_1}\wedge\cdots\wedge dy^{\sigma(p)}_{\alpha_p}\otimes dx^\mu$$

for all permutations $\sigma\in B_{p+1}$, all multi-indices $\beta,\alpha_1,\ldots,\alpha_{p+1}\in(\mathbb{N}\cup\{0\})^n$ with $|\alpha_1|\le r,\ldots,|\alpha_{p+1}|\le r$, where (x^i,y^j_α) is the induced coordinate system on $J^r(\mathbb{R}^{n,m})$ and $dx^\mu = dx^1\wedge\cdots\wedge dx^n$. Moreover, if we denote by W the vector space spanned by all λ of the form (10), then (9) for $\lambda\in W$ depends linearly on λ.

Then by the invariance of D with respect to $(\tau^1x^1,\ldots,\tau^nx^n,y^1,\ldots,y^m)$ for $\tau^i\neq 0$ we get

$$\begin{aligned}
(11)\qquad &D(x^\beta y^{\sigma(p+1)}_{\alpha_{p+1}}dy^{\sigma(1)}_{\alpha_1}\wedge\cdots\wedge dy^{\sigma(p)}_{\alpha_p}\otimes dx^\mu)_{\Theta}\left(\frac{d}{dt}\bigg|_0 (tj^q_0(e_1)),\ldots,\right.\\
&\qquad\qquad\left.\frac{d}{dt}\bigg|_0 (tj^q_0(e_p)),\frac{\partial}{\partial y^{p+1}}\bigg|_0\right)=0
\end{aligned}$$

provided $\beta\neq\alpha_1+\cdots+\alpha_{p+1}$ for all $\beta,\alpha_1,\ldots,\alpha_{p+1},\sigma$ as above.

Define $\alpha=\alpha_1+\dots+\alpha_{p+1}=(\alpha^1,\dots,\alpha^n)$. Suppose that $\alpha^i\neq 0$ for some $i=1,\dots,n$. The map

$$\psi=(x^1,\dots,x^n,y^1,\dots,y^{p+1}+x^iy^{p+1},\dots,y^m)^{-1}$$

is an $\mathcal{FM}_{m,n}$-map near 0. It preserves $x^1,\dots,x^n$, Θ, $\frac{d}{dt}\big|_0(tj^q_0(e_1)),\dots,\frac{d}{dt}\big|_0(tj^q_0(e_p))$ and $\frac{\partial}{\partial y^{p+1}}\big|_0$ and sends y^{p+1}_ϱ (for $|\varrho|\le r$) into

$$y^{p+1}_\varrho+x^iy^{p+1}_\varrho+y^{p+1}_{\varrho-1_i}$$

(see the proof of Proposition 1) and preserves the other y^j_σ (and then sends dy^{p+1}_ϱ into $dy^{p+1}_\varrho+x^idy^{p+1}_\varrho+dy^{p+1}_{\varrho-1_i}$ as $dx^i=0$ on $V\mathbb{R}^{n,m}$). Then using the invariance of D with respect to ψ, from (11) for $\beta=\alpha-1_i$ we deduce that the evaluation (9) for λ as in (10) and $\beta=\alpha$ is a linear combination of evaluations (9) for λ as in (10) and $\beta=\alpha-1_i$ and $\alpha-1_i$ playing the role of α.

Continuing this process we deduce that the evaluations (9) for λ as in (10) and $\beta=\alpha$ are determined by the evaluations (9) for all λ as in (10) with $\beta=\alpha=(0)$.

The above considerations show that D is uniquely determined by the evaluations (9) for all λ as in (10) for $\beta=\alpha=(0)$ and all $\sigma\in B_{p+1}$. Then using the invariance of D with respect to the permutations of first p fibred coordinates (preserving Θ and $\frac{\partial}{\partial y^{p+1}}\big|_0$ and sending $\frac{d}{dt}\big|_0(tj^q_0(e_1))\wedge\dots\wedge\frac{d}{dt}\big|_0(tj^q_0(e_p))$ to $\varepsilon\frac{d}{dt}\big|_0(tj^q_0(e_1))\wedge\dots\wedge\frac{d}{dt}\big|_0(tj^q_0(e_p))$ for $\varepsilon=+1$ or -1), we see that D is determined by the values (9) for $\lambda=\lambda_1,\lambda_2$, where $\lambda_1=y^{p+1}_{(0)}dy^1_{(0)}\wedge\dots\wedge dy^p_{(0)}\otimes dx^\mu$ and $\lambda_2=y^1_{(0)}dy^2_{(0)}\wedge\dots\wedge dy^{p+1}_{(0)}\otimes dx^\mu$.

Using the invariance of D with respect to the (locally defined) $\mathcal{FM}_{n,m}$-map

$$(x^1,\dots,x^n,y^1+y^1y^{p+1},y^2,\dots,y^m)^{-1}$$

preserving Θ, $\frac{\partial}{\partial y^{p+1}}\big|_0$ and $\frac{d}{dt}\big|_0(tj^q_0(e_1))\wedge\dots\wedge\frac{d}{dt}\big|_0(tj^q_0(e_p))$, from

$$D(dy^1_{(0)}\wedge\dots\wedge dy^{p-1}_{(0)}\wedge dy^p\otimes dx^\mu)_\Theta\biggl(\frac{d}{dt}\bigg|_0(tj^q_0(e_1)),\dots,\frac{d}{dt}\bigg|_0(tj^q_0(e_p)),\frac{\partial}{\partial y^{p+1}}\bigg|_0\biggr)=0$$

(a consequence of the invariance of D with respect to the fibre homotheties $(x^1,\dots,x^n,y^1,\dots,ty^{p+1},\dots,y^m)$), we deduce that the value (9) for $\lambda=\lambda_1$ is plus or minus the value (9) for $\lambda=\lambda_2$.

Thus the vector space of all D in question is 1-dimensional. ∎

5. Uniqueness of the extended Helmholtz operator. We have the extended Helmholtz operator $\widetilde{H}=I\circ d_V:E^{p,n}_0\to E^{p+1,n}_0$, where $d_V:E^{p,n}_0\to E^{p+1,n}_0$ is the vertical differential and $I:E^{p+1,n}_0\to E^{p+1,n}_0$ is the interior Euler operator. (The name is because $\widetilde{H}$ is really an extension of the Helmholtz operator $H : \mathrm{Im}(I) \to E_0^{p+1,n}$, where $I : E_0^{p,n} \to E_0^{p,n}$.) The operator $\widetilde{H}$ satisfies appropriate modifications of properties 1 and 2 from Definition 1. We say that an operator $G : E_0^{p,n} \to E_0^{p+1,n}$ is of the *extended Helmholtz operator type* if it has those modified properties. Then Proposition 3 implies the following corollary.

COROLLARY 1. *Let $\pi : E \to M$ be a fibred manifold with n-dimensional basis and m-dimensional fibres. Let $p \geq 1$ be an integer. If $m \geq p+1$ then any operator $G : E_0^{p,n} \to E_0^{p+1,n}$ of the extended Helmholtz operator type is of the form $G = c\widetilde{H}$, $c \in \mathbb{R}$.*

Of course, Corollary 1 for $p = 0$ is also true because of the result of [2].

We observe that Corollary 1 with no assumption on G is false (the vertical differential $d_V : E_0^{p,n} \to E_0^{p+1,n}$ is a counterexample).

Acknowledgements. The author thanks R. Vitolo for suggesting the problem and helpful discussions.

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Received 4.6.2008
and in final form 21.7.2008 (1886)