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Extension from linear subvarieties for the Bergman scale of spaces on convex domains

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Abstract. We study the problem of extending functions from linear affine subvarieties for the Bergman scale of spaces on convex finite type domains. Our results solve the problem for $H^1(D)$. For other Bergman spaces the result is ϵ-optimal.

1. Introduction. In this paper we investigate the problem of extending functions from a linear subvariety for Bergman spaces. A domain D on which the function spaces are defined is assumed to be smoothly bounded and convex of finite type. We explain what this means.

The domain is defined by a function r,

$$D = \{z\colon r(z) < 0\},$$

which is assumed to be smooth on some open neighbourhood of $\bar{D}$ and satisfy $dr \neq 0$ on bD (we say that r is *non-degenerate* on bD). We use the symbol ϱ to denote $|r|$.

The finite type assumption means that the maximal order of contact of bD with germs of complex analytic sets is finite. Such domains have been studied since the discovery of their importance in the $\overline{\partial}$-Neumann problem ([24], [25], [12], [9], [10]). Our goal is to provide an important element of function theory on convex finite type domains.

Let us recall that the *Bergman space* $H^p(D)$, $1 \leq p < \infty$, is the space of all holomorphic functions F in D such that

$$\int\limits_D |F|^p \, dV < \infty.$$

The symbol dV stands for the volume measure in $\mathbb{C}^n$.

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Let $\omega = (\omega_1, \ldots, \omega_n) \in D$, $e = (e_1, \ldots, e_n) \in \mathbb{C}^n$ and set

$$A = A(\omega, e) := \{z \in \mathbb{C}^n \colon h_{\omega,e}(z) := \langle z - \omega, e\rangle = 0\},$$

where

$$\langle z - \omega, e\rangle = \sum_{j=1}^{n} (z_j - \omega_j)\bar{e}_j.$$

The operator $R_{D\cap A}$ of restriction to the subvariety A is defined for all holomorphic functions in D in the following way:

$$R_{D\cap A} \colon H(D) \ni f \mapsto f|_{D\cap A} \in H(D \cap A).$$

In this paper we examine for which functions f holomorphic in $D \cap A$ there is a function F in the Bergman space $H^p(D), 1 \le p < \infty$, such that

$$f = R_{D\cap A} F.$$

The answer we give is the following: There exist positive Borel measures μ and μ_ϵ, $\epsilon > 0$, supported on $D \cap A$ such that if a function f holomorphic in $D \cap A$ is integrable with respect to μ, then it extends to a function which belongs to $H^1(D)$. A holomorphic function which is p-integrable with respect to μ_ϵ for some $\epsilon > 0$ admits an extension to a function which belongs to $H^p(D)$, $1 < p < \infty$. For $p = 1$ the result is sharp, i.e. the condition which guarantees the existence of an extension is also necessary. When p is between 1 and ∞ there is an ϵ-gap between a condition which suffices for an extension with values in $H^p(D)$ and the one which is necessary.

In both cases we prove that there exist linear extension operators with values in $H^p(D), 1 \le p < \infty$. If $p = 1$ the corresponding operator is defined on the space of holomorphic functions which are integrable with respect to the measure μ. When $1 < p < \infty$ the extension operators are defined on spaces of holomorphic functions which are p-integrable with respect to one of the measures μ_ϵ. What is important is that the measure μ and the measures $\mu_\epsilon, \epsilon > 0$, depend on the specific non-isotropic geometry of the convex domain of finite type and are independent of p.

The celebrated Ohsawa–Takegoshi extension theorem [30] states that in the case of pseudoconvex domains each holomorphic $L^2(D \cap A)$-function admits an extension to a holomorphic function in $H^2(D)$. Our results concern the whole scale of Bergman spaces $H^p(D)$, $1 \le p < \infty$. In the specific case of $H^2(D)$ they say that under the additional assumption that D is convex and of finite type the class of functions which admit an extension to a function in $H^2(D)$ is larger than $H^2(D \cap A)$. There appears the so called ‘gain of regularity’. This seems interesting since K. Diederich and E. Mazzilli proved in [16] that there are finite type domains and subvarieties (non-linear and of higher codimension) with no ‘gain of regularity’ as far as the extension problem is concerned.

Let us now introduce some notation and formulate the results. Let ν be a positive Borel measure on $D \cap A(\omega, e)$. Then

$$H^p(D \cap A(\omega, e), d\nu) := \Big\{ f \in H(D \cap A(\omega, e)) \colon \int\limits_{D \cap A(\omega,e)} |f|^p \, d\nu < \infty \Big\},$$

for $1 \le p < \infty$, is the space of all functions holomorphic in $D \cap A(\omega, e)$ and p-integrable with respect to the measure ν.

We can present the results now.

THEOREM 1.1. *Assume that D is a bounded convex domain of finite type defined by a function r which is smooth on some neighbourhood of $\bar{D}$ and $dr \neq 0$ on bD. Let $e \in \mathbb{C}^n$ be a unit vector and $\omega \in D$. There exists a bounded operator*

$$E_{A(\omega,e)} \colon H^1\big(D \cap A(\omega, e), |\partial h_{\omega,e}|^2_{\mathcal{N}} \, dV_{A(\omega,e)}\big) \to H^1(D)$$

such that

$$R_{D \cap A} \circ E_{A(\omega,e)} = \mathrm{id}_{H^1(D \cap A(\omega,e), |\partial h_{\omega,e}|^2_{\mathcal{N}} \, dV_{A(\omega,e)})} .$$

As already stated, we solve the extension and restriction problem for $H^1(D)$ completely. The condition which suffices for the extension is also necessary.

THEOREM 1.2. *Assume that D is a bounded convex domain of finite type defined by a function r which is smooth on some neighbourhood of $\bar{D}$ and $dr \neq 0$ on bD. Let $\omega \in D$ and $e \in \mathbb{C}^n$ with $e \neq 0$. A function $f \in H(D \cap A(\omega, e))$ admits an extension which belongs to the Bergman space $H^1(D)$ if and only if*

$$\int\limits_{D \cap A(\omega,e)} |f| \, |\partial h_{\omega,e}|^2_{\mathcal{N}} \, dV_{D \cap A} < \infty.$$

In other words,

$$R_{D \cap A}[H^1(D)] = H^1\big(D \cap A(\omega, e), |\partial h_{\omega,e}|^2_{\mathcal{N}} \, dV_{D \cap A}\big).$$

In order to prove Theorem 1.2 we invoke the following result proved by the author in [23]:

THEOREM 1.3. *Assume that D is a bounded convex domain of finite type defined by a function r which is smooth on some neighbourhood of $\bar{D}$ and $dr \neq 0$ on bD. Then for any $1 \le p < \infty$,*

$$R_{D \cap A}[H^p(D)] \subset H^p\big(D \cap A(\omega, e), |\partial h_{\omega,e}|^2_{\mathcal{N}} \, dV_{D \cap A}\big).$$

For other Bergman spaces we obtain an ϵ-optimal result.

THEOREM 1.4. *Assume that D is a bounded convex domain of finite type defined by a function r which is smooth on some neighbourhood of $\bar{D}$ and*

$dr \neq 0$ on bD. Let $\omega \in D$ and $e \in \mathbb{C}^n$ with $e \neq 0$. For any $\epsilon > 0$ and $1 < p < \infty$ there exists a bounded operator

$$E_{A(\omega,e)} \colon H^p\big(D \cap A(\omega,e), |\partial h_{\omega,e}|_{\mathcal{N}}^{2-\epsilon}\, dV_{D\cap A(\omega,e)}\big) \to H^p(D)$$

such that

$$R_{D\cap A} \circ E_{A(\omega,e)} = \mathrm{id}_{H^p(D\cap A(\omega,e)), |\partial h_{\omega,e}|_{\mathcal{N}}^{2-\epsilon}\, dV_{D\cap A(\omega,e)}}.$$

We now explain the notation used in Theorems 1.1–1.4. The symbol $dV_{D\cap A}$ stands for the measure induced by the volume element of the complex subvariety $A = A(\omega,e)$, and $|\partial h_{\omega,e}|_{\mathcal{N}}$ is a special non-isotropic norm of the $(1,0)$-covector $\partial h_{\omega,e}$. It is important to realize that although $h_{\omega,e}$ is affine linear, the function $z \mapsto |\partial h_{\omega,e}|^2_{\mathcal{N}}(z) = |\partial h_{\omega,e}(z)|^2_{\mathcal{N}}$ is not constant. To define the norm we first recall an object which is of key importance in convex finite type domains.

The complex boundary distance is defined for z in U, a small neighbourhood of bD, $v \in \mathbb{C}^n$ and $\varepsilon > 0$ in the following way:

$$\tau(z,v,\varepsilon) := \sup\{c \colon |r(z+\lambda v) - r(z)| < \varepsilon,\ |\lambda| < c\}.$$

This notion was introduced by J. D. McNeal [28], [29].

Assume that Ω is a $(1,0)$-covector at a point ζ. We define its non-isotropic norm by

$$|\Omega|_{\mathcal{N}}(\zeta) := \sup\{|\Omega(v)|\tau(\zeta,v,|r(\zeta)|) \colon v \neq 0\}.$$

The norm $|\cdot|_{\mathcal{N}}$ was introduced by J. Bruna, P. Charpentier and Y. Dupain [7] when constructing zero varieties of Nevanlinna class functions. Roughly speaking, its role in Theorems 1.1–1.4 is to compensate in Wirtinger's formula for a lacking variable in the volume form $dV_{D\cap A}$.

In the proof of Theorems 1.1 and 1.4 we use extension operators constructed by B. Berndtsson [4]. The construction is based on integral formulae developed by M. Andersson and B. Berndtsson [5]. The fundamental element of the method is the support function constructed for convex finite type domains by K. Diederich and J. E. Fornæss [14]. In order to prove our results we first obtain certain non-isotropic estimates (Lemma 3.1). This suffices to prove Theorem 1.1 and, as a result, also Theorem 1.2. In order to prove Theorem 1.4 we have to modify Schur's test (Proposition 3.2). The proof of Theorem 1.4 is completed once we prove additional non-isotropic estimates (Lemma 3.3).

One may wonder what is the reason for the ϵ-gap between the conditions in Theorems 1.4 and 1.1. The gap is a consequence of the fact that in Schur's test (and its modified version) one needs to integrate in both variables in order to have boundedness on L^p spaces when $1 < p < \infty$. In the L^1 case, control over only one variable is needed.

The problem of extension from linear subvarieties for bounded holomorphic functions was considered by K. Diederich and E. Mazzilli [15] who generalized to convex finite type domains the result by G. Henkin [22]. The same problem for hypersurfaces which are not affine linear was investigated in [1]. Extension problems were also considered by E. Amar [3], [2], A. Cumenge [11] and E. Mazzilli [27]. Other aspects of function theory on convex finite type domains have also been studied. We list only those most relevant to our work. Regularity of the $\overline{\partial}$-equation on convex finite type domains was investigated by K. Diederich, J. E. Fornæss and B. Fischer [13], B. Fischer [18], [19] and T. Hefer [20], [21]. The Henkin–Skoda problem for convex finite type domains was solved by J. Bruna, P. Charpentier and Y. Dupain [7]. Duality problems were investigated by S. G. Krantz and S.-Y. Li [26]. The present author also studied extension problems for varieties which are not linear [23]. The elementary example of the domain $D = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon |z_1|^2 + |z_2|^4 + |z_3|^2 < 1\}$ and the subvariety $A = \{z_3 = 0\}$ shows that neither Theorem 1.1 nor Theorem 1.4 is a consequence of the results of [23].

The paper is divided into three sections. In the next one we provide background on analysis on convex finite type domains, and construction of the support function and the extension operator. In order to make the paper self-contained we recall both definitions and properties of the objects we use. The last section contains the proofs of Theorems 1.1, 1.2 and 1.4.

We write $A \lesssim B$ if there exists a constant c such that $A \leq cB$. If both $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

2. Preliminaries

2.1. Geometry of convex domains of finite type. In this section we recall basic information concerning smoothly bounded convex domains of finite type in $\mathbb{C}^n, n > 1$. Such a domain will be denoted by D throughout the paper. The domain D is assumed to be defined by a smooth function r with $dr \neq 0$ on the boundary of D. Without loss of generality we may assume that D contains 0 and $r = p_D - 1$ with p_D standing for the Minkowski functional of D. As a result, we may assume that r is convex on some neighbourhood of bD. Notice that with this choice of r, sublevels of the defining function are just dilations of the boundary. Thus, for small $|t|$ the domains $D_t := \{z \in \mathbb{C}^n \colon r(z) < t\}$ are convex as well. We write $\varrho(\zeta)$ to denote $|r(\zeta)|$.

Recall that $p \in bD$ is said to be *of finite type* if the maximal order of contact at p of the hypersurface bD with germs of non-singular analytic sets is finite. The domain D is said to be of finite type if each of its boundary points is of finite type. Under the assumption that D is convex it suffices to take into account only germs of complex lines. This was proved in [6] and [31]. The maximal order of contact of points $p \in bD$ with complex lines is called the *type* of the domain.

We shall make use of concepts introduced by J. D. McNeal [28], [29], H. P. Boas and E. Straube [6], J. Bruna, P. Charpentier and Y. Dupain [7], K. Diederich and J. E. Fornæss [14] and others. In order to make the paper self-contained we recall the concepts we will use.

The symbol U stands for a sufficiently small open neighbourhood of bD. The *complex boundary distance* is defined for $z \in U$, $v \in \mathbb{C}^n$ and $\varepsilon > 0$ by

$$\tau(z, v, \varepsilon) := \sup\{c\colon |r(z + \lambda v) - r(z)| < \varepsilon,\ |\lambda| < c\}.$$

The function τ may also be considered to be defined by the condition

$$\sum_{1 \le i+j \le M} |a_{ij}(z, v)| \tau(z, v, \varepsilon)^{i+j} = \varepsilon, \tag{1}$$

where

$$a_{ij}(z, v) = \frac{\partial^{i+j}}{\partial\lambda^i \partial\bar{\lambda}^j} r(z + \lambda v)\bigg|_{\lambda=0}$$

and M in (1) stands for the type of the domain. As a result,

$$\tau(z, v, \varepsilon) \sim \min_{1 \le k \le M} \left\{ \left(\frac{\varepsilon}{\sum_{i+j=k} |a_{ij}(z, v)|} \right)^{1/k} \right\}. \tag{2}$$

Formula (1) implies in particular that $\tau(z, v, \varepsilon) \lesssim \varepsilon^{1/M}$ and the estimate is uniform for $z \in U$ and $|v| = 1$.

Definition 2.1 (ε-extremal basis). Let $D = \{r < 0\}$ be a smooth convex domain of finite type M, with r convex and smooth. For $\zeta \in \bar{D}$ and $\varepsilon > 0$ the *ε-extremal basis*

$$u_1^{\zeta,\varepsilon}, \ldots, u_n^{\zeta,\varepsilon}$$

at ζ is defined as follows: Choose a point q_1 with $r(q_1) = r(\zeta) + \varepsilon$ in the direction of the line given by the gradient $\partial r(\zeta)$. Then $|q_1 - \zeta|$ is comparable to the distance from ζ to the level set $bD_{\zeta,\varepsilon} := \{z\colon r(z) = r(\zeta) + \varepsilon\}$. Let $u_1^{\zeta,\varepsilon}$ be the unit vector in the direction of $q_1 - \zeta$. Then choose a unit vector $u_2^{\zeta,\varepsilon}$ orthogonal to $\mathrm{Span}_{\mathbb{C}}\{u_1^{\zeta,\varepsilon}\}$ such that the maximal distance from ζ to $bD_{\zeta,\varepsilon}$ along directions orthogonal to $\mathrm{Span}_{\mathbb{C}}\{u_1^{\zeta,\varepsilon}\}$ is achieved along the line given by $u_2^{\zeta,\varepsilon}$ at a point q_2. Now continue by choosing a unit vector $u_3^{\zeta,\varepsilon}$ orthogonal to $\mathrm{Span}_{\mathbb{C}}\{u_1^{\zeta,\varepsilon}, u_2^{\zeta,\varepsilon}\}$ etc. until the basis is constructed.

Extremal bases are used to define certain polydisks. Namely, fix $\zeta \in U$ and let $\varepsilon > 0$ be small enough. Let, for $j = 1, \ldots, n$,

$$\tau_j(\zeta, \varepsilon) := |q_j - \zeta| = \tau(\zeta, u_j^{\zeta,\varepsilon}, \varepsilon).$$

For $C \in \mathbb{R}_0^+$, define polydisks

$$CP_\varepsilon(\zeta) := \Big\{ z = \zeta + \sum_{j=1}^{n} z_{\zeta,\varepsilon,j} u_j^{\zeta,\varepsilon} \in \mathbb{C}^n : |z_{\zeta,\varepsilon,j}| < C\tau_j^e(\zeta,\varepsilon) \text{ for } j = 1,\dots,n \Big\}.$$

We now recall properties of these objects. First of all, if $z \in P_\varepsilon(\zeta)$, then

$$\tau(\zeta, v, \varepsilon) \sim \tau(z, v, \varepsilon). \tag{3}$$

This is Proposition 2.3 in [29]. This implies that if $z \in P_\varepsilon(\zeta)$ then

$$V(P_\varepsilon(z)) \sim V(P_\varepsilon(\zeta)). \tag{4}$$

Proposition 2.2 ([13, Proposition 1.3]). *Let U be a sufficiently small neighbourhood of bD.*

(i) *There exists a constant C_1 such that if $\zeta_1, \zeta_2 \in U$ and $\varepsilon > 0$ and*

$$P_\varepsilon(\zeta_1) \cap P_\varepsilon(\zeta_2) \neq \emptyset$$

then

$$P_\varepsilon(\zeta_1) \subset P_{C_1\varepsilon}(\zeta_2) \quad \textit{and} \quad P_\varepsilon(\zeta_2) \subset P_{C_1\varepsilon}(\zeta_1).$$

(ii) *For each constant K there are constants $c(K), C(K)$ such that for each $\zeta \in U$ and $\varepsilon > 0$,*

$$P_{c(K)\varepsilon}(\zeta) \subset KP_\varepsilon(\zeta) \subset P_{C(K)\varepsilon}(\zeta),$$
$$c(K)P_\varepsilon(\zeta) \subset P_{K\varepsilon}(\zeta) \subset C(K)P_\varepsilon(\zeta).$$

(iii) *There are constants $c_2 < 1$, $C_2 > 1$ such that*

$$\tfrac{1}{2}P_\varepsilon(\zeta) \subset C_2 P_{\varepsilon/2}(\zeta),$$
$$C_2 P_t(\zeta) \subset P_\varepsilon(\zeta) \quad \textit{if } t < c_2\varepsilon.$$

(iv) *There exists a constant c_3 such that for each $\zeta \in D$,*

$$c_3 P_{|r(\zeta)|}(\zeta) \subset D.$$

For $i \in \mathbb{N}_0$ we introduce the polyannuli

$$P_\varepsilon^i(z) := CP_{2^i\varepsilon}(z) \setminus \tfrac{1}{2}P_{2^i\varepsilon}(z)$$

with a suitably defined constant $C > 0$. Proposition 2.2 implies that if $\varepsilon_0 > 0$ is sufficiently small, then

$$P_{\varepsilon_0}(z) \setminus P_\varepsilon(z) \subset \bigcup_{i=0}^{C_5\lceil \log(\varepsilon_0/\varepsilon)\rceil} P_\varepsilon^i(z)$$

with a uniform constant C_5.

We will need to compare $\tau(z, v, \varepsilon)$ for different values of ε.

Lemma 2.3 ([15, Proposition 3.3]). *Let D be a convex domain of finite type defined by a smooth function r such that $dr \neq 0$ on bD. If $\varepsilon_1 \geq \varepsilon_2$ then*

$$\tau(z, v, \varepsilon_1) \lesssim \frac{\varepsilon_1}{\varepsilon_2}\tau(z, v, \varepsilon_2).$$

2.2. Support functions. In [14] K. Diederich and J. E. Fornæss constructed support functions for convex domains of finite type in $\mathbb{C}^n$. We now recall elements of this construction.

Assume that $D = \{z\colon r(z) < 0\}$, where r is chosen in such a way that $|\mathrm{grad}\, r| > 1/2$ in an open set U containing bD and

$$D_t := \{z\colon r(z) < t\}$$

are convex domains of type M for $t = r(\zeta)$, $\zeta \in U$. Let n_ζ be the unit outer normal vector to the hypersurface $\{z\colon r(z) = r(\zeta)\}$ at the point ζ. There exists a family of unitary transformations $\Phi(\zeta)\colon \mathbb{C}^n \to \mathbb{C}^n$, $\zeta \in U$,

$$\Phi(\zeta)^\star \Phi(\zeta) = \Phi(\zeta)\Phi(\zeta)^\star = I$$

such that $\Phi(\zeta) n_\zeta = (1, 0, \ldots, 0)$ for all $\zeta \in U$. The family $\Phi(\zeta)$, $\zeta \in U$, may be chosen to depend smoothly on ζ (cf. [14]). Define

$$\begin{aligned} &r_\zeta(w) := r(\zeta + \Phi(\zeta)^\star w),\\ &S_\zeta(w) = 3w_1 + K_1 w_1^2 - c\sum_{j=2}^{M} K_2 \sigma_j \sum_{|\alpha|=j,\, \alpha_1=0} \frac{1}{\alpha!}\frac{\partial^j r_\zeta}{\partial w^\alpha}(0) w^\alpha, \end{aligned}$$

for K_1, K_2 suitably large and $c > 0$ suitably small. The only symbol which needs an explanation is σ_j:

$$\sigma_j := \begin{cases} 1 & \text{if } j \equiv 0 \bmod 4,\\ -1 & \text{if } j \equiv 2 \bmod 4,\\ 0 & \text{otherwise.} \end{cases}$$

Define

$$S(z, \zeta) := S_\zeta(\Phi(\zeta)(z - \zeta)).$$

The following fundamental fact was proved by K. Diederich and J. E. Fornæss:

Theorem 2.4 (Diederich–Fornæss [14]). *Let $D \subset\subset \mathbb{C}^n$ be a smooth convex domain of finite type M and r a convex defining function of D in a neighbourhood U of bD. Then the function $S = S(z, \zeta) \in C^\infty(\bar{D} \times \mathbb{C}^n)$, holomorphic in z, constructed in* [14]*, has the following property:*

Let $\zeta \in U$, let n_ζ denote the outer unit normal to the level set $\{\eta : r(\eta) = r(\zeta)\}$ and let v be any unit vector complex tangential to this level set at ζ. Define

$$a_{\alpha\beta} := \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha \partial\bar{\lambda}^\beta} r(\zeta + \lambda v)\Big|_{\lambda=0}.$$

Then there are constants $K, c, d > 0$ such that for all $z = \zeta + \mu n_\zeta + \lambda v$ with

$\lambda, \mu \in \mathbb{C}$, *we have*

$$\Re S(z,\zeta) \\ \leq -\left|\frac{\Re\mu}{2}\right| - \frac{K}{2}(\Im\mu)^2 - c\sum_{j=2}^{M}\sum_{\alpha+\beta=j}|a_{\alpha\beta}|\,|\lambda|^j + d\sup\{0, r(z)-r(\zeta)\}.$$

The existence of support functions was proved in [14]. We repeated however the formulation of Theorem 2.4 after [17]. We can replace the function $S(z,\zeta)$ by $\frac{1}{2d}S(z,\zeta)$. Therefore, we may assume that

$$\Re S(z,\zeta) \\ \leq -\left|\frac{\Re\mu}{2}\right| - \frac{K}{2}(\Im\mu)^2 - c\sum_{j=2}^{M}\sum_{\alpha+\beta=j}|a_{\alpha\beta}|\,|\lambda|^j + \frac{1}{2}\sup\{0, r(z)-r(\zeta)\}.$$

We now recall some crucial properties of the function S.

Lemma 2.5 ([18, Lemma 3.2]). *Let U be a sufficiently small neighbourhood of bD and $\varepsilon_0 > 0$ sufficiently small. For all $z, \zeta \in U$ and $\varepsilon < \varepsilon_0$ we have*

$$|S(z,\zeta)| \gtrsim \varepsilon$$

for all $\zeta \in P^0_\varepsilon(z)$ or $z \in P^0_\varepsilon(\zeta)$.

Notice that for $z, \zeta \in D$,

$$r(\zeta) + \tfrac{1}{2}\max\{0, r(z)-r(\zeta)\} = \begin{cases} r(\zeta) & \text{if } r(z)-r(\zeta) < 0, \\ \tfrac{1}{2}r(z) + \tfrac{1}{2}r(\zeta) & \text{if } r(z)-r(\zeta) \geq 0, \end{cases} \\ \leq \tfrac{1}{2}r(\zeta).$$

In view of Proposition 2.4, this suffices to prove the following fact:

Lemma 2.6. *For each $z, \zeta \in D$,*

$$|r(\zeta) + S(z,\zeta)| \gtrsim \varrho(\zeta).$$

We will also use the following lemma:

Lemma 2.7. *There exists an open set $U \supset bD$ and a constant $c > 0$ such that if $z, \zeta \in D \cap U$ and $\zeta \in P_c(z) \setminus P_{2^i\varrho(z)}(z)$ with $2^i\varrho(z) < c$, then*

$$|r(\zeta) + S(z,\zeta)| \gtrsim 2^i\varrho(z).$$

The proof of Lemma 2.7 is exactly the same as the proof of Lemma 4.2 in [13] or Lemma 3.3 in [17], therefore we omit it. Next one defines n functions $Q_1, \ldots, Q_n\colon (U \cap D) \times U \to \mathbb{C}$ holomorphic in the first variable and C^∞ in the second such that

$$S(z,\zeta) = \sum_{j=1}^{n} Q_j(z,\zeta)(z_j - \zeta_j).$$

This is accomplished in the following way:

$$Q(z,\zeta) := \Phi(\zeta)^T Q_\zeta(\Phi(\zeta)(z-\zeta)), \tag{5}$$

where

$$Q^1_\zeta(w) := 3 + K_1 w_1,$$

$$Q^k_\zeta(w) := -c\sum_{j=2}^{M} K_2^{2^j}\sigma_j \sum_{|\alpha|=j,\,\alpha_1=0,\,\alpha_k>0} \frac{\alpha_k}{j\alpha!}\frac{\partial^j r_\zeta}{\partial w^\alpha}(0)\frac{w^\alpha}{w_k}, \qquad k=2,\dots,n.$$

With the function $Q=(Q_1,\dots,Q_n)$ defined in (5) one associates the $(1,0)$-form

$$\sum_{j=1}^{n} Q_j(z,\zeta)d\zeta_j.$$

Fix $\zeta_0 \in D\cap U$ and choose an ε-extremal basis $(u_1,\dots,u_n)$ at ζ_0. Write

$$z = \zeta_0 + \sum_{j=1}^{n} w_j u_j, \qquad \zeta = \zeta_0 + \sum_{j=1}^{n} \eta_j u_j.$$

Let Φ be a unitary transformation such that $w=\Phi(z-\zeta_0)$ and $\eta=\Phi(\zeta-\zeta_0)$ and set

$$\varphi(w) := \zeta_0 + \Phi^\star w.$$

Consider the pull-back of the form $\sum_{j=1}^n Q_j(z,\zeta)d\zeta_j$,

$$\varphi^*\Big(\sum_{j=1}^{n} Q_j(z,\zeta)d\zeta_j\Big) = \sum_{j=1}^{n} Q_j(z,\zeta_0+\Phi^\star\eta)\varphi^*(d\zeta_j) = \sum_{j=1}^{n}\tilde{Q}_j(w,\eta)d\eta_j, \tag{6}$$

with $\tilde{Q}$ defined by

$$\tilde{Q}(w,\eta) := \bar{\Phi}Q(\zeta_0+\Phi^\star w, \zeta_0+\Phi^\star\eta).$$

We use the following estimates of the function $\tilde{Q}$.

PROPOSITION 2.8 ([18, Lemma 3.3]). *For all η with $|\eta_j| < \tau_j(\zeta_0,\varepsilon)$,*

$$|\tilde{Q}_k(0,\eta)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)},$$

$$\left|\frac{\partial}{\partial\bar{\eta}_j}\tilde{Q}_k(0,\eta)\right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0,\varepsilon)\tau_k(\zeta_0,\varepsilon)},$$

and the constants involved are independent of ζ_0 and ε.

PROPOSITION 2.9 ([18, Lemma 3.4]). *For all w with $|w_j| < \tau_j(\zeta_0,\varepsilon)$,*

$$|\tilde{Q}_k(w,0)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0,\varepsilon)}, \qquad \left|\frac{\partial}{\partial\bar{\eta}_j}\tilde{Q}_k(w,0)\right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0,\varepsilon)\tau_k(\zeta_0,\varepsilon)}$$

and the constants involved are independent of ζ_0 and ε.

2.3. Extension operator. Assume as before that D is a smoothly bounded convex domain of finite type defined by a smooth function r such that $dr \neq 0$ on bD. We intend to recall the construction of an extension operator. Although our results concern affine linear hypersurfaces, we consider a more general setting now. Namely, let $V \supset \bar{D}$ be an open set and $h\colon V \to \mathbb{C}$ a holomorphic function. Define

$$A := \{z \in V\colon h(z) = 0\}.$$

The variety A is assumed to be non-singular and to cut the boundary bD transversally. In other words, we assume that $\partial h \neq 0$ on A and $\partial h \wedge \partial r \neq 0$ at each point of $A \cap bD$.

In order to be able to use integral formulae from [4] we need to define

$$\mathcal{Q}^1(\zeta, z) := \frac{1}{r(\zeta)}\bigg((1-\phi(\zeta))\sum_{j=1}^{n} Q_j(z,\zeta) + \phi(\zeta)\sum_{j=1}^{n}\frac{\partial r}{\partial \zeta_j}d\zeta_j\bigg).$$

The symbol ϕ stands for a smooth cut-off function supported on a compact subset of D.

With this definition we are ready to recall the construction developed in [4].

Proposition 2.10. *For any $N > 1$ the integral operator E_A^N defined on any function $f \in H^\infty(D\cap A)$ for any $z \in D$ by*

$$E_A^N f(z) := \int\limits_{D\cap A} f(\zeta)$$
$$\cdot \bigg[dV^{\#}\rfloor\bigg(\frac{r^{N+n-1}(\zeta)}{(r(\zeta)+(1-\phi(\zeta))S(z,\zeta)+\phi(\zeta)\sum_{j=1}^{n}\frac{\partial r}{\partial \zeta_j}(\zeta)(z_j-\zeta_j))^{N+n-1}}$$
$$\cdot(\overline{\partial}\mathcal{Q}^1(\zeta,z))^{n-1} \wedge \frac{\overline{\partial h(\zeta)} \wedge (\sum_{j=1}^{n} h_j(\zeta,z)d\zeta_j)}{\|\partial h\|^2}\bigg)\bigg]\, dV_{D\cap A}$$

is a linear extension operator. Furthermore, the function $E_A^N f(\,\cdot\,)$ is continuous on $\bar{D}\setminus(bD\cap A)$.

The holomorphic functions $h_j\colon V\times V \to \mathbb{C}$ are defined by

$$h(z) - h(\zeta) = \sum_{j=1}^{n} h_j(\zeta,z)(z_j-\zeta_j).$$

We now explain the meaning of the symbols used in Proposition 2.10. For an (n,n)-form Ω the symbol $dV^{\#}\rfloor\,\Omega$ stands for the contraction of the (n,n)-vector $dV^{\#}$, which is dual to the volume form, with the form Ω. The duality between the exterior algebras of the complexified tangent and cotangent bundles is induced by the duality between vectors and covectors. In order to make this statement clear, observe that once holomorphic coordinates

$(\zeta_1, \dots, \zeta_n)$ are chosen in such a way that

$$dV = c d\zeta_1 \wedge d\bar{\zeta}_1 \wedge \dots \wedge d\zeta_n \wedge d\bar{\zeta}_n$$

for some constant c, which is the case in our situation, and

$$\Omega = \Omega_n d\zeta_1 \wedge d\bar{\zeta}_1 \wedge \dots \wedge d\zeta_n \wedge d\bar{\zeta}_n,$$

then

$$dV^{\#} \rfloor \Omega = c\Omega_n. \tag{7}$$

Thus, for an (n,n)-form Ω the expression $dV^{\#} \rfloor \Omega$ is a smooth function, and consequently

$$\Big[dV^{\#} \rfloor \Big(\frac{r^{N+n-1}(\zeta)}{(r(\zeta) + (1-\phi(\zeta))S(z,\zeta) + \phi(\zeta)\sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j}(\zeta)(z_j - \zeta_j))^{N+n-1}} \\ \cdot (\bar{\partial}\mathcal{Q}^1(\zeta,z))^{n-1} \wedge \frac{\overline{\partial h(\zeta)} \wedge (\sum_{j=1}^{n} h_j(\zeta,z) d\zeta_j)}{\|\partial h\|^2} \Big) \Big] \, dV_{D\cap A}$$

is an $(n-1,n-1)$-form. Thus the integral in Proposition 2.10 is meaningful.

One more comment is in order. The operator E_A^N in Proposition 2.10 is defined for $f \in H^\infty(D \cap A)$ and under this assumption is an extension operator. It is easy to observe however that for an appropriate N the operator E_A^N is well-defined and is also an extension operator for those holomorphic functions which we consider. This will also follow from estimates which we prove in the next section.

3. Proofs. We shall deal now with extension operators. But first we need some auxiliary remarks. Assume that A and B are (p,q)-forms,

$$A = \sum_{\alpha,\beta} A_{\alpha\beta} Z^{*\alpha} \wedge \bar{Z}^{*\beta} = \sum_{\alpha,\beta} A_{\alpha\beta} Z^*_{\alpha_1} \wedge \dots \wedge Z^*_{\alpha_p} \wedge \bar{Z}^*_{\beta_1} \wedge \dots \wedge \bar{Z}^*_{\beta_q},$$
$$B = \sum_{\alpha,\beta} B_{\alpha\beta} Z^{*\alpha} \wedge \bar{Z}^{*\beta} = \sum_{\alpha,\beta} B_{\alpha\beta} Z^*_{\alpha_1} \wedge \dots \wedge Z^*_{\alpha_p} \wedge \bar{Z}^*_{\beta_1} \wedge \dots \wedge \bar{Z}^*_{\beta_q},$$

and φ is a differentiable map. The symbols $Z_1^*, \dots, Z_n^*$ stand for the frame of $(1,0)$-differential forms, which is dual to a frame $Z_1, \dots, Z_n$ of $(1,0)$-vectors. Thus

$$A^{\#} = \sum_{\alpha} A_{\alpha\beta} Z_{\alpha_1} \wedge \dots \wedge Z_{\alpha_p} \wedge \bar{Z}_{\beta_1} \wedge \dots \wedge \bar{Z}_{\beta_q}.$$

The pull-back of $A^{\#} \rfloor B$ by the map φ satisfies

$$\varphi^*(A^{\#} \rfloor B) = (\varphi^* A)^{\#} \rfloor (\varphi^* B).$$

We will use this fact below in the proofs of Lemmas 3.1 and 3.3.

LEMMA 3.1. *Assume that $\omega \in D$ and $e \in \mathbb{C}^n$ is a unit vector. Let, as before,*

$$A(\omega, e) := \{z \in \mathbb{C}^n \colon h_{\omega,e}(z) = \langle z-\omega, e\rangle = 0\}$$

be the complex affine linear hypersurface determined by ω and e. For the variety $A(\omega, e)$ consider the operator $E^N_{A(\omega,e)}$ from Proposition 2.10. Denote by $E^N_{A(\omega,e)}(\,\cdot\,,\,\cdot\,)$ the kernel function of this operator. If $N > 2$ then there exists a constant $C > 0$, which depends neither on ω nor on e, such that

$$\int_D |E^N_{A(\omega,e)}(\zeta, z)|\, dV(z) < C|\partial h_{\omega,e}|^2_{\mathcal{N}}(\zeta). \tag{8}$$

Proof. Notice that

$$\overline{\partial}\mathcal{Q}^1(\zeta,z) = -\frac{\overline{\partial} r}{r^2(\zeta)} \wedge \sum_{j=1}^n Q_j(z,\zeta)d\zeta_j + \frac{1}{r(\zeta)}\sum_{j=1}^n \overline{\partial} Q_j(z,\zeta)\wedge d\zeta_j.$$

As a result,

$$\begin{aligned}(\overline{\partial}\mathcal{Q}^1(\zeta,z))^{n-1} &= \frac{1}{r^{n-1}(\zeta)}\Big(\sum_{j=1}^n \overline{\partial} Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1} \\ &\quad - (n-1)\frac{\overline{\partial} r(\zeta)}{r^n(\zeta)} \wedge \Big(\sum_{j=1}^n Q_j(z,\zeta)d\zeta_j\Big) \wedge \Big(\sum_{j=1}^n \overline{\partial} Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-2}.\end{aligned}$$

Thus

$$\begin{aligned} & r^{N+n-1}(\zeta)(\mathcal{Q}^1(\zeta,z))^{n-1} \wedge \Omega[h_{\omega,e}](\zeta,z) \\ &\quad = r^N(\zeta)\Big(\sum_{j=1}^n \overline{\partial} Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1} \wedge \Omega[h_{\omega,e}](\zeta,z) \\ &\quad\quad - (n-1)r^{N-1}(\zeta)\overline{\partial} r(\zeta) \wedge \Big(\sum_{j=1}^n Q_j(z,\zeta)d\zeta_j\Big) \\ &\quad\quad \wedge \Big(\sum_{j=1}^n \overline{\partial} Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-2} \wedge \Omega[h_{\omega,e}](\zeta,z), \end{aligned} \tag{9}$$

where

$$\Omega[h_{\omega,e}](\zeta,z) := \frac{\overline{\partial h_{\omega,e}(\zeta)} \wedge \big(\sum_{j=1}^n h_{\omega,e,j}(\zeta,z)d\zeta_j\big)}{\|\partial h_{\omega,e}\|^2}.$$

We will deal with the part E^N_1 of the kernel $E^N_A(\zeta,z)$ which corresponds to the first term on the right-hand side of (9).

We may assume that $\zeta \in U \cap D$, since otherwise the estimate is trivial. For such a fixed ζ consider a cover $\{c_3 P_{\varrho(\zeta)}(\zeta), P^i_{\varrho(\zeta)}(\zeta)\}_{i\in\mathbb{N}}$ of D (the constant c_3

was defined in Proposition 2.2). It follows from Lemma 2.6 that

$$|r(\zeta)+S(z,\zeta)| \gtrsim \varrho(\zeta). \tag{10}$$

Furthermore, there exists $i_0 \in \mathbb{N}$ such that if $i \geq i_0$ and $z \in P^i_{\varrho(\zeta)}(\zeta)$, then

$$|r(\zeta)+S(z,\zeta)| \geq |S(z,\zeta)| - \varrho(\zeta) \gtrsim 2^i \varrho(\zeta), \tag{11}$$

by Lemma 2.5. Since estimate (11) may only fail for $i \leq i_0$ with a uniform i_0, we may assume in view of (10) that it holds for each $i \in \mathbb{N}$.

As a consequence, for each $\zeta, z \in D$,

$$\begin{aligned}(12)\quad &|E_1^N(\zeta,z)| \\ &\lesssim \varrho^{-(n-1)}(\zeta)\Big| dV^{\#}\rfloor\Big(\Big(\sum_{j=1}^n \overline{\partial}Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1} \wedge \Omega[h_{\omega,e}](\zeta,z)\Big)\Big|,\end{aligned}$$

and if additionally $z \in P^i_{\varrho(\zeta)}(\zeta)$, then

$$\begin{aligned}(13)\quad &|E_1^N(\zeta,z)| \\ &\lesssim (2^i)^{-N}(2^i\varrho(\zeta))^{-(n-1)}\Big|(dV^{\#})\rfloor\Big(\sum_{j=1}^n \overline{\partial}Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1} \wedge \Omega[h_{\omega,e}](\zeta,z)\Big)\Big|.\end{aligned}$$

In order to estimate the right-hand side in (12) and (13), we change coordinates. Choose a $2^i\varrho(\zeta)$-extremal basis $(u_1^{\zeta,2^i\varrho(\zeta)},\dots,u_n^{\zeta,2^i\varrho(\zeta)})$ at the point ζ and let $(w_1(z),\dots,w_n(z))$ be the corresponding coordinates of a point z. Then

$$w = (\Phi^{\zeta,2^i\varrho(\zeta)})^{-1}(z-\zeta)$$

with $\Phi^{\zeta,2^i\varrho(\zeta)}$ standing for a unitary transformation and

$$z = \Phi^{\zeta,2^i\varrho(\zeta)}(w)+\zeta = \varphi^{\zeta,2^i\varphi(\zeta)}(w).$$

Naturally,

$$P_{2^i\varrho(\zeta)}(\zeta) = \varphi^{\zeta,2^i\varrho(\zeta)}\big(\{w\in\mathbb{C}^n\colon |w_j| < \tau(\zeta,u_j^{\zeta,2^i\varrho(\zeta)},2^i\varrho(\zeta)),\ j=1,\dots,n\}\big).$$

To simplify the notation we write φ,Φ for $\varphi^{\zeta,2^i\varrho(\zeta)},\Phi^{\zeta,2^i\varrho(\zeta)}$, respectively, if ζ and i are fixed.

Denote

$$\mathcal{E}(\zeta,z) := \Big(\sum_{j=1}^n \overline{\partial}Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1}\wedge \Omega[h_{\omega,e}](\zeta,z).$$

With this notation we can write

$$\begin{aligned}(dV^{\#}\rfloor\mathcal{E})(\zeta,z) &= (dV^{\#}\rfloor\mathcal{E})(\varphi\varphi^{-1}(\zeta),\varphi\varphi^{-1}(z)) \\ &= \varphi^*\big(dV^{\#}\rfloor\mathcal{E}(\,\cdot\,,\varphi(w))\big)(\eta) \\ &= \big((\varphi^* dV)^{\#}\rfloor\varphi^*\mathcal{E}(\,\cdot\,,\varphi(w))\big)(\eta).\end{aligned}$$

Notice that

$$\varphi^*(\partial h_{\omega,e}) = \sum_{j=1}^{n} \frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j} d\eta_j$$

and

$$h_{\omega,e}(z) - h_{\omega.e}(\zeta) = \sum_{j=1}^{n} \frac{\partial h_{\omega,e}}{\partial\zeta_j}(z_j - w_j) = \sum_{j=1}^{n} h_{\omega,e,j}(\zeta,z)(z_j-\zeta_j)$$

with $h_{\omega,e,j} = \frac{\partial h_{\omega.e}}{\partial\zeta_j}$. This implies that

$$\sum_{j=1}^{n} h_{\omega,e,j}(\zeta,z)d\zeta_j = \partial_\zeta h_{\omega,e}$$

and, as a result,

$$\begin{aligned}(14)\qquad \varphi^*\Big(\overline{\partial h_{\omega,e}} \wedge \sum_{j=1}^{n} h_{\omega,e,j}(\zeta,z)d\zeta_j\Big) &= \overline{\partial(h_{\omega,e}\circ\varphi)} \wedge \partial(h_{\omega,e}\circ\varphi) \\ &= \sum_{j,l=1}^{n} \overline{\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}} \frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_l} d\bar\eta_j \wedge d\eta_l.\end{aligned}$$

Thus we obtain

$$\begin{aligned}(\varphi^*\mathcal{E})(0,\varphi(w)) = &\Big(\sum_{i,j=1}^{n} \frac{\partial\tilde Q_j}{\partial\bar\eta_i}(w,0)d\bar\eta_i \wedge d\eta_j\Big)^{n-1} \\ &\wedge \sum_{k,l=1}^{n} \overline{\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_k}}\,\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_l} d\bar\eta_k \wedge d\eta_l\end{aligned}$$

with the notation of Propositions 2.8 and 2.9. We can now complete the estimates of (12) and (13):

$$\begin{aligned}&\Big|(dV^{\#})\rfloor\Big(\sum_{j=1}^{n} \overline\partial Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1} \wedge \varOmega[h_{\omega,e}](\zeta,z)\Big)\Big| \\ &\qquad\le \sum_{\alpha,\beta}\prod_{j=1}^{n-1}\Big|\frac{\partial\tilde Q_{\alpha_j}}{\partial\bar\eta_{\beta_j}}(w,0)\Big|\Big|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_{\alpha_n}}\overline{\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_{\beta_n}}}\Big|,\end{aligned}$$

where the sum is over all permutations α,β of $1,\dots,n$. In order to estimate derivatives of the function $\tilde Q$ we apply Proposition 2.9. Eventually, we obtain for $z \in P^i_{\varrho(\zeta)}(\zeta)$ the estimate

$$\begin{aligned}&|E_1^N(\zeta,z)| \\ &\quad\lesssim 2^{-iN}(2^i\varrho(\zeta))^{-(n-1)}\Big|(dV^{\#})\rfloor\Big(\Big(\sum_{j=1}^{n}\overline\partial Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1} \wedge \varOmega[h_{\omega,e}](\zeta,z)\Big)\Big|\end{aligned}$$

$$\lesssim 2^{-iN}\sum_{j,l=1}^{n}\frac{1}{\prod_{\alpha\neq j,\,\beta\neq l}\tau_\alpha(\zeta,2^i\varrho(\zeta))\tau_\beta(\zeta,2^i\varrho(\zeta))}\left|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}\,\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_l}\right|$$
$$\lesssim 2^{-iN}\sum_{j=1}^{n}\frac{1}{\prod_{\alpha\neq j}\tau_\alpha^2(\zeta,2^i\varrho(\zeta))}\left|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}\right|^2,$$

and a similar one for $z\in P_{\varrho(\zeta)}(\zeta)$.

As a consequence, it follows from Lemma 2.3 that

$$\int\limits_{P^i_{\varrho(\zeta)}(\zeta)}|E_1^N(\zeta,z)|\,dV(z)\le 2^{-iN}\sum_{j=1}^{n}\left|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}\right|^2\tau(\zeta,u_j^{\zeta,2^i\varrho(\zeta)},2^i\varrho(\zeta))^2$$
$$\lesssim 2^{-i(N-2)}|\partial h_{\omega,e}|^2_{\mathcal{N}}(\zeta),$$

since

$$V(P^i_{\varrho(\zeta)})\le V(P_{2^i\varrho(\zeta)}(\zeta))\sim\prod_{j=1}^{n}\tau^2(\zeta,u_j^{\zeta,2^i\varrho(\zeta)},2^i\varrho(\zeta)).$$

This completes the proof of the lemma. ∎

The estimate in Lemma 3.1 suffices to prove Theorem 1.1.

Proof of Theorem 1.1. The proof follows immediately from estimate (8) in Lemma 3.1 and Fubini's Theorem. ∎

Observe that Theorems 1.1 and 1.3 immediately imply Theorem 1.2.

We now proceed to prove Theorem 1.4. Recall that Schur's test is a tool which allows one to prove that an integral operator

$$Tf(x)=\int\limits_X K(x,y)f(y)\,d\mu(y)$$

is bounded from $L^p(X,d\mu)$ into $L^p(X,d\nu)$ when μ and ν are positive measures on X, and K is a non-negative kernel. However, the situation which we come across in Theorem 1.4 does not fit exactly into the pattern of Schur's test. The measure which appears in the definition of the operator $E^N_{A(\omega,e)}$ differs from the measures which define the L^p spaces. We therefore need to modify Schur's criterion:

PROPOSITION 3.2. *Let μ,ν be positive Borel measures on X and let m be a positive weight function. If there exist non-negative functions h_1,h_2 such that*

$$\int\limits_X K(x,y)h_1(y)^q m(y)^{-q/p}d\,\mu(y)\le C_1h_2(x)^q,$$
$$\int\limits_X K(x,y)h_2(x)^p\,d\nu(x)\le C_2h_1(y)^p,$$

then the operator

$$Tf(x) = \int_X f(y)K(x,y)\,d\mu(y)$$

is bounded between $L^p(X, md\mu)$ *and* $L^p(X, d\nu)$.

For the sake of completeness we include the proof which is a minor modification of the standard case.

Proof. We can write

$$\begin{aligned} Tf(x) &= \int_X K(x,y)h_1(y)h_1(y)^{-1}f(y)m(y)^{-1/p}m(y)^{1/p}\,d\mu(y) \\ &\le \Big(\int_X K(x,y)h_1(y)^q m(y)^{-q/p}\,d\mu(y)\Big)^{1/q} \\ &\quad\times \Big(\int_X K(x,y)h_1(y)^{-p}m(y)|f(y)|^p d\mu(y)\Big)^{1/p} \\ &\le C_1^{1/q}h_2(x)\Big(\int_X K(x,y)h_1(y)^{-p}m(y)|f(y)|^p\,d\mu(y)\Big)^{1/p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_X |Tf(x)|^p\,d\nu(x) &\le C_1^{p/q}\int_X h_2(x)^p\int_X K(x,y)h_1(y)^{-p}m(y)|f(y)|^p\,d\mu(y)\,d\nu(x) \\ &= C_1^{p/q}\int_X |f(y)|^p h_1(y)^{-p}m(y)\int_X K(x,y)h_2(x)^p\,d\nu(x)\,d\mu(y) \\ &\le C_1^{p/q}C_2\int_X |f(y)|^p m(y)\,d\mu(y). \end{aligned}$$

Thus

$$\Big(\int_X |Tf(x)|^p d\nu(x)\Big)^{1/p} \le C_1^{1/q}C_2^{1/p}\Big(\int_X |f(x)|^p m(x)d\mu(x)\Big)^{1/p}. \quad \blacksquare$$

We now apply Proposition 3.2 with $d\nu = dV$, $d\mu = dV_{D\cap A(\omega,e)}$ and $m = |\partial h_{\omega,e}|_{\mathcal{N}}^{2-\epsilon}$ for some $\epsilon > 0$. Set $h_2 \equiv 1$ and $h_1(\zeta) = |\partial h_{\omega,e}|_{\mathcal{N}}^{2/p}(\zeta)$. Lemma 3.1 says that the second estimate in Proposition 3.2 holds true with this choice of measures and functions. In order to prove that

$$E^N_{A(\omega,e)}\colon H^p(D\cap A(\omega,e), |\partial h_{\omega,e}|_{\mathcal{N}}^{2-\epsilon}) \to H^p(D)$$

with $1 < p < \infty$ and $\epsilon > 0$ we therefore need to show that for some constant C_ϵ,

$$\int_{D\cap A} |E^N_{A(\omega,e)}(\zeta,z)|\,|\partial h_{\omega,e}|_{\mathcal{N}}^{2q/p}|\partial h_{\omega,e}|_{\mathcal{N}}^{-q/p(2-\varepsilon)}\,dV_{D\cap A(\omega,e)} \le C_\epsilon.$$

This means that the proof of Theorem 1.4 will be completed once we prove the following lemma:

LEMMA 3.3. *Assume that $\omega \in D$ and $e \in \mathbb{C}^n$ is a unit vector. Let, as before,*

$$A(\omega,e) := \{z \in \mathbb{C}^n : h_{\omega,e}(z) = \langle z-\omega, e\rangle = 0\}$$

be the complex affine linear hypersurface determined by ω and e. For the variety $A(\omega,e)$ consider the operator $E^N_{A(\omega,e)}$ from Proposition 2.10. Denote by $E^N_{A(\omega,e)}(\,\cdot\,,\,\cdot\,)$ the kernel function of this operator. If $N > 2$ and $\epsilon > 0$ then there exists a constant $C_\epsilon > 0$, which depends neither on ω nor on e, such that

$$\int\limits_{D\cap A(\omega,e)} |E^N_{A(\omega,e)}(\zeta,z)|\,|\partial h_{\omega,e}|^\epsilon_{\mathcal{N}}(\zeta)\,dV_{D\cap A_{\omega,e}}(\zeta) < C_\epsilon. \tag{15}$$

Proof. Fix $z \in D$. It is enough to provide estimates under the assumption that $\zeta \in P_c(z)$ for some small fixed $c > 0$ which is independent of z. Indeed, it follows from Lemmas 2.5 and 2.6 that

$$\int\limits_{(D\cap A(\omega,e))\setminus P_c(z)} |E^N_{A(\omega,e)}(\zeta,z)|\,|\partial h_{\omega,e}|^\epsilon_{\mathcal{N}}(\zeta)\,dV_{D\cap A(\omega,e)} \lesssim 1.$$

Consider a family $P^i_{\varrho(z)}(z)$ with $2^i\varrho(z) < Cc$ with a uniform $C > 0$ which is chosen in such a way that

$$P_c(z) \setminus \tfrac{1}{2}P_{\varrho(z)}(z) \subset \bigcup_{i=0}^{\lceil \log_2 \frac{Cc}{\varrho(z)}\rceil} P^i_{\varrho(z)}(z).$$

In order to prove the lemma we need to deal with the following integrals:

$$\int\limits_{D\cap A(\omega,e)\cap \frac{1}{2}P_{\varrho(z)}(z)} |E^N_{A(\omega,e)}(\zeta,z)|\,|\partial h_{\omega,e}|^\epsilon_{\mathcal{N}}(\zeta)\,dV_{D\cap A}(\zeta),$$

$$\int\limits_{D\cap A(\omega,e)\cap P^i_{\varrho(z)}(z)} |E^N_{A(\omega,e)}(\zeta,z)|\,|\partial h_{\omega,e}|^\epsilon_{\mathcal{N}}(\zeta)\,dV_{D\cap A}(\zeta),$$

for $i = 0,\ldots,\lceil \log_2 \frac{Cc}{\varrho(z)}\rceil$.

As in Lemma 3.1, we estimate the kernel function $E^N_1(\zeta,z)$. This time however z is fixed.

We may assume that if $\zeta \in \frac{1}{2}P_{\varrho(z)}(z)$ then $\varrho(\zeta) \lesssim \varrho(z)$. It then follows from Lemma 2.6 that

$$|r(\zeta) + S(z,\zeta)| \gtrsim \varrho(z).$$

As a consequence, for $\zeta \in \frac{1}{2}P_{\varrho(z)}(z)$,

$$
\begin{aligned}
(16)\qquad & |E_1^N(\zeta,z)| \\
&\lesssim \varrho^{-(n-1)}(z)\Big|dV^{\#}\rfloor\Big(\Big(\sum_{j=1}^{n}\overline{\partial}Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1}\wedge \Omega[h_{\omega,e}](\zeta,z)\Big)\Big|.
\end{aligned}
$$

If $\zeta \in P^i_{\varrho(z)}(z)$ then it follows from Lemmas 2.6 and 2.7 that

$$
|E_1^N(\zeta,z)| \lesssim \frac{1}{(2^i\varrho(z))^{n-1}}\Big|dV^{\#}\rfloor\Big(\Big(\sum_{j=1}^{n}\overline{\partial}Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1}\wedge \Omega[h_{\omega,e}](\zeta,z)\Big)\Big|.
$$

Also,

$$
\begin{aligned}
|\partial h_{\omega,e}|_{\mathcal{N}}(\zeta) &= \sup\{|\partial h_{\omega,e}(v)|\tau(\zeta,v,\varrho(\zeta))\colon v\neq 0\}\\
&= \sup\Big\{\Big|\partial h_{\omega,e}\Big(|v|\frac{v}{|v|}\Big)\Big|\tau(\zeta,v,\varrho(\zeta))\colon v\neq 0\Big\}\\
&= \sup\Big\{\Big|\partial h_{\omega,e}\Big(\frac{v}{|v|}\Big)\Big|\tau\Big(\zeta,\frac{v}{|v|},\varrho(\zeta)\Big)\colon v\neq 0\Big\}
\end{aligned}
$$

and, as a result, uniformly

$$
|\partial h_{\omega,e}|_{\mathcal{N}}(\zeta) = \sup\{|\partial h_{\omega,e}(v)|\tau(\zeta,v,\varrho(\zeta))\colon |v|=1\}\lesssim \varrho(\zeta)^{1/M}.
$$

If $\zeta \in \frac{1}{2}P_{\varrho(z)}(z)$ then $\varrho(\zeta)\lesssim\varrho(z)$, and if $\zeta\in P^i_{\varrho(z)}(z)$ then $\varrho(\zeta)\lesssim 2^i\varrho(z)$. Thus

$$
\begin{aligned}
&\int\limits_{D\cap A(\omega,e)\cap\frac{1}{2}P_{\varrho(z)}(z)} |E_1^N(\zeta,z)|\,|\partial h_{\omega,e}|^{\epsilon}_{\mathcal{N}}(\zeta)\,dV_{D\cap A}(\zeta)\\
&\quad\lesssim (\varrho(z))^{\epsilon/M}\frac{1}{(\varrho(z))^{n-1}}\int\limits_{D\cap A(\omega,e)\cap\frac{1}{2}P_{\varrho(z)}(z)} |((dV^{\#})\rfloor\mathcal{E})(\zeta,z)|\,dV_{D\cap A}(\zeta),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\limits_{D\cap A(\omega,e)\cap P^i_{\varrho(z)}(z)} |E_1^N(\zeta,z)|\,|\partial h_{\omega,e}|^{\epsilon}_{\mathcal{N}}(\zeta)\,dV_{D\cap A}(\zeta)\\
&\quad\lesssim (2^i\varrho(z))^{\epsilon/M}\frac{1}{(2^i\varrho(z))^{n-1}}\int\limits_{D\cap A(\omega,e)\cap P^i_{\varrho(z)}(z)} |((dV^{\#})\rfloor\mathcal{E})(\zeta,z)|\,dV_{D\cap A}(\zeta),
\end{aligned}
$$

where, as in Lemma 3.1,

$$
\mathcal{E}(\zeta,z) := \Big(\sum_{j=1}^{n}\overline{\partial}Q_j(z,\zeta)\wedge d\zeta_j\Big)^{n-1}\wedge\Omega[h_{\omega,e}](\zeta,z).
$$

We have therefore to estimate

$$\int\limits_{D\cap A(\omega,e)\cap\frac{1}{2}P_{\varrho(z)}(z)} |((dV^{\#})\rfloor\mathcal{E})(\zeta,z)|\,dV_{D\cap A}(\zeta)$$

and

$$\int\limits_{D\cap A(\omega,e)\cap P^i_{\varrho(z)}(z)} |((dV^{\#})\rfloor\mathcal{E})(\zeta,z)|\,dV_{D\cap A}(\zeta).$$

We consider the latter integral; the former is treated in the same way. Choose a $2^i\varrho(z)$-extremal basis at z and as in Proposition 2.8 let

$$\eta = \Phi(\zeta - z)$$

for a unitary transformation Φ, and

$$\zeta = \varphi(\eta) = z + \Phi^{\star}\eta.$$

We change variables and put $z = \varphi(0)$ to obtain

$$\begin{aligned}
&\int\limits_{D\cap A(\omega,e)\cap P^i_{\varrho(z)}(z)} |(dV^{\#}\rfloor\mathcal{E})(\zeta,z)|\,dV_{D\cap A}(\zeta)\\
&\quad\le \int\limits_{D\cap A(\omega,e)\cap\varphi(\{\eta\colon\,|\eta_j|<\tau_j(z,2^i\varrho(z)),\,j=1,\dots,n\})} |(dV^{\#}\rfloor\mathcal{E})(\zeta,z))|\,dV_{D\cap A}(\zeta)\\
&\quad= \int\limits_{\{\eta\colon\,|\eta_j|<\tau_j(z,2^i\varrho(z)),\,j=1,\dots,n\}\cap\varphi^{-1}(D\cap A)} \big|\varphi^*\big((dV^{\#}\rfloor\mathcal{E}(\zeta,z))\big)\big|\varphi^*(dV_{D\cap A})(\eta)\\
&\quad= \int\limits_{\{\eta\colon\,|\eta_j|<\tau_j(z,2^i\varrho(z)),\,j=1,\dots,n\}\cap\varphi^{-1}(D\cap A)} \big|\big((\varphi^*\,dV^{\#})\rfloor(\varphi^*\mathcal{E})\big)(\eta,\varphi(0))\big|\\
&\qquad\qquad\cdot\varphi^*(dV_{D\cap A})(\eta).
\end{aligned}$$

Since

$$\varphi^* dV \sim d\eta_1\wedge d\bar{\eta}_1\wedge\cdots\wedge d\eta_n\wedge d\bar{\eta}_n,$$

the function $(\varphi^* dV^{\#})\rfloor(\varphi^*\mathcal{E})$ is, up to a constant, equal to the coefficient of

$$d\eta_1\wedge d\bar{\eta}_1\wedge\cdots\wedge d\eta_n\wedge d\bar{\eta}_n$$

in $\varphi^*\mathcal{E}$ (cf. comments preceding (7)). With the same notation as in (6) and (14) we obtain

$$\big|\big((\varphi^* dV^{\#})\rfloor(\varphi^*\mathcal{E})\big)(\eta,\varphi(0))\big| \le \sum_{\alpha,\beta}\prod_{j=1}^{n-1}\bigg|\frac{\partial\tilde{Q}_{\alpha_j}}{\partial\eta_{\beta_j}}(0,\eta)\bigg|\,\bigg|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_{\alpha_n}}\overline{\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_{\beta_n}}}\bigg|$$

where the sum is over all permutations α,β of $1,\dots,n$. Therefore it follows from Proposition 2.8 that

$$\int_{D\cap A(\omega,e)\cap P^i_{\varrho(z)}(z)} \big|(dV^{\#}\rfloor\mathcal{E})(\zeta,z)\big| dV_{D\cap A} \tag{17}$$

$$\lesssim \sum_{j=1}^{n} \frac{(2^i\varrho(z))^{n-1}}{\prod_{k\neq j}\tau_k^2(z,2^i\varrho(z))} \int_{\{\eta:|\eta_j|<\tau_j(z,2^i\varrho(z))\}\cap\varphi^{-1}(D\cap A)} \left|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}\right|^2 \varphi^*(dV_{D\cap A}).$$

Thus we need to estimate $\varphi^*(dV_{D\cap A})$. Consider a d-dimensional complex submanifold $N\subset D\subset\subset\mathbb{C}^n$. According to Wirtinger's formula, if

$$\sum_{j,l=1}^{n} h_{j\bar{l}}dz_j\otimes\overline{dz_l}$$

is a Hermitian metric in $D\subset\mathbb{C}^n$ (in general on an n-dimensional complex manifold), then

$$\left(\frac{\sqrt{-1}}{2d!}\right)^d\Big(\sum_{j,l=1}^{n} h_{j\bar{l}}dz_j\wedge d\bar{z}_l\Big)^d$$

restricted to N is equal to the volume form on N induced by the metric in $\mathbb{C}^n$ on N. This means that

$$dV_{D\cap A}\sim\Big(\sum_{j=1}^{n} d\zeta_j\wedge d\bar{\zeta}_j\Big)^{n-1}_{|D\cap A}$$

and consequently

$$\varphi^*(dV_{D\cap A})\sim\Big(\sum_{j=1}^{n} d\eta_j\wedge d\bar{\eta}_j\Big)^{n-1}_{|\varphi^{-1}(D\cap A)}.$$

Since $A(\omega,e)=\{\zeta\colon h_{\omega,e}(\zeta)=0\}$, we have

$$0=\sum_{j=1}^{n}\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}\,d\eta_j$$

on $\varphi^{-1}(A(\omega,e))$. As a result, if $\frac{\partial(h_{\omega,\varepsilon}\circ\varphi)}{\partial\eta_j}\neq 0$, then

$$\varphi^*(dV_{D\cap A}) \tag{18}$$

$$\lesssim(\sqrt{-1})^{n-1}\frac{d\eta_1\wedge d\bar{\eta}_1\wedge\ldots\wedge d\eta_{j-1}\wedge d\bar{\eta}_{j-1}\wedge d\eta_{j+1}\wedge d\bar{\eta}_{j+1}\wedge\ldots\wedge d\eta_n\wedge d\bar{\eta}_n}{\left|\frac{\partial(h_{\omega,e}\circ\varphi)}{\partial\eta_j}\right|^2}$$

with a uniform constant.

Therefore it follows from (17) that

$$\begin{aligned}
&\int\limits_{D\cap A\cap P^i_{\varrho(z)}(z)} |E_1^N(\zeta,z)|\,|\partial h_{\omega,e}|^{\epsilon}_{\mathcal{N}}(\zeta)\,dV_{D\cap A}(\zeta)\\
&\quad\lesssim \sum_{j=1}^{n}(2^i\varrho(z))^{\epsilon/M}\frac{1}{\prod_{k\neq j}\tau_k^2(z,2^i\varrho(z))}\\
&\qquad\cdot\int\limits_{P^i_j} d\Re\eta_1\,d\Im\eta_1\ldots d\Re\eta_{j-1}\,d\Im\eta_{j-1}\,d\Re\eta_{j+1}\,d\Im\eta_{j+1}\ldots d\Re\eta_n\,d\Im\eta_n\\
&\quad\lesssim (2i\varrho(z))^{\epsilon/M},
\end{aligned}$$

where we used the symbol P^i_j to denote the projection of $\varphi^{-1}(P_{2^i\varrho(z)}(z))$ onto the hyperplane $\{\eta_{j=0}\}\cong\mathbb{C}^{n-1}$. Consequently,

$$\int\limits_{D\cap A}|E_1^N(\zeta,z)|\,|\partial h_{\omega,e}|^{\epsilon}_{\mathcal{N}}\,dV_{D\cap A}(\zeta)\lesssim 1+\varrho(z)^{\epsilon/M}\sum_{i=0}^{\lceil\log_2\frac{Cc}{\varrho(z)}\rceil}2^{\epsilon i/M}\lesssim 1,$$

which completes the proof. ■

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