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Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions

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Abstract. We consider an almost Kenmotsu manifold M^{2n+1} with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ and we prove that M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, provided that M^{2n+1} is ξ-Riemannian-semisymmetric. Moreover, if M^{2n+1} is a ξ-Riemannian-semisymmetric almost Kenmotsu manifold such that ξ belongs to the (k,μ)-nullity distribution, we prove that M^{2n+1} is of constant sectional curvature -1.

1. Introduction. The notion of k-nullity distribution was first introduced by A. Gray [11] and S. Tanno [21] in the study of Riemannian manifolds (M,g), and it is defined for any $p \in M$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.1}$$

where X, Y denote arbitrary vectors in T_pM and $k \in \mathbb{R}$. We also refer the reader to [3, 15] for some related results on k-nullity distribution.

Recently, D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [2] introduced a generalized notion of the k-nullity distribution named the (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = k[&g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{1.2}$$

where $h = \frac{1}{2}\mathcal{L}_\xi \phi$, $\mathcal{L}$ denotes the Lie differentiation and $(k,\mu) \in \mathbb{R}^2$.

Later, G. Dileo and A. M. Pastore [8] introduced another generalized notion of the k-nullity distribution named the $(k,\mu)'$-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any

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$p \in M^{2n+1}$ as follows:

$$N_p(k,\mu)' = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \tag{1.3}$$

where $h = \frac{1}{2}\mathcal{L}_\xi\phi$, $h' = h \circ \phi$, and $(k,\mu) \in \mathbb{R}^2$.

Results on contact metric and Sasakian manifolds satisfying the (k,μ)-nullity conditions have been obtained by many authors. For more details, we refer the reader to [2, 10, 17, 20]. On the other hand, Kenmotsu manifolds [14], another special class of almost contact metric manifolds, were also investigated by many authors. For example, U. C. De, A. Yıldız and A. F. Yalınız obtained some results on ϕ-recurrent Kenmotsu manifolds [5] and 3-dimensional locally ϕ-recurrent normal almost contact metric manifolds [6]. Ricci semisymmetric and W_2-semisymmetric Kenmotsu manifolds were investigated by J. B. Jun et al. [13] and A. Yıldız et al. [24] respectively. Also, H. Öztürk, N. Aktan and C. Murathan [16] studied α-Kenmotsu manifolds under some conditions such as Ricci recurrence or Ricci semisymmetry.

However, the results mentioned above all concern Kemotsu manifolds; recently, however, G. Dileo and A. M. Pastore [7] obtained some important classification theorems on locally symmetric almost Kenmotsu manifolds. Moreover, almost Kenmotsu manifolds satisfying some nullity conditions [8] and requiring a condition of η-parallelism [9] were also investigated by G. Dileo and A. M. Pastore. Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions were studied by the present authors in [22]. It is worth pointing out that almost Kenmotsu pseudo-metric manifolds satisfying the (k,μ) or $(k,\mu)'$-nullity conditions were studied by the present authors in [23]. For some results on k-nullity distributions, and generalized $(k,\mu)'$- and (k,μ)-nullity distributions on almost Kenmotsu manifolds, we refer the reader to A. M. Pastore and V. Saltarelli [18, 19].

A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold (see [12]), thus, it is of interest to generalize some results on Kenmotsu manifolds to almost Kenmotsu manifolds. In fact, in this paper, we classify almost Kenmotsu manifolds satisfying the nullity conditions and the Riemannian semisymmetry conditions as follows.

THEOREM 1.1. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold of dimension $2n+1$ such that M^{2n+1} is ξ-Riemannian-semisymmetric. If the characteristic vector field ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$, then $k=-2$, $\mu=-2$ and hence M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

THEOREM 1.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension $(2n+1)$ such that M^{2n+1} is ξ-Riemannian-semisymmetric. If the characteristic vector field ξ belongs to the (k, μ)-nullity distribution, then M^{2n+1} is of constant sectional curvature -1.*

This paper is organized in the following way. In Section 2, we provide some basic formulas and properties of almost Kenmotsu manifolds due to G. Dileo and A. M. Pastore [7] and K. Kenmotsu [14]. Later, Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2 with their corollaries, respectively.

2. Almost Kenmotsu manifolds. We first recall some basic notions and properties of almost Kenmotsu manifolds following [7, 14]. An *almost contact structure* [1] on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity mapping, which imply that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. A Riemannian metric g on M^{2n+1} is said to be *compatible* with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y on M^{2n+1}. An almost contact structure endowed with a compatible Riemannian metric is said to be an *almost contact metric structure* (see [1]). The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M^{2n+1}. An almost contact metric manifold with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is called an *almost Kenmotsu manifold*. It is known [1] that the normality of an almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Consequently, the normality of an almost Kenmotsu manifold is expressed by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ for any vector fields X, Y. A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (see [12]).

We put $l = R(\cdot\,, \xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi \phi$ on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. Thus, the two $(1,1)$-type tensor fields l and h are symmetric and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad \mathrm{tr}(h) = 0, \quad \mathrm{tr}(h\phi) = 0, \quad h\phi + \phi h = 0. \tag{2.3}$$

We also have the following formulas presented in [7, 8]:

$$\nabla_X\xi = -\phi^2 X - \phi h X \ (\Rightarrow \nabla_\xi \xi = 0), \tag{2.4}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$\operatorname{tr} l = S(\xi,\xi) = g(Q\xi,\xi) = -2n - \operatorname{tr} h^2, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where $h' = h \circ \phi$ and S, Q, ∇ and $\Gamma(TM)$ denote the Ricci tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the Lie algebra of all vector fields on M^{2n+1}, respectively.

Throughout this paper, by *Riemannian semisymmetry* of a Riemannian manifold M we mean that $R(X,Y) \cdot R = 0$ for any $X, Y \in \Gamma(TM)$. Next we give the notion of ξ-Riemannian-semisymmetry:

DEFINITION 2.1. An almost Kenmotsu manifold is said to be *ξ-Riemannian-semisymmetric* if

$$R(\xi, X) \cdot R = 0 \tag{2.8}$$

for any vector field X, where $R(\xi, X)$ acts on R as a derivation.

3. ξ belongs to the $(k,\mu)'$-nullity distribution. In this section we deal with almost Kenmotsu manifolds for which ξ belongs to the $(k,\mu)'$-nullity distribution. From (1.3) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.1}$$

where k and μ are both constants on M^{2n+1}. Throughout the paper, we denote by $\mathcal{D}$ the contact distribution defined by $\mathcal{D} = \ker(\eta) = \operatorname{Im}(\phi)$. Replacing Y by ξ in (3.1) gives $lX = k(X - \eta(X)\xi) + \mu h'X$; using (2.1) and (2.3) in this equation we obtain $\phi l \phi X = -k(X - \eta(X)\xi) + \mu h'X$. Substituting the above equation into (2.5) gives

$$h'^2 = (k+1)\phi^2 \ (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{3.2}$$

Now let $X \in \mathcal{D}$ be the eigenvector field of h' corresponding to the eigenvalue λ, thus from (3.2) we see that $\lambda^2 = -(k+1)$. It follows that $k \le -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote the eigenspaces associated with h' by $[\lambda]'$ and $[-\lambda]'$ corresponding to the eigenvalue $\lambda \neq 0$ and $-\lambda$ of h' respectively.

LEMMA 3.1 ([8, Proposition 4.2]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k,\mu)'$-nullity distribution. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the*

Riemannian curvature tensor satisfies:

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \tag{3.3}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0, \tag{3.4}$$

$$R(X_\lambda, Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \tag{3.5}$$

$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \tag{3.6}$$

$$R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \tag{3.7}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}] \tag{3.8}$$

and $\mu = -2$.

Proof of Theorem 1.1. Suppose that an almost Kenmotsu manifold M^{2n+1} is ξ-Riemannian semisymmetric and ξ belongs to the $(k,\mu)'$-nullity distribution. Thus, using the symmetries of the Riemannian curvature, from (3.1) it follows that

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X] \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$. Also, replacing X by ξ in (3.1) gives

$$R(X,\xi)\xi = kX - k\eta(X)\xi + \mu h'X \tag{3.10}$$

for any $X \in \Gamma(TM)$. On the other hand, the assumption that M^{2n+1} is ξ-Riemannian semisymmetric implies $(R(\xi, X)\cdot R)(U, V, W) = 0$ for any $X, U, V, W \in \Gamma(TM)$. Substituting $U = \xi$ into this equation we have

$$\begin{aligned} &R(X,\xi)R(\xi,V)W \\ &\quad = R(R(X,\xi)\xi, V)W + R(\xi, R(X,\xi)V)W + R(\xi,V)R(X,\xi)W \end{aligned} \tag{3.11}$$

for any $X, V, W \in \Gamma(TM)$. Now we calculate every term in equation (3.11). Making use of (3.9) and (3.10) we obtain

$$\begin{aligned} &R(X,\xi)R(\xi,V)W \\ &\quad = kg(V,W)R(X,\xi)\xi - k\eta(W)R(X,\xi)V \\ &\qquad + \mu g(h'V,W)R(X,\xi)\xi - \mu\eta(W)R(X,\xi)h'V \\ &\quad = \big[-k^2 g(V,W)\eta(X) + k^2\eta(W)g(X,V) + 2k\mu\eta(W)g(h'V,X) \\ &\qquad - k\mu\eta(X)g(h'V,W) + \mu^2\eta(W)g(h'^2X,V)\big]\xi + k^2 g(V,W)X \\ &\qquad + k\mu g(V,W)h'X - k^2\eta(V)\eta(W)X - k\mu\eta(V)\eta(W)h'X \\ &\qquad + k\mu g(h'V,W)X + \mu^2 g(h'V,W)h'X \end{aligned} \tag{3.12}$$

for any $X, V, W \in \Gamma(TM)$. Making use of (3.10) we get

$$\begin{aligned}(3.13)\quad & R(R(X,\xi)\xi, V)W \\ &= kR(X,V)W - k\eta(X)R(\xi,V)W + \mu R(h'X,V)W \\ &= -[k^2\eta(X)g(V,W) + k\mu\eta(X)g(h'V,W)]\xi \\ &\quad + kR(X,V)W + \mu R(h'X,V)W + k^2\eta(X)\eta(W)V \\ &\quad + k\mu\eta(X)\eta(W)h'V\end{aligned}$$

for any $X, V, W \in \Gamma(TM)$. Similarly, it follows from (3.9) and (3.10) that

$$\begin{aligned}(3.14)\quad & R(\xi, R(X,\xi)V)W \\ &= k\eta(V)R(\xi,X)W + \mu\eta(V)R(\xi,h'X)W \\ &= \big[k^2\eta(V)g(X,W) + k\mu\eta(V)g(h'X,W) + k\mu\eta(V)g(h'X,W) \\ &\quad + \mu^2\eta(V)g(h'^2X,W)\big]\xi - k^2\eta(V)\eta(W)X - 2k\mu\eta(V)\eta(W)h'X \\ &\quad - \mu^2\eta(V)\eta(W)h'^2X\end{aligned}$$

for any $X, V, W \in \Gamma(TM)$. Moreover, from (3.9) and (3.10) we obtain

$$\begin{aligned}(3.15)\quad & R(\xi,V)R(X,\xi)W \\ &= -kg(X,W)R(\xi,V)\xi + k\eta(W)R(\xi,V)X \\ &\quad - \mu g(h'X,W)R(\xi,V)\xi + \mu\eta(W)R(\xi,V)h'X \\ &= \big[-k^2\eta(V)g(X,W) + k^2\eta(W)g(X,V) + 2k\mu\eta(W)g(h'X,V) \\ &\quad - k\mu\eta(V)g(h'X,W) + \mu^2\eta(W)g(h'^2X,V)\big]\xi + k^2g(X,W)V \\ &\quad + k\mu g(X,W)h'V - k^2\eta(X)\eta(W)V - k\mu\eta(X)\eta(W)h'V \\ &\quad + k\mu g(h'X,W)V + \mu^2 g(h'X,W)h'V\end{aligned}$$

for any $X, V, W \in \Gamma(TM)$. Finally, substituting (3.12)–(3.15) into (3.11) gives

$$\begin{aligned}(3.16)\quad & k^2g(V,W)X + k\mu g(V,W)h'X + k\mu g(h'V,W)X \\ &= kR(X,V)W + \mu R(h'X,V)W - k\mu\eta(V)\eta(W)h'X \\ &\quad - \mu^2\eta(V)\eta(W)h'^2X + [k\mu\eta(V)g(h'X,W) + \mu^2\eta(V)g(h'^2X,W)]\xi \\ &\quad + k^2g(X,W)V + k\mu g(X,W)h'V + k\mu g(h'X,W)V \\ &\quad + \mu^2g(h'X,W)h'V - \mu^2g(h'V,W)h'X\end{aligned}$$

for any $X, V, W \in \Gamma(TM)$. G. Dileo and A. M. Pastore [8] proved that if ξ belongs to the $(k,\mu)'$-nullity distribution then $\mu = -2$. Using this relation and letting $X, W \in [\lambda]'$ and $V \in [-\lambda]'$ in (3.16) implies that

$$(3.17)\quad kR(X,V)W - 2\lambda R(X,V)W + k^2g(X,W)V - 4\lambda^2g(X,W)V = 0.$$

Applying Lemma 3.1 to (3.17) we obtain

$$(3.18)\qquad [(k-2\lambda)(k+2) + k^2 - 4\lambda^2]g(X,W)V = 0.$$

By using the relationship $\lambda = \pm\sqrt{-k-1}$ in (3.18) we obtain

$$2\lambda(\lambda-1)(\lambda+1)^2 = 0. \tag{3.19}$$

Noticing that the assumption $h' \neq 0$ implies that $k \neq -1$ from (3.2) and hence $\lambda \neq 0$, then it follows from (3.19) that $\lambda^2 = 1$ and hence $k = -2$. Without losing generality we now choose $\lambda = 1$. Then (3.7) and (3.8) can be written respectively as follows:

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Furthermore, using $\mu = -2$ and from (3.10) it follows that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. As shown in [8], the distribution $[\xi]\oplus[\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves since $H = -(1-\lambda)\xi$, where H is the mean curvature vector field of the leaves of $[-\lambda]'$ immersed in M^{2n+1}. As $\lambda = 1$, we know that the two orthogonal distributions $[\xi]\oplus[\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves. This completes the proof. ∎

COROLLARY 3.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension $2n+1$ such that the characteristic vector field ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If M^{2n+1} is Riemannian semisymmetric, then M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Since the almost Kenmotsu manifold M^{2n+1} is Riemannian semisymmetric, it is ξ-Riemannian-semisymmetric. Now the assertion follows from Theorem 1.1. ∎

4. ξ belongs to the (k,μ)-nullity distribution. In this section we shall deal with almost Kenmotsu manifolds for which the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. From (1.2) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{4.1}$$

where k and μ are both constants on M^{2n+1}.

Before giving the detailed proof of Theorem 1.2, we need the following key lemma.

LEMMA 4.1 ([8, Theorem 4.1]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a warped product of an open interval and an almost Kähler manifold.*

Proof of Theorem 1.2. Suppose that an almost Kenmotsu manifold M^{2n+1} is ξ-Riemannian semisymmetric and ξ belongs to the (k,μ)-nullity distribution. It follows from (4.1) and Lemma 4.1 that

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X \tag{4.2}$$

for any $X, Y \in \Gamma(TM)$. Also, replacing Y by ξ in (4.1) gives

$$R(X,\xi)\xi = -X + \eta(X)\xi \tag{4.3}$$

for any $X \in \Gamma(TM)$. Next we again calculate every term in (3.11). Making use of (4.2) and (4.3) we obtain

$$\begin{aligned} R(X,\xi)R(\xi,V)W &= -g(V,W)R(X,\xi)\xi + \eta(W)R(X,\xi)V \\ &= -g(V,W)\eta(X)\xi + \eta(W)g(X,V)\xi + g(V,W)X - \eta(V)\eta(W)X \end{aligned} \tag{4.4}$$

for any $X, V, W \in \Gamma(TM)$. By using (4.3) we also get

$$\begin{aligned} R(R(X,\xi)\xi,V)W &= -R(X,V)W + \eta(X)R(\xi,V)W \\ &= -R(X,V)W + \eta(X)\eta(W)V - \eta(X)g(V,W)\xi \end{aligned} \tag{4.5}$$

for any $X, V, W \in \Gamma(TM)$. Similarly, it follows from (4.2) and (4.3) that

$$R(\xi,R(X,\xi)V)W = -\eta(V)R(\xi,X)W = \eta(V)g(X,W)\xi - \eta(V)\eta(W)X \tag{4.6}$$

for any $X, V, W \in \Gamma(TM)$. Finally, from (4.2) and (4.3) we obtain

$$\begin{aligned} R(\xi,V)R(X,\xi)W &= g(X,W)R(\xi,V)\xi - \eta(W)R(\xi,V)X \\ &= -\eta(V)g(X,W)\xi + \eta(W)g(X,V)\xi + g(X,W)V - \eta(X)\eta(W)V \end{aligned} \tag{4.7}$$

for any $X, V, W \in \Gamma(TM)$. Substituting (4.4)–(4.7) into (3.11) we have

$$R(X,V)W = -g(V,W)X + g(X,W)V \tag{4.8}$$

for any $X, V, W \in \Gamma(TM)$. Thus, the conclusion follows from (4.8). ∎

G. Dileo and A. M. Pastore [8] proved that a locally symmetric almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is of constant sectional curvature -1. Applying Theorem 1.2 we now give the following corollary which extends the related result shown in [8].

COROLLARY 4.2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold of dimension $2n+1$ such that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. Then the following statements are equivalent:*

(i) *M^{2n+1} is of constant sectional curvature -1.*
(ii) *M^{2n+1} is locally symmetric, that is, $\nabla R = 0$.*
(iii) *M^{2n+1} is Riemannian semisymmetric, that is, $R \cdot R = 0$.*
(iv) *M^{2n+1} is ξ-Riemannian-semisymmetric.*

Proof. It is easy to see (i)⇒(ii)⇒(iii)⇒(iv). On the other hand, Theorem 1.2 implies that (iv)⇒(i) under the given hypotheses. ∎

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