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General uniform approximation theory by multivariate singular integral operators

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Abstract. We study the uniform approximation properties of general multivariate singular integral operators on $\mathbb{R}^N$, $N \geq 1$. We establish their convergence to the unit operator with rates. The estimates are pointwise and uniform. The established inequalities involve the multivariate higher order modulus of smoothness. We list the multivariate Picard, Gauss–Weierstrass, Poisson–Cauchy and trigonometric singular integral operators to which this theory can be applied directly.

1. Introduction. The rate of convergence of univariate singular integral operators has been studied earlier in [A], [AM1], [AM2], [AM3], [G], [MR], and these articles motivate the current work. Here we consider some very general multivariate singular integral operators on $\mathbb{R}^N$, $N \geq 1$, and we study the degree of approximation to the unit operator with rates over smooth functions. We establish related inequalities involving the multivariate higher modulus of smoothness with respect to $\|\cdot\|_\infty$. The estimates are pointwise and uniform. See Theorems 3.1, 3.3. We mention particular operators that are covered by our theory. The discussed linear operators are not in general positive. Other motivations for our research come from [ADu1], [ADu2].

2. Technical results. For $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m} & \text{if } j = 1, \dots, r, \\ 1 - \displaystyle\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m} & \text{if } j = 0, \end{cases}$$

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and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \qquad k = 1, \ldots, m.$$

Notice that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int\limits_{\mathbb{R}^N} f(x_1 + s_1 j, \ldots, x_N + s_N j)\, d\mu_{\xi_n}(s),$$

where $s := (s_1, \ldots, s_N)$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers, and μ_{ξ_n} is a probability Borel measure on $\mathbb{R}^N$.

Example 2.1. The operators $\theta_{r,n}^{[m]}$ are not in general positive. For example consider the function $\varphi(u_1, \ldots, u_N) = \sum_{i=1}^{N} u_i^2$ and take $r = 2$, $m = 3$, and $x_i = 0$, $i = 1, \ldots, N$. Observe that $\varphi \geq 0$, but

$$\begin{aligned}
\theta_{2,n}^{[3]}(\varphi; 0, 0, \ldots, 0) &= \Big(\sum_{j=1}^{2} j^2 \alpha_{j,2}^{[3]}\Big) \int\limits_{\mathbb{R}^N} \Big(\sum_{i=1}^{N} s_i^2\Big)\, d\mu_{\xi_n}(s) \\
&= (\alpha_{1,2}^{[3]} + 4\alpha_{2,2}^{[3]}) \int\limits_{\mathbb{R}^N} \Big(\sum_{i=1}^{N} s_i^2\Big)\, d\mu_{\xi_n}(s) \\
&= \Big(-2 + \frac{1}{2}\Big) \int\limits_{\mathbb{R}^N} \Big(\sum_{i=1}^{N} s_i^2\Big)\, d\mu_{\xi_n}(s) < 0,
\end{aligned}$$

assuming that $\int_{\mathbb{R}^N} (\sum_{i=1}^{N} s_i^2)\, d\mu_{\xi_n}(s) < \infty$.

Lemma 2.2. *The operators $\theta_{r,n}^{[m]}$ preserve the constant functions in N variables.*

Proof. Let $f(x_1, \ldots, x_N) = c$. Then

$$\theta_{r,n}^{[m]}(c; x_1, \ldots, x_n) = \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int\limits_{\mathbb{R}^n} c\, d\mu_{\xi_n}(s) = c. \quad \blacksquare$$

Definition 2.3. Let $f \in C_B(\mathbb{R}^N)$, the space of all bounded and continuous functions on $\mathbb{R}^N$. Then the rth *multivariate modulus of smoothness* of

f is given by (see, e.g., [AG])

$$\omega_r(f;h) := \sup_{\sqrt{u_1^2+\cdots+u_N^2}\le h} \|\Delta^r_{u_1,\ldots,u_N}(f)\|_\infty < \infty, \qquad h>0,$$

where $\|\cdot\|_\infty$ is the sup-norm and

$$\begin{aligned}\Delta^r_u f(x) &:= \Delta^r_{u_1,\ldots,u_N} f(x_1,\ldots,x_N)\\ &= \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} f(x_1+ju_1,\ldots,x_N+ju_N).\end{aligned}$$

Let $m\in\mathbb{N}$ and let $f\in C^m(\mathbb{R}^N)$. In this article we assume that all partial derivatives of f of order m are bounded, i.e.

$$\left\|\frac{\partial^m f(\cdot,\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}}\right\|_\infty < \infty,$$

for all $\alpha_j\in\mathbb{Z}^+$, $j=1,\ldots,N$, with $\sum_{j=1}^N \alpha_j = m$.

REMARK 2.4. We will write partial derivatives of f as $f_\alpha := \frac{\partial^{|\alpha|}f}{\partial x^\alpha}$; here and throughout,

$$\alpha := (\alpha_1,\ldots,\alpha_N), \qquad \alpha_i\in\mathbb{Z}^+,\ i=1,\ldots,N \quad \text{and} \quad |\alpha| := \sum_{i=1}^N \alpha_i.$$

Consider $g_z(t) := f(x_0+t(z-x_0))$, $t\ge 0$, $x_0, z\in\mathbb{R}^N$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^N (z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_{01}+t(z_1-x_{01}),\ldots,x_{0N}+t(z_N-x_{0N}))$$

for all $j=0,1,\ldots,m$.

We have the multivariate Taylor's formula ([ADr])

$$\begin{aligned}f(z_1,\ldots,z_N) &= g_z(1)\\ &= \sum_{j=0}^m \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1}(g_z^{(m)}(\theta)-g_z^{(m)}(0))\,d\theta.\end{aligned}$$

Notice $g_z(0)=f(x_0)$. Also for $j=0,1,\ldots,m$, we have

$$g_z^{(j)}(0) = \sum_{|\alpha|=j} \frac{j!}{\prod_{i=1}^N \alpha_i!}\left(\prod_{i=1}^N (z_i-x_{0i})^{\alpha_i}\right) f_\alpha(x_0).$$

Furthermore

$$g_z^{(m)}(\theta) = \sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!}\left(\prod_{i=1}^N (z_i-x_{0i})^{\alpha_i}\right) f_\alpha(x_0+\theta(z-x_0)), \qquad 0\le\theta\le 1.$$

We apply the above for

$$z=(z_1,\ldots,z_N)=(x_1+s_1j,\ldots,x_N+s_Nj)=x+sj$$

and

$$x_0=(x_{01},\ldots,x_{0N})=(x_1,\ldots,x_N)=x$$

to get

$$\begin{aligned}
&f(x_1+s_1j,\ldots,x_N+s_Nj)=g_{x+sj}(1)\\
&\quad=\sum_{\widetilde{j}=0}^{m}\frac{g_{x+sj}^{(\widetilde{j})}(0)}{\widetilde{j}!}+\frac{1}{(m-1)!}\int_0^1(1-\theta)^{m-1}(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0))\,d\theta,
\end{aligned}$$

where $g_{x+sj}(t):=f(x+tsj)$. Notice $g_{x+sj}(0)=f(x)$.

Also for $\widetilde{j}=0,1,\ldots,m$ we have

$$g_{x+sj}^{(\widetilde{j})}(0)=\sum_{|\alpha|=\widetilde{j}}\frac{\widetilde{j}!}{\prod_{i=1}^N\alpha_i!}\Big(\prod_{i=1}^N(s_ij)^{\alpha_i}\Big)f_\alpha(x).$$

Furthermore we get, for $0\le\theta\le1$,

$$g_{x+sj}^{(m)}(\theta)/m!=\sum_{|\alpha|=m}\frac{1}{\prod_{i=1}^N\alpha_i!}\Big(\prod_{i=1}^N(s_ij)^{\alpha_i}\Big)f_\alpha(x+\theta(sj)).$$

For $\widetilde{j}=1,\ldots,m$ and $|\alpha|=\widetilde{j}$, it will be proved in (2.2) that

$$c_{\alpha,n,\widetilde{j}}:=c_{\alpha,n}:=\int_{\mathbb{R}^N}\prod_{i=1}^N s_i^{\alpha_i}\,d\mu_{\xi_n}(s_1,\ldots,s_N)\in\mathbb{R}.$$

Consequently,

$$\sum_{\widetilde{j}=1}^{m}\frac{\int_{\mathbb{R}^N}g_{x+sj}^{(\widetilde{j})}(0)\,d\mu_{\xi_n}(s)}{\widetilde{j}!}=\sum_{\widetilde{j}=1}^{m}j^{\widetilde{j}}\sum_{|\alpha|=\widetilde{j}}\frac{1}{\prod_{i=1}^N\alpha_i!}c_{\alpha,n}f_\alpha(x).$$

Next we observe that

$$\begin{aligned}
&\frac{1}{(m-1)!}\int_{\mathbb{R}^N}\Big(\int_0^1(1-\theta)^{m-1}(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0))\,d\theta\Big)\,d\mu_{\xi_n}(s)\\
&\qquad=mj^m\sum_{|\alpha|=m}\frac{1}{\prod_{i=1}^N\alpha_i!}\\
&\qquad\quad\times\int_{\mathbb{R}^N}\Big(\prod_{i=1}^N s_i^{\alpha_i}\int_0^1(1-\theta)^{m-1}[f_\alpha(x+\theta(sj))-f_\alpha(x)]\,d\theta\Big)\,d\mu_{\xi_n}(s).
\end{aligned}$$

Remark 2.5. We further notice that

$$\theta_{r,n}^{[m]}(f;x)-f(x)=\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}(f(x+sj)-f(x))\,d\mu_{\xi_n}(s)$$

$$=\sum_{j=0}^{r}\alpha_{j,r}^{[m]}$$

$$\times\int_{\mathbb{R}^N}\left[\sum_{\widetilde{j}=1}^{m}\frac{g_{x+sj}^{(\widetilde{j})}(0)}{\widetilde{j}!}+\frac{1}{(m-1)!}\int_0^1(1-\theta)^{m-1}(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0))\,d\theta\right]d\mu_{\xi_n}(s).$$

That is,

$$\Delta_{r,n}^{[m]}(x):=\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(\int_{\mathbb{R}^N}\left(\sum_{\widetilde{j}=1}^{m}\frac{g_{x+sj}^{(\widetilde{j})}(0)}{\widetilde{j}!}\right)d\mu_{\xi_n}(s)\right)$$

$$=\frac{1}{(m-1)!}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}\left(\int_0^1(1-\theta)^{m-1}(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0))\,d\theta\right)d\mu_{\xi_n}(s)=:R_{r,n}^{[m]}.$$

We observe that

$$\begin{aligned}\Delta_{r,n}^{[m]}(x)&=\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{j=1}^{r}\alpha_{j,r}^{[m]}\sum_{\widetilde{j}=1}^{m}j^{\widetilde{j}}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n}f_\alpha(x)}{\prod_{i=1}^N\alpha_i!}\right)\\&=\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\left(\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^{\widetilde{j}}\right)\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n}f_\alpha(x)}{\prod_{i=1}^N\alpha_i!}\right)\\&=\theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{\widetilde{j}=1}^{m}\delta_{\widetilde{j},r}^{[m]}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n}f_\alpha(x)}{\prod_{i=1}^N\alpha_i!}\right).\end{aligned}$$

Remark 2.6. We see that

$$\begin{aligned}R_{r,n}^{[m]}&=m\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^m\sum_{|\alpha|=m}\frac{1}{\prod_{i=1}^N\alpha_i!}\\&\quad\times\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\int_0^1(1-\theta)^{m-1}(f_\alpha(x+\theta(sj))-f_\alpha(x))\,d\theta\right)d\mu_{\xi_n}(s)\\&=m\sum_{|\alpha|=m}\frac{1}{\prod_{i=1}^N\alpha_i!}\\&\quad\times\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\int_0^1(1-\theta)^{m-1}\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^m(f_\alpha(x+\theta(sj))-f_\alpha(x))\,d\theta\right)d\mu_{\xi_n}(s)\end{aligned}$$

$$= m \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1}$$

$$\times \left[\sum_{j=1}^r (-1)^{r-j} \binom{r}{j} f_\alpha(x+\theta(sj)) + (-1)^r \binom{r}{0} f_\alpha(x) \right] d\theta \Bigg) \, d\mu_{\xi_n}(s)$$

$$= m \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1}$$

$$\times \left[\sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f_\alpha(x+\theta(sj)) \right] d\theta \Bigg) \, d\mu_{\xi_n}(s).$$

We have proved that the error or remainder of approximation takes the form

$$R_{r,n}^{[m]} = m \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \times \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} (\Delta_{\theta s}^r f_\alpha(x)) \, d\theta \Bigg) \, d\mu_{\xi_n}(s). \tag{2.1}$$

We further make

Remark 2.7. We see that

$$|R_{r,n}^{[m]}| \overset{(2.1)}{\leq} m \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \times \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} |\Delta_{\theta s}^r f_\alpha(x)| \, d\theta \Bigg) \, d\mu_{\xi_n}(s)$$

$$\leq m \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \|\Delta_{\theta s}^r f_\alpha\|_\infty \, d\theta \Bigg) \, d\mu_{\xi_n}(s)$$

$$\leq m \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \times \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \omega_r(f_\alpha, \theta\|s\|_2) \, d\theta \Bigg) \, d\mu_{\xi_n}(s)$$

$$\leq \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Bigg(\prod_{i=1}^N |s_i|^{\alpha_i} \omega_r(f_\alpha, \|s\|_2) \Bigg) \, d\mu_{\xi_n}(s).$$

So far we have proved

$$|R_{r,n}^{[m]}| \leq \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i} \omega_r(f_\alpha, \|s\|_2) \Big) \, d\mu_{\xi_n}(s)$$

$$= \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \bigg(\prod_{i=1}^N |s_i|^{\alpha_i} \omega_r \Big(f_\alpha, \xi_n \frac{\|s\|_2}{\xi_n} \Big) \bigg) \, d\mu_{\xi_n}(s).$$

Using the fact that $\omega_r(f;\lambda u) \leq (1+\lambda)^r \omega_r(f;u)$ for $\lambda, u > 0$, we get

$$|R_{r,n}^{[m]}| \leq \sum_{|\alpha|=m} \frac{\omega_r(f_\alpha, \xi_n)}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i} \Big) \Big(1 + \frac{\|s\|_2}{\xi_n} \Big)^r \, d\mu_{\xi_n}(s).$$

Remark 2.8. Notice that for $|\alpha| = m$,

$$\int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i} \Big) \, d\mu_{\xi_n}(s) \leq \int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i} \Big) \Big(1 + \frac{\|s\|_2}{\xi_n} \Big)^r \, d\mu_{\xi_n}(s) < \infty,$$

under the assumption of Theorem 3.1 below. Hence

$$|c_{\alpha,n}| \leq \int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i} \Big) \, d\mu_{\xi_n}(s) < \infty$$

for $|\alpha| = m$. Hence also

$$\int_{\mathbb{R}^N} |s_i|^m \, d\mu_{\xi_n}(s) < \infty, \qquad i = 1, \ldots, N. \tag{2.2}$$

Let $1 \leq \widetilde{j} \leq m-1$. Then

$$\int_{\mathbb{R}^N} |s_i|^{\widetilde{j}} \, d\mu_{\xi_n}(s) \leq \Big(\int_{\mathbb{R}^N} (|s_i|^{\widetilde{j}})^{m/\widetilde{j}} \, d\mu_{\xi_n}(s) \Big)^{\widetilde{j}/m} = \Big(\int_{\mathbb{R}^N} |s_i|^m \, d\mu_{\xi_n}(s) \Big)^{\widetilde{j}/m} < \infty.$$

For $\widetilde{j} = 1, \ldots, m-1$ and $|\alpha| = \widetilde{j}$ we have $\sum_{i=1}^N \alpha_i/\widetilde{j} = 1$. Hence

$$\int_{\mathbb{R}^N} \prod_{i=1}^N |s_i|^{\alpha_i} \, d\mu_{\xi_n}(s) \leq \prod_{i=1}^N \Big(\int_{\mathbb{R}^N} (|s_i|^{\alpha_i})^{\widetilde{j}/\alpha_i} \, d\mu_{\xi_n}(s) \Big)^{\alpha_i/\widetilde{j}}$$

$$= \prod_{i=1}^N \Big(\int_{\mathbb{R}^N} |s_i|^{\widetilde{j}} \, d\mu_{\xi_n}(s) \Big)^{\alpha_i/\widetilde{j}} < \infty.$$

3. Main results. Based on the above Remarks 2.4–2.8 we present

Theorem 3.1. *Let $m \in \mathbb{N}$, $f \in C^m(\mathbb{R}^N)$, $N \geq 1$, $x \in \mathbb{R}^N$. Assume $\big\| \frac{\partial^m f(\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \big\|_\infty < \infty$ for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N$, with $|\alpha| := \sum_{j=1}^N \alpha_j = m$.*

Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}}$ a bounded sequence of positive real numbers.

Assume that for all α with $|\alpha| = m$ we have

$$u_{\xi_n} := \int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i}\Big)\Big(1 + \frac{\|s\|_2}{\xi_n}\Big)^r d\mu_{\xi_n}(s) < \infty.$$

For $\widetilde{j} = 1, \ldots, m$ and $|\alpha| = \widetilde{j}$, set

$$c_{\alpha,n} := c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i}\, d\mu_{\xi_n}(s_1, \ldots, s_N).$$

Then

(i)

$$E_{r,n}^{[m]}(x) := \Big|\theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]}\Big(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod_{i=1}^N \alpha_i!}\Big)\Big| \tag{3.1}$$
$$\le \sum_{|\alpha|=m} \frac{\omega_r(f_\alpha, \xi_n)}{\prod_{i=1}^N \alpha_i!} \int_{\mathbb{R}^N} \Big(\prod_{i=1}^N |s_i|^{\alpha_i}\Big)\Big(1 + \frac{\|s\|_2}{\xi_n}\Big)^r d\mu_{\xi_n}(s)$$

for all $x \in \mathbb{R}^N$.

(ii) $\|E_{r,n}^{[m]}\|_\infty \le$ *Right hand side of* (3.1).

If $\xi_n \to 0$, as $n \to \infty$, and u_{ξ_n} is uniformly bounded, then we obtain $\|E_{r,n}^{[m]}\|_\infty \to 0$ with rates of convergence.

(iii) *Moreover*

$$\|\theta_{r,n}^{[m]}(f) - f\|_\infty \le \sum_{\widetilde{j}=1}^m |\delta_{\widetilde{j},r}^{[m]}|\Big(\sum_{|\alpha|=\widetilde{j}} \frac{|c_{\alpha,n,\widetilde{j}}|\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!}\Big) + \textit{Right hand side of } (3.1)$$

if $\|f_\alpha\|_\infty < \infty$ for all α with $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, \ldots, m$. Furthermore, if $\xi_n \to 0$ as $n \to \infty$, assuming that $c_{\alpha,n,\widetilde{j}} \to 0$, while u_{ξ_n} is uniformly bounded, we conclude that

$$\|\theta_{r,n}^{[m]}(f) - f\|_\infty \to 0$$

with rates.

Now we consider the case $m = 0$.

Remark 3.2. Here $f \in C_B(\mathbb{R}^N)$ (bounded and continuous functions). We observe that

$$\theta_{r,n}^{[0]}(f;x) - f(x) = \sum_{j=0}^r \alpha_{j,r}^{[0]} \int_{\mathbb{R}^N} (f(x + js) - f(x))\, d\mu_{\xi_n}(s)$$
$$= \int_{\mathbb{R}^N} \Big(\sum_{j=0}^r \alpha_{j,r}^{[0]} f(x + js) - \Big(\sum_{j=0}^r \alpha_{j,r}^{[0]}\Big) f(x)\Big)\, d\mu_{\xi_n}(s)$$

$$= \int_{\mathbb{R}^N} \left(\sum_{j=1}^{r} \alpha_{j,r}^{[0]} f(x+js) - \left(\sum_{j=1}^{r} \alpha_{j,r}^{[0]} \right) f(x) \right) d\mu_{\xi_n}(s)$$
$$= \int_{\mathbb{R}^N} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x+js) - \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \right) f(x) \right) d\mu_{\xi_n}(s)$$
$$= \int_{\mathbb{R}^N} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x+js) + (-1)^r \binom{r}{0} f(x) \right) d\mu_{\xi_n}(s)$$
$$= \int_{\mathbb{R}^N} \left(\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+js) \right) d\mu_{\xi_n}(s) = \int_{\mathbb{R}^N} (\Delta_s^r f(x))\, d\mu_{\xi_n}(s).$$

Consequently,

$$|\theta_{r,n}^{[0]}(f;x) - f(x)| \le \int_{\mathbb{R}^N} |\Delta_s^r f(x)|\, d\mu_{\xi_n}(s) \le \int_{\mathbb{R}^N} \|\Delta_s^r f\|_\infty \, d\mu_{\xi_n}(s)$$
$$\le \int_{\mathbb{R}^N} \omega_r(f, \|s\|_2)\, d\mu_{\xi_n}(s) = \int_{\mathbb{R}^N} \omega_r\left(f, \xi_n \frac{\|s\|_2}{\xi_n}\right) d\mu_{\xi_n}(s)$$
$$\le \omega_r(f,\xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s).$$

Based on the above we present

THEOREM 3.3. *Let $f \in C_B(\mathbb{R}^N)$, $N \ge 1$. Then*

$$\|\theta_{r,n}^{[0]} f - f\|_\infty \le \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) \right) \omega_r(f,\xi_n),$$

under the assumption

$$\Phi_{\xi_n} := \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) < \infty.$$

If $\xi_n \to 0$ as $n \to \infty$, and Φ_{ξ_n} are uniformly bounded, we obtain

$$\|\theta_{r,n}^{[0]} f - f\|_\infty \to 0$$

with rates.

Let all entities be as above. We define the following specific operators:

(i) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f;x_1,\ldots,x_N) := \frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \times \int_{\mathbb{R}^N} f(x_1 + s_1 j, \ldots, x_N + s_N j) e^{-\sum_{i=1}^N |s_i|/\xi_n}\, ds_1 \ldots ds_N.$$

Notice that
$$\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} e^{-\sum_{i=1}^N |s_i|/\xi_n}\,ds_1\ldots ds_N = 1$$
(see [A]).

(ii) The general multivariate Gauss–Weierstrass singular integral operators:
$$W_{r,n}^{[m]}(f;x_1,\ldots,x_N) := \frac{1}{(\sqrt{\pi\xi_n})^N}\sum_{j=0}^r \alpha_{j,r}^{[m]} \times \int_{\mathbb{R}^N} f(x_1+s_1j,\ldots,x_N+s_Nj)e^{-\sum_{i=1}^N s_i^2/\xi_n}\,ds_1\ldots ds_N.$$
Notice that
$$\frac{1}{(\sqrt{\pi\xi_n})^N}\int_{\mathbb{R}^N} e^{-\sum_{i=1}^N s_i^2/\xi_n}\,ds_1\ldots ds_N = 1$$
(see [AM1]).

(iii) The general multivariate Poisson–Cauchy singular integral operators:
$$U_{r,n}^{[m]}(f;x_1,\ldots,x_N) := W_n^N \sum_{j=0}^r \alpha_{j,r}^{[m]} \times \int_{\mathbb{R}^N} f(x_1+s_1j,\ldots,x_N+s_Nj)\prod_{i=1}^N \frac{1}{(s_i^{2\alpha}+\xi_n^{2\alpha})^\beta}\,ds_1\ldots ds_N,$$
with $\alpha\in\mathbb{N}$, $\beta>\frac{1}{2\alpha}$, and
$$W_n := \frac{\Gamma(\beta)\alpha\xi_n^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}$$
(see [AM2]). Notice that
$$W_n^N\int_{\mathbb{R}^N}\prod_{i=1}^N \frac{1}{(s_i^{2\alpha}+\xi_n^{2\alpha})^\beta}\,ds_1\ldots ds_N = 1$$
(see [AM2], [Z, p. 397, formula 595].

(iv) The general multivariate trigonometric singular integral operators:
$$T_{r,n}^{[m]}(f;x_1,\ldots,x_N) := \lambda_n^{-N}\sum_{j=0}^r \alpha_{j,r}^{[m]} \times \int_{\mathbb{R}^N} f(x_1+s_1j,\ldots,x_N+s_Nj)\prod_{i=1}^N\left(\frac{\sin(s_i/\xi_n)}{s_i}\right)^{2\beta}ds_1\ldots ds_N,$$
where $\beta\in\mathbb{N}$, and
$$\lambda_n := 2\xi_n^{1-2\beta}\pi(-1)^\beta\beta\sum_{k=1}^\beta(-1)^k\frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}$$

(see [AM3], [E, p. 210, item 1033]). Notice that

$$\lambda_n^{-N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} \left(\frac{\sin(s_i/\xi_n)}{s_i} \right)^{2\beta} ds_1 \ldots ds_N = 1.$$

One can apply Theorems 3.1 and 3.3 to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results. We intend to do that in a future article.

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