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Subextension of plurisubharmonic functions without changing the Monge–Ampère measures and applications

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Abstract. The aim of the paper is to investigate subextensions with boundary values of certain plurisubharmonic functions without changing the Monge–Ampère measures. From the results obtained, we deduce that if a given sequence is convergent in C_{n-1}-capacity then the sequence of the Monge–Ampère measures of subextensions is weakly*-convergent. As an application, we investigate the Dirichlet problem for a nonnegative measure μ in the class $\mathcal{F}(\Omega, g)$ without the assumption that μ vanishes on all pluripolar sets.

1. Introduction. Let $\Omega \subset \widetilde{\Omega}$ be domains in $\mathbb{C}^n$ and let u be a plurisubharmonic function on Ω (briefly, $u \in \mathrm{PSH}(\Omega)$). A function $\widetilde{u} \in \mathrm{PSH}(\widetilde{\Omega})$ is said to be a *subextension* of u if $\widetilde{u}(z) \le u(z)$ for all $z \in \Omega$. The subextension problem in the class $\mathcal{F}(\Omega)$ has been studied by Cegrell and Zeriahi [CeZe], who proved that if $\Omega, \widetilde{\Omega}$ are bounded hyperconvex domains in $\mathbb{C}^n$ with $\Omega \Subset \widetilde{\Omega}$ and $u \in \mathcal{F}(\Omega)$, then there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega})$ such that $\widetilde{u} \le u$ on Ω and

$$\int_{\widetilde{\Omega}} (dd^c \widetilde{u})^n \le \int_{\Omega} (dd^c u)^n.$$

For the class $\mathcal{E}_p(\Omega), p > 0$, the subextension problem was investigated by P. H. Hiep [H2], who proved that if $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}^n$ are hyperconvex domains and $u \in \mathcal{E}_p(\Omega), p > 0$, then there exists a function $\widetilde{u} \in \mathcal{E}_p(\widetilde{\Omega})$ such that $\widetilde{u} \le u$ on Ω and $e_p(\widetilde{u}) = \int_{\widetilde{\Omega}} (-\widetilde{u})^p (dd^c \widetilde{u})^n \le \int_{\Omega} (-u)^p (dd^c u)^n$. The subextension problem involving boundary values was considered by Czyż and Hed [CH], who showed that if Ω_1 and Ω_2 are two bounded hyperconvex domains such that $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n, n \ge 1$ and $u \in \mathcal{F}(\Omega_1, F)$ with $F \in \mathcal{E}(\Omega_1)$, then u has a subextension $v \in \mathcal{F}(\Omega_2, G)$ with $G \in \mathcal{E}(\Omega_2) \cap \mathrm{MPSH}(\Omega_2)$ and

$$\int_{\Omega_2} (dd^c v)^n \le \int_{\Omega_1} (dd^c u)^n,$$

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under the assumption that $F \geq G$ on Ω_1, where $\mathcal{F}(\Omega)$ (resp. $\mathcal{E}(\Omega)$) are the classes of unbounded plurisubharmonic functions defined in Section 2 and $\mathrm{MPSH}(\Omega)$ denotes the set of maximal plurisubharmonic functions on Ω. It should be remarked that in [CeZe] and [CH], only estimates on the total Monge–Ampère mass of the subextension and of the given function are obtained.

In this paper, besides establishing the existence of a subextension with given boundary values in the class $\mathcal{F}(\Omega, f)$, we prove that the Monge–Ampère measures of the subextension and of the given function do not change. Namely, we prove the following:

THEOREM 3.4. *Let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$ and $g \in \mathcal{E}(\widetilde{\Omega}) \cap \mathrm{MPSH}(\widetilde{\Omega})$ with $f \geq g$ on Ω. Then for every $u \in \mathcal{F}(\Omega, f)$ with $\int_\Omega (dd^c u)^n < \infty$ there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $\widetilde{u} \leq u$ on Ω and $(dd^c \widetilde{u})^n = 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$.*

Note that from our results, it is easy to obtain estimates on the total Monge–Ampère mass, appearing in [CeZe] and [CH]. Next, from the above result, we deduce a result on the weak*-convergence of the sequence of the Monge–Ampère measures of subextensions if we assume that the given sequence is convergent in C_{n-1}-capacity (see Corolary 3.5 below). Also, using the result obtained, we investigate the Dirichlet problem for a nonnegative measure μ in the class $\mathcal{F}(\Omega, g)$ with $g \in \mathcal{E}(\Omega) \cap \mathrm{MPSH}(\Omega)$. It should be noticed that the above problem was considered earlier by Åhag, who solved the Dirichlet problem in the class $\mathcal{F}(\Omega, g)$ for a nonnegative measure μ under the assumption that μ vanishes on pluripolar sets. In our note, we omit this assumption.

The paper is organized as follows. In Section 2, we give some elements of pluripotential theory which are necessary for our results. We recall some classes of unbounded plurisubharmonic functions introduced and investigated by Cegrell. Note that by the results of Cegrell [Ce1], [Ce2], the Monge–Ampère operator is well defined on these classes as a nonnegative Radon measure. Next, in Section 3, we prove the main results of the paper. We give new proofs for subextension of plurisubharmonic functions with boundary values and show the equality between the Monge–Ampère measures of the subextension and of the given function. In Section 4 we apply the above result to investigating the Dirichlet problem.

2. Preliminaries. The elements of pluripotential theory that will be used in this paper can be found in [ÅCCH], [BT], [Kl], [Ko1], [Ko2] and [Xi]. Now we recall some Cegrell's classes of plurisubharmonic functions (see [Ce1] and [Ce2]) and classes of plurisubharmonic functions with generalized boundary values connected with Cegrell's classes. Let Ω be an open set in $\mathbb{C}^n$.

By $\mathrm{PSH}^-(\Omega)$, we denote the set of negative plurisubharmonic functions on Ω.

2.1. Now we assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$. This means that Ω is a bounded domain in $\mathbb{C}^n$ and there exists a plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that $\Omega_c = \{z \in \Omega : \varphi(z) < c\} \Subset \Omega$ for every $c < 0$. As in [Ce1], we define the following subclasses of $\mathrm{PSH}^-(\Omega)$:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\Big\},$$

and

$$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \forall z_0 \in \Omega,\ \exists \text{ a neighbourhood } \omega \ni z_0,$$
$$\mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } \omega,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\Big\}.$$

The following inclusions are obvious: $\mathcal{E}_0 \subset \mathcal{F} \subset \mathcal{E}$.

Next, we recall classes of plurisubharmonic functions with generalized boundary values in $\mathcal{E}$. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}\}$. Then we say that a plurisubharmonic function u defined on Ω is in $\mathcal{K}(\Omega, G)$ for some $G \in \mathcal{E}$ if there exists a function $\varphi \in \mathcal{K}$ such that

$$\varphi + G \le u \le G$$

on Ω. For a systematic and complete study of classes of plurisubharmonic functions with generalized boundary values in other classes, we refer the readers to [ÅhC2]. Note that functions in $\mathcal{K}(\Omega, G)$ need not have finite total Monge–Ampère mass (see [ÅhC1]).

2.2. Because in this note we also need the class of maximal plurisubharmonic functions, we recall the following definition given in [Bł1].

Definition 2.1. A plurisubharmonic function u on Ω is said to be *maximal plurisubharmonic* (briefly, $u \in \mathrm{MPSH}(\Omega)$) if for every compact set $K \Subset \Omega$ and every $v \in \mathrm{PSH}(\Omega)$, if $v \le u$ on $\Omega \setminus K$ then $v \le u$ on Ω.

It is well known (see, e.g., [Kl]) that locally bounded plurisubharmonic functions are maximal if and only if they satisfy the homogeneous Monge–Ampère equation $(dd^cu)^n = 0$. In [Bł2] Blocki extended the above result to the class $\mathcal{E}(\Omega)$.

2.3. We now recall the notion of C_n-capacity of a Borel set and extensions of this notion, as well as the convergence in capacity of a sequence of plurisubharmonic functions. For related definitions, we refer the readers to [Xi] and [Ce3].

Let Ω be an open set in $\mathbb{C}^n$ and $E \subset \Omega$ a Borel subset. Following [BT] we define the C_n-*capacity* of E as follows:

$$C_n(E) = C_n(E, \Omega) = \sup\Big\{ \int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 < u < 0 \Big\}.$$

Extending this notion, Xing [Xi] introduced the notion of C_{n-1}-capacity. Let $E \subset \Omega$ be a Borel subset. The C_{n-1}-*capacity* of E is defined by

$$C_{n-1}(E) = C_{n-1}(E, \Omega) = \sup\Big\{ \int_E (dd^c u)^{n-1} \wedge dd^c |z|^2 : u \in \mathrm{PSH}(\Omega),\ -1 < u < 0 \Big\}.$$

In [Xi], it is remarked that there is a constant $A_\Omega > 0$ such that $C_{n-1}(E) \leq A_\Omega C_n(E)$ for all Borel subsets $E \subset \Omega$.

Next, we deal with the convergence of a sequence of plurisubharmonic functions in capacity and recall an important result due to Cegrell [Ce3] which we need in our proofs later.

Let u_j, u be plurisubharmonic functions in an open set Ω of $\mathbb{C}^n$. We say that u_j *converges to* u *in* C_s-*capacity*, $s = n, n-1$, if for every compact subset K of Ω and every $\delta > 0$,

$$\lim_{j\to\infty} C_s(\{z \in K : |u_j(z) - u(z)| > \delta\}) = 0.$$

From the inequality $C_{n-1}(E) \leq A_\Omega C_n(E)$ for all $E \subset \Omega$ it follows that convergence in C_n-capacity implies convergence in C_{n-1}-capacity. Moreover, by using the quasi-continuity of plurisubharmonic functions [BT], it is not difficult to prove that if a sequence $\{u_j\}$ of plurisubharmonic functions is increasing (or decreasing) and converges to a plurisubharmonic function u then it converges to u in C_n-capacity.

An important result proved recently by Cegrell [Ce3] is as follows. Assume that $u_0 \in \mathcal{E}$ and $\{u_j\} \subset \mathcal{E}$ is a sequence with $u_0 \leq u_j$ for $j \geq 1$. If u_j converges to a plurisubharmonic function $u \in \mathcal{E}$ in C_{n-1}-capacity then the sequence of measures $(dd^c u_j)^n$ converges to $(dd^c u)^n$ in the weak*-topology as $j \to \infty$. We shall use this result several times in our proofs in the next section.

3. Subextension with boundary values without changing the Monge–Ampère measures. The aim of this section is to study subextensions of plurisubharmonic functions in the class $\mathcal{F}(\Omega, f)$, $f \in \mathcal{E}$. Our results differ those in [CH] in that we control the Monge–Ampère mesures of subextensions and of the given function. In order to arrive at the main result of this note (Theorem 3.4 below) we need some auxiliary results.

The following proposition is the main tool in the proof of Theorem 3.4 and in Section 4.

Proposition 3.1. *Let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $g \in \mathcal{E}(\widetilde{\Omega}) \cap \mathrm{MPSH}(\widetilde{\Omega})$. Assume that $u \in \mathcal{E}(\Omega)$ satisfies the following conditions:*

(a) $\int_\Omega (dd^c u)^n < \infty$,
(b) $(dd^c v)^n \le 1_\Omega (dd^c u)^n$ *on* $\widetilde{\Omega}$ *with some* $v \in \mathcal{F}(\widetilde{\Omega}, g)$ *and* $v \le u$ *on* Ω.

Then there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $\widetilde{u} \le v$ on $\widetilde{\Omega}$ and $(dd^c\widetilde{u})^n = 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$.

Proof. Since $(dd^c v)^n \le 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$, we have

$$1_{\{v=-\infty\}}(dd^c v)^n \le 1_{\Omega\cap\{u=-\infty\}}(dd^c u)^n$$

on $\widetilde{\Omega}$. Moreover, since $v \le u$ on Ω, Lemma 4.1 in [ÅCCH] implies that

$$1_{\Omega\cap\{v=-\infty\}}(dd^c v)^n \ge 1_{\Omega\cap\{u=-\infty\}}(dd^c u)^n$$

on Ω. Therefore $1_{\{v=-\infty\}}(dd^c v)^n = 1_{\Omega\cap\{u=-\infty\}}(dd^c u)^n$ on $\widetilde{\Omega}$. Put $\mu = 1_\Omega (dd^c u)^n - (dd^c v)^n$ on $\widetilde{\Omega}$. We notice that μ is a nonnegative measure vanishing on all pluripolar subsets of $\widetilde{\Omega}$. We split the proof into three steps.

Step 1. We prove that there exists $w \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $w \le v$ and

$$(dd^c w)^n \ge \mu + 1_{\{v>-\infty\}}(dd^c v)^n.$$

Indeed, since μ vanishes on all pluripolar sets in $\widetilde{\Omega}$ and

$$\mu(\widetilde{\Omega}) \le \int_\Omega (dd^c u)^n < \infty,$$

Lemma 5.14 in [Ce1] implies that there exists $v_0 \in \mathcal{F}(\widetilde{\Omega})$ such that $(dd^c v_0)^n = \mu$. Put $w = v + v_0$. We have $w \in \mathcal{F}(\widetilde{\Omega}, g)$, $w \le v$ and

$$(dd^c w)^n \ge (dd^c v_0)^n + (dd^c v)^n \ge \mu + 1_{\{v>-\infty\}}(dd^c v)^n.$$

Step 2. Put

$$\widetilde{u} = (\sup\{\varphi \in \mathcal{E}(\widetilde{\Omega}) : \varphi \le v \text{ and } (dd^c\varphi)^n \ge \mu + 1_{\{v>-\infty\}}(dd^c v)^n\})^*.$$

It is clear that $w \le \widetilde{u} \le v$. Hence, $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega}, g)$. We will prove that

$$(dd^c\widetilde{u})^n \ge 1_\Omega (dd^c u)^n \quad \text{on } \widetilde{\Omega}.$$

Indeed, by using the Choquet lemma we infer that there exists a sequence $\{\varphi_j\} \subset \mathcal{E}(\widetilde{\Omega})$ such that $\varphi_j \le v$, $(dd^c\varphi_j)^n \ge \mu + 1_{\{v>-\infty\}}(dd^c v)^n$ and

$$\widetilde{u} = \Big(\sup_{j\in\mathbb{N}^*} \varphi_j\Big)^*.$$

By Proposition 4.3 in [KH] we can replace φ_j by $\max\{w, \varphi_1, \ldots, \varphi_j\}$. Hence, we can assume that $w \le \varphi_j$ and $\varphi_j \nearrow \widetilde{u}$ a.e. Therefore, $\varphi_j \to \widetilde{u}$ in

C_n-capacity and by [Ce3] we have $(dd^c\varphi_j)^n \to (dd^c\widetilde{u})^n$ weakly. Thus,

$$(dd^c\widetilde{u})^n \geq \mu + 1_{\{v>-\infty\}}(dd^c v)^n.$$

The above inequality is equivalent to

$$1_{\{\widetilde{u}>-\infty\}}(dd^c\widetilde{u})^n \geq \mu + 1_{\{v>-\infty\}}(dd^c v)^n. \tag{3.1}$$

Moreover, since $\widetilde{u} \leq v$ on $\widetilde{\Omega}$, Lemma 4.1 in [ÅCCH] implies that

$$1_{\{\widetilde{u}=-\infty\}}(dd^c\widetilde{u})^n \geq 1_{\{v=-\infty\}}(dd^c v)^n. \tag{3.2}$$

Combining (3.1) and (3.2) we infer that

$$(dd^c\widetilde{u})^n \geq \mu + (dd^c v)^n = 1_\Omega (dd^c u)^n.$$

STEP 3. For each $j = 1, 2, \ldots$, put

$$\widetilde{u}_j = (\sup\{\varphi \in \mathcal{E}(\widetilde{\Omega}) : \varphi \leq \max(v, g-j),\ (dd^c\varphi)^n \geq \mu + 1_{\{v>g-j\}}(dd^c v)^n\})^*.$$

It is easy to see that $\widetilde{u} \leq \widetilde{u}_{j+1} \leq \widetilde{u}_j$ for every j and $\widetilde{u} = \lim_{j\to\infty} \widetilde{u}_j$. Now, since $\mu + (dd^c \max(v, g-j))^n$ vanishes on all pluripolar sets, by Theorem 3.9 in [Ce2] there exists a function $w_j \in \mathcal{F}(\widetilde{\Omega}, g)$ such that

$$(dd^c w_j)^n = \mu + (dd^c \max(v, g-j))^n.$$

Since $(dd^c w_j)^n \geq (dd^c \max(v, g-j))^n$, Theorem 3.8 in [Ce2] implies that $w_j \leq \max(v, g-j)$. Moreover, by Theorem 4.1 in [KH] we have

$$(dd^c w_j)^n \geq \mu + 1_{\{v>g-j\}}(dd^c \max(v, g-j))^n = \mu + 1_{\{v>g-j\}}(dd^c v)^n.$$

Hence, $w_j \leq \widetilde{u}_j$. From Proposition 2.2 in [CH] it follows that

$$\begin{aligned}\int_\Omega (dd^c u)^n \leq \int_{\widetilde{\Omega}} (dd^c\widetilde{u})^n &= \lim_{j\to\infty} \int_{\widetilde{\Omega}} (dd^c\widetilde{u}_j)^n \\ &\leq \limsup_{j\to\infty} \int_{\widetilde{\Omega}} (dd^c w_j)^n \\ &\leq \limsup_{j\to\infty} \int_{\widetilde{\Omega}} [\mu + (dd^c \max(v, g-j))^n] = \int_\Omega (dd^c u)^n.\end{aligned}$$

Hence,

$$(dd^c\widetilde{u})^n = 1_\Omega (dd^c u)^n. \ \blacksquare$$

The next lemma is important for the proof of Theorem 3.4.

LEMMA 3.2. *Let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$. Assume that $f \in \mathcal{E}(\Omega)$ and $g \in \mathcal{E}(\widetilde{\Omega}) \cap \mathrm{MPSH}(\widetilde{\Omega})$ are such that $f \geq g + \delta$ on Ω with some $\delta > 0$. Then for every $u \in \mathcal{E}_0(\Omega, f)$ such that the function*

$$w := (\sup\{\varphi \in \mathrm{PSH}^-(\widetilde{\Omega}) : \varphi \leq g \text{ on } \widetilde{\Omega}\setminus\Omega \text{ and } \varphi \leq \min(u, g) \text{ on } \Omega\})^*$$

is in $\mathcal{E}(\widetilde{\Omega})$, we have $(dd^c w)^n \leq 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$.

Proof. We split the proof into two steps.

STEP 1. We prove that $(dd^c w)^n = 0$ on $\widetilde{\Omega}\setminus\Omega$. Indeed, since $u \in \mathcal{E}_0(\Omega, f)$, there exists $\psi \in \mathcal{E}_0(\Omega)$ such that $\psi + f \le u \le f$ on Ω. Put $U := \overline{\{\psi < -\delta\}} \Subset \Omega$. Then $\psi \ge -\delta$ on $\Omega\setminus U$ so it is easy to see that $\min(u, g) = g$ on $\Omega\setminus U$. Since $\widetilde{\Omega}\setminus\overline{U}$ is an open set and $g \in \mathrm{MPSH}(\widetilde{\Omega}\setminus\overline{U})$, we have $w \in \mathrm{MPSH}(\widetilde{\Omega}\setminus\overline{U})$. Indeed, let $v \in \mathrm{PSH}^-(\widetilde{\Omega}\setminus\overline{U})$ and $v \le w$ outside $K \Subset \widetilde{\Omega}\setminus\overline{U}$. Put

$$v_1 = \begin{cases} \max(v, w) & \text{on } \widetilde{\Omega}\setminus U, \\ w & \text{on } U. \end{cases}$$

Then $v_1 \in \mathrm{PSH}^-(\widetilde{\Omega})$ and $v_1 \le w \le g$ outside K in $\widetilde{\Omega}$. By the maximality of g it follows that $v_1 \le g$ on $\widetilde{\Omega}$. It is easy to see that $v_1 \le \min(u, g)$ on Ω. Hence, by definition of w it follows that $v_1 \le w$ on $\widetilde{\Omega}$ and the desired conclusion follows. Thus $(dd^c w)^n = 0$ on $\widetilde{\Omega}\setminus\Omega$.

STEP 2. We will prove $(dd^c w)^n \le (dd^c u)^n$ on Ω. First, we prove that $(dd^c w)^n = 0$ on $\{w < \min(u, g)\}\cap\Omega$. It is easy to see that

$$\begin{aligned}\{w < \min(u,g)\}\cap\Omega &= \bigcup_{a\in\mathbb{Q}^-} (\{w < a < \min(u,g)\}\cap\Omega) \\ &\subset \bigcup_{a,b\in\mathbb{Q}^-}\bigcup_{\varepsilon>0} \big((\{w < a < u-\varepsilon < b < g\}\cap\Omega) \\ &\qquad\qquad \cup (\{w < a < g-\varepsilon < b < u\})\cap\Omega\big).\end{aligned}$$

Hence, it suffices to prove that $(dd^c w)^n = 0$ on $\{w < a < u-\varepsilon < b < g\}\cap\Omega$; the proof for $\{w < a < g-\varepsilon < b < u\}$ is similar. Let $\{u_j\} \subset \mathcal{E}_0(\Omega)\cap\mathcal{C}(\Omega)$ with $u_j \searrow u$ on Ω and $\{g_j\} \subset \mathcal{E}_0(\widetilde{\Omega})\cap\mathcal{C}(\widetilde{\Omega})$ be such that $g_j \searrow g$ on $\widetilde{\Omega}$. Put

$$w_j := (\sup\{\varphi \in \mathrm{PSH}^-(\widetilde{\Omega}) : \varphi \le g_j \text{ on } \widetilde{\Omega}\setminus\Omega \text{ and } \varphi \le \min(u_j, g_j) \text{ on } \Omega\})^*.$$

Them $w_j \in \mathcal{E}(\widetilde{\Omega})$. We have $w_j \searrow w$ as $j \nearrow \infty$ and $\{w < a\} = \bigcup_{k=1}^{\infty}\{w_k < a\}$. Hence, it suffices to show that $(dd^c w)^n = 0$ on $\{w_k < a < u-\varepsilon < b < g\}\cap\Omega$. By Corollary 9.2 in [BT] it is easy to see that

$$(dd^c w_j)^n = 0 \quad \text{on } \{w_j < \min(u_j, g_j)\}\cap\Omega.$$

Moreover, $\{w_k < a < u-\varepsilon < b < g\}\cap\Omega \subset \{w_j < \min(u_j, g_j)\}\cap\Omega$ for every $j \ge k$. Hence,

$$(dd^c w_j)^n = 0 \quad \text{on } \{w_k < a < u-\varepsilon < b < g\}\cap\Omega$$

for every $j \ge k$. Therefore,

$$\max(g-b, 0)(dd^c w_j)^n = 0 \quad \text{on } \{w_k < a < u-\varepsilon < b\}\cap\Omega$$

for every $j \ge k$. Hence,

$$\max(u-\varepsilon-a, 0)\max(g-b, 0)(dd^c w_j)^n = 0 \quad \text{on } \{w_k < a\}\cap\{u < b+\varepsilon\}\cap\Omega$$

for every $j \geq k$. This is equivalent to

$$\big[(\max(u-\varepsilon-a,0)+\max(g-b,0))^2-\max(u-\varepsilon-a,0)^2 \\ -\max(g-b,0)^2\big](dd^c w_j)^n = 0$$

on $\{w_k < a\} \cap \{u < b+\varepsilon\} \cap \Omega$ for every $j \geq k$. Therefore, by Corollary 3.3 in [Ce3] we get

$$\big[(\max(u-\varepsilon-a,0)+\max(g-b,0))^2-\max(u-\varepsilon-a,0)^2 \\ -\max(g-b,0)^2\big](dd^c w)^n = 0$$

on $\{w_k < a\} \cap \{u < b+\varepsilon\} \cap \Omega$. Thus Lemma 4.2 in [KH] implies that

$$(dd^c w)^n = 0 \quad \text{ on } \{w < a < u-\varepsilon < b < g\} \cap \Omega.$$

Now we prove that $(dd^c w)^n \leq (dd^c u)^n$ on $\{w = \min(u,g)\} \cap \Omega$. Indeed, since $\{w = \min(u,g)\} \subset \{w = g\} \cup \{w = u\}$ so it suffices to prove that $(dd^c w)^n \leq (dd^c u)^n$ on $\{w = g\} \cup \{w = u\}$. Let K be a compact set in $\{w = g\}$. Since $K \Subset \{w + 1/j > g\}$ for every j so by Theorem 4.1 in [KH] we have

$$\int_K (dd^c w)^n = \lim_{j\to\infty} \int_K (dd^c \max(w+1/j,g))^n \\ \leq \int_K (dd^c \max(w,g))^n = \int_K (dd^c g)^n.$$

Hence, $(dd^c w)^n \leq (dd^c g)^n = 0 \leq (dd^c u)^n$ on $\{w = g\}$. Similarly, $(dd^c w)^n \leq (dd^c u)^n$ on $\{w = u\}$. Therefore, $(dd^c w)^n \leq (dd^c u)^n$ on Ω. ∎

Remark 3.3. From the above proof we have the following. Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$, μ is a nonnegative measure in Ω, and $u, v \in \mathcal{E}(\Omega)$ are such that $(dd^c u)^n \leq 1_A \mu$, $(dd^c v)^n \leq 1_B \mu$, where $A \cap B = \emptyset$ and

$$w := (\sup\{\varphi \in \mathrm{PSH}^-(\Omega) : \varphi \leq \min(u,v) \text{ on } \Omega\})^* \in \mathcal{E}(\Omega).$$

Then $(dd^c w)^n \leq 1_{A\cup B}\mu$.

Now we are in a position to state the main result of the paper. Note that in our result, in contrast to [CeZe], the assumption that $\Omega \Subset \widetilde{\Omega}$ is not necessary. At the same time, compared with Theorem 1.1 in [CH], we obtain a better relation between the Monge–Ampère measures of a subextension and the given function.

Theorem 3.4. *Let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$ and let $f \in \mathcal{E}(\Omega)$ and $g \in \mathcal{E}(\widetilde{\Omega}) \cap \mathrm{MPSH}(\widetilde{\Omega})$ with $f \geq g$ on Ω. Then for every $u \in \mathcal{F}(\Omega, f)$ with $\int_\Omega (dd^c u)^n < \infty$ there exists $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $\widetilde{u} \leq u$ on Ω and $(dd^c \widetilde{u})^n = 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$.*

Proof. By Proposition 3.1 it suffices to construct a function $v \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $v \le u$ on Ω and $(dd^c v)^n \le 1_\Omega (dd^c u)^n$ on $\widetilde{\Omega}$.

First, we prove that there exists $w \in \mathcal{F}(\widetilde{\Omega}, g)$ such that $w \le u$ on Ω. Indeed, since $u \in \mathcal{F}(\Omega, f)$, there exists $u_0 \in \mathcal{F}(\Omega)$ such that $u_0 + f \le u \le f$ on Ω. By Lemma 4.5 in [H2] there exist $\widetilde{u}_0 \in \mathcal{F}(\widetilde{\Omega})$ such that $\widetilde{u}_0 \le u_0$ on Ω. Put $w := \widetilde{u}_0 + g$. Then $w \in \mathcal{F}(\widetilde{\Omega}, g)$. Moreover, since $f \ge g$ on Ω, we have $w \le u_0 + f \le u$ on Ω.

Now, since $u \in \mathcal{F}(\Omega, f)$, there exists $\{u_j\} \subset \mathcal{E}_0(\Omega, f)$ such that $u_j \searrow u$ as $j \nearrow \infty$. Indeed, from the definition of $\mathcal{F}(\Omega, f)$ there exists $\psi \in \mathcal{F}$ such that

$$\psi + f \le u \le f$$

on Ω. Take a sequence $\{\psi_j\} \subset \mathcal{E}_0(\Omega)$ such that $\psi_j \searrow \psi$ on Ω. Put $u_j = \max(\psi_j + f, u)$. Then $u_j \in \mathcal{E}_0(\Omega, f)$ and $u_j \searrow u$ on Ω as $j \nearrow \infty$, as desired.

Choose a sequence $\delta_j \searrow 0$. Put $g_j = g - \delta_j$ and

$$v_{j,k} := (\sup\{\varphi \in \mathrm{PSH}^-(\widetilde{\Omega}) : \varphi \le g_k \text{ on } \widetilde{\Omega} \setminus \Omega \text{ and } \varphi \le \min(u_j, g_k) \text{ on } \Omega\})^*.$$

It is easy to see that $w - \delta_k \le v_{j,k}$ on $\widetilde{\Omega}$, so $v_{j,k} \in \mathcal{E}(\widetilde{\Omega})$. Moreover, $\{v_{j,k}\}_{k \ge 1}$ is increasing as $k \nearrow \infty$. Lemma 3.2 implies that $(dd^c v_{j,k})^n \le 1_\Omega (dd^c u_j)^n$ on $\widetilde{\Omega}$. Put $v_j = (\lim_{k \to \infty} v_{j,k})^*$. We have $v_{j,k} \nearrow v_j$ a.e. on $\widetilde{\Omega}$, hence by [Ce3] we get $(dd^c v_j)^n \le 1_\Omega (dd^c u_j)^n$ on $\widetilde{\Omega}$.

On the other hand, it is easy to see that $v_j \searrow v \in \mathcal{E}(\widetilde{\Omega})$ so again by [Ce3] we get $(dd^c v_j)^n \to (dd^c v)^n$ weakly in $\widetilde{\Omega}$. Moreover, by Lemma 3.1 in [Ce1] and Corollary 3.4 in [ÅCCH] it follows that $1_\Omega (dd^c u_j)^n \to 1_\Omega (dd^c u)^n$ weakly in $\widetilde{\Omega}$. Therefore,

$$(dd^c v)^n \le 1_\Omega (dd^c u)^n \quad \text{on } \widetilde{\Omega}.$$

Finally, since $w \le v \le g$ on $\widetilde{\Omega}$, we have $v \in \mathcal{F}(\widetilde{\Omega}, g)$ and $v \le u$ on Ω, and the desired conclusion follows. ∎

From the above theorem we have the following corollary which deals with the weak*-convergence of the sequence of the Monge–Ampère measures of subextensions when the given sequence is convergent in C_{n-1}-capacity.

Corollary 3.5. *Let $\Omega \subset \widetilde{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$, and $f \in \mathcal{E}(\Omega)$ and $g \in \mathcal{E}(\widetilde{\Omega}) \cap \mathrm{MPSH}(\widetilde{\Omega})$ be such that $f \ge g$ on Ω. Then for every sequence $u_j, u_0 \subset \mathcal{F}(\Omega, f)$ such that $u_j \ge u_0$ for all $j \ge 1$ and $\int_\Omega (dd^c u_j)^n < \infty$, $\int_\Omega (dd^c u_0)^n < \infty$, and u_j is convergent to u_0 in C_{n-1}-capacity on Ω, the subextensions $\widetilde{u}_j, \widetilde{u}_0 \subset \mathcal{F}(\widetilde{\Omega}, g)$ of u_j, u_0 as in Theorem* 3.4 *are such that $(dd^c \widetilde{u}_j)^n$ is weakly*-convergent to $(dd^c \widetilde{u}_0)^n$ as $j \to \infty$.*

Proof. First we show that for every $\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{j \to \infty} \int_\Omega \varphi (dd^c u_j)^n = \int_\Omega \varphi (dd^c u_0)^n. \tag{3.3}$$

Indeed, from the hypothesis it follows that u_j converges to u_0 on Ω in measure with respect to the Lebesgue measure dV_n on $\mathbb{C}^n$. Hence, there exists a subsequence of $\{u_j\}_{j\geq 1}$ that converges to u_0 a.e. on Ω. Without loss of generality we may assume that u_j is convergent to u_0 a.e. on Ω. Put $v_j := (\sup\{u_s : s \geq j\})^*$. We have $u_j \leq v_j$, and hence $v_j \in \mathcal{F}(\Omega, f)$ and $v_j \searrow u_0$ as $j \to \infty$. Corollary 3.4 in [ÅCCH] implies that if $\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$ then

$$\lim_{j\to\infty} \int_\Omega \varphi (dd^c v_j)^n = \int_\Omega \varphi (dd^c u_0)^n.$$

Moreover, since $u_0 \leq u_j \leq v_j$ on Ω, Lemma 3.3 in [ÅCCH] yields

$$\int_\Omega \varphi (dd^c v_j)^n \geq \int_\Omega \varphi (dd^c u_j)^n \geq \int_\Omega \varphi (dd^c u_0)^n.$$

Hence, we get

$$\lim_{j\to\infty} \int_\Omega \varphi (dd^c v_j)^n = \lim_{j\to\infty} \int_\Omega \varphi (dd^c u_j)^n = \int_\Omega \varphi (dd^c u_0)^n,$$

and (3.3) is proved.

Now, by Theorem 3.4 there exist subextensions $\widetilde{u}_j, \widetilde{u}_0 \in \mathcal{F}(\widetilde{\Omega}, g)$ of u_j, u_0 such that $\widetilde{u}_j \leq u_j$ and $\widetilde{u}_0 \leq u_0$ on Ω and $(dd^c\widetilde{u}_j)^n = 1_\Omega (dd^c u_j)^n$ and $(dd^c\widetilde{u}_0)^n = 1_\Omega (dd^c u_0)^n$. We prove that $(dd^c\widetilde{u}_j)^n$ is weakly*-convergent to $(dd^c\widetilde{u}_0)^n$. Indeed, assume that $\chi \in \mathcal{C}_0^\infty(\widetilde{\Omega})$. By Lemma 3.1 in [Ce1] there exist $\varphi_1, \varphi_2 \in \mathcal{E}_0(\widetilde{\Omega})$ such that $\chi = \varphi_1 - \varphi_2$. We have

$$\begin{aligned}\lim_{j\to\infty} \int_{\widetilde{\Omega}} \chi (dd^c\widetilde{u}_j)^n &= \lim_{j\to\infty} \int_\Omega \varphi_1 (dd^c u_j)^n - \lim_{j\to\infty} \int_\Omega \varphi_2 (dd^c u_j)^n \\ &= \int_\Omega \varphi_1 (dd^c u_0)^n - \int_\Omega \varphi_2 (dd^c u_0)^n = \int_{\widetilde{\Omega}} \chi (dd^c\widetilde{u}_0)^n,\end{aligned}$$

and we get the required conclusion. ∎

4. Applications. In this section we give an application of Theorem 3.4 to solving the Dirichlet problem in the class $\mathcal{F}(\Omega, g)$, $g \in \mathrm{MPSH}(\Omega) \cap \mathcal{E}(\Omega)$, without the assumption that the measure μ vanishes on all pluripolar sets.

PROPOSITION 4.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let μ be a nonnegative measure in Ω with $\mu(\Omega) < \infty$. Assume that $g \in \mathcal{E}(\Omega) \cap \mathrm{MPSH}(\Omega)$. Then there exists $u \in \mathcal{F}(\Omega, g)$ with $(dd^c u)^n = \mu$ in Ω if and only if for every hyperconvex domain $U \Subset \Omega$ there exists $u_U \in \mathcal{E}(U)$ such that $(dd^c u_U)^n = \mu$ on U.*

Proof. Necessity follows from the local property of the class $\mathcal{E}(\Omega)$. To prove sufficiency, let $\{\Omega_j\}_{j\geq 1}$ be hyperconvex domains in Ω such that $\Omega_j \Subset$

$\Omega_{j+1} \Subset \Omega$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Put $\Omega_0 = \emptyset$ and

$$\varphi_j := (\sup\{\varphi \in \mathrm{PSH}^-(\Omega_{j+2}) : \varphi \le u_{\Omega_{j+2}} \text{ on } \Omega_{j+1}\})^*.$$

Since $u_{\Omega_{j+2}} \in \mathcal{E}(\Omega_{j+2})$, we have $\varphi_j \in \mathcal{F}(\Omega_{j+2})$. Moreover,

$$(dd^c\varphi_j)^n \ge 1_{\Omega_j \setminus \Omega_{j-1}}\mu.$$

Hence, by Theorem 4.14 in [ÅCCH] there exists $\psi_j \in \mathcal{F}(\Omega_{j+2}, g)$ such that $(dd^c\psi_j)^n = 1_{\Omega_j\setminus\Omega_{j-1}}\mu$ on Ω_{j+2}. Therefore, by Theorem 3.4 there exists $w_j \in \mathcal{F}(\Omega, g)$ such that $w_j \le \psi_j$ on Ω_{j+2} and $(dd^c w_j)^n = 1_{\Omega_{j+2}}(dd^c\psi_j)^n = 1_{\Omega_{j+2}}1_{\Omega_j\setminus\Omega_{j-1}}\mu = 1_{\Omega_j\setminus\Omega_{j-1}}\mu$ on Ω.

Now, we claim that there exists a decreasing sequence $\{u_j\} \subset \mathcal{F}(\Omega, g)$ with $(dd^c u_j)^n = 1_{\Omega_j}\mu$ on Ω. Put $u_1 := w_1$. Assume by induction that we have determined u_j. We find u_{j+1} as follows. Put

$$v_{j+1} := (\sup\{\varphi \in \mathrm{PSH}^-(\Omega) : \varphi \le \min(u_j, w_{j+1}) \text{ on } \Omega\})^*.$$

Note that $u_j + w_{j+1} \le v_{j+1}$ on Ω, so $v_{j+1} \in \mathcal{F}(\Omega, g)$. By Remark 3.3 we have $(dd^c v_{j+1})^n \le 1_{\Omega_{j+1}}\mu$. Therefore, by the proof of Proposition 3.1 there exists $u_{j+1} \in \mathcal{F}(\Omega, g)$ such that $u_{j+1} \le v_{j+1} \le u_j$ and $(dd^c u_{j+1})^n = 1_{\Omega_{j+1}}\mu$, and the claim follows.

Put $u := \lim_{j\to\infty} u_j$. Since

$$\sup_j \int_\Omega (dd^c u_j)^n \le \mu(\Omega) < \infty,$$

Proposition 2.3 in [CH] implies that $u \in \mathcal{F}(\Omega, g)$. It is clear that $(dd^c u)^n = \mu$ and the desired conclusion follows. ∎

COROLLARY 4.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $g \in \mathcal{E}(\Omega) \cap \mathrm{MPSH}(\Omega)$. Assume that $u \in \mathcal{E}(\Omega)$ with $\int_\Omega (dd^c u)^n < \infty$. Then there exists $v \in \mathcal{F}(\Omega, g)$ such that $(dd^c v)^n = (dd^c u)^n$.*

Proof. Apply the above proposition to the measure $\mu = (dd^c u)^n$. ∎

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