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Correspondence between diffeomorphism groups and singular foliations

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Abstract. It is well-known that any isotopically connected diffeomorphism group G of a manifold determines a unique singular foliation $\mathcal{F}_G$. A one-to-one correspondence between the class of singular foliations and a subclass of diffeomorphism groups is established. As an illustration of this correspondence it is shown that the commutator subgroup $[G,G]$ of an isotopically connected, factorizable and non-fixing C^r diffeomorphism group G is simple iff the foliation $\mathcal{F}_{[G,G]}$ defined by $[G,G]$ admits no proper minimal sets. In particular, the compactly supported e-component of the leaf preserving C^∞ diffeomorphism group of a regular foliation $\mathcal{F}$ is simple iff $\mathcal{F}$ has no proper minimal sets.

1. Introduction. Throughout by a *foliation* we mean a singular foliation (Sussmann [17], Stefan [15]), and by a *regular foliation* we mean a foliation whose leaves have the same dimension. Introducing the notion of foliations, Sussmann and Stefan emphasized that they play a role of collections of "accessible" sets. Alternatively, they regarded foliations as integrable smooth distributions. Another point of view is to treat foliations as by-products of non-transitive geometric structures (cf. [2], [20] and examples in [10]). In Molino's approach some types of singular foliations constitute collections of closures of leaves of certain regular foliations ([7], [21]). In this note we regard foliations as a special type of diffeomorphism groups.

Given a C^∞ smooth paracompact boundaryless manifold M, $\mathrm{Diff}^r(M)_0$ (resp. $\mathrm{Diff}^r_c(M)_0$), where $1 \le r \le \infty$, is the subgroup of the group of all C^r diffeomorphisms $\mathrm{Diff}^r(M)$ on M consisting of diffeomorphisms that can be joined to the identity through a C^r isotopy (resp. compactly supported C^r isotopy) on M. A diffeomorphism group $G \le \mathrm{Diff}^r(M)$ is called *isotopically connected* if any element f of G can be joined to id_M through a C^r isotopy in G. That is, there is a mapping $\mathbb{R} \times M \ni (t,x) \mapsto f_t(x) \in M$ of class C^r

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with $f_t \in G$ for all t and such that $f_0 = \mathrm{id}$ and $f_1 = f$. It is well-known that any isotopically connected group $G \le \mathrm{Diff}^r(M)_0$ defines a unique foliation of class C^r, designated by $\mathcal{F}_G$ (see Sect. 2).

Our first aim is to establish a correspondence between the class $\mathfrak{F}^r(M)$ of all C^r foliations on M and a subclass of diffeomorphism groups on M, and, by using it, to interpret some results and some open problems concerning non-transitive diffeomorphism groups. The second aim is to prove new results (Theorems 1.1 and 1.2) illustrating this correspondence.

A group $G \le \mathrm{Diff}^r(M)$ is called *factorizable* if for every open cover $\mathcal{U}$ and every $g \in G$ there are $g_1, \dots, g_r \in G$ with $g = g_1 \dots g_r$ and such that $g_i \in G_{U_i}$, $i = 1, \dots, r$, for some $U_1, \dots, U_r \in \mathcal{U}$. Here for $U \subset M$ and $G \le \mathrm{Diff}^r(M)$, G_U stands for the identity component of the group of all diffeomorphisms from G compactly supported in U. Next, G is said to be *non-fixing* if $G(x) \neq \{x\}$ for every $x \in X$.

Theorem 1.1. *Assume that $G \le \mathrm{Diff}^r_c(M)_0$, $1 \le r \le \infty$, is an isotopically connected, non-fixing and factorizable group of diffeomorphisms of a smooth manifold M. Then the commutator group $[G,G]$ is simple if and only if the corresponding foliation $\mathcal{F}_{[G,G]}$ admits no proper* (*i.e. not equal to M*) *minimal set.*

In early 1970's Thurston and Mather proved that the group $\mathrm{Diff}^r_c(M)_0$, where $1 \le r \le \infty$, $r \neq \dim(M) + 1$, is perfect and simple (see [18], [6], [1]). Next, similar results were proved for classical diffeomorphism groups of class C^∞ ([1], [13]). For the significance of these simplicity theorems, see, e.g., [1], [13] and references therein.

Let $(M_i, \mathcal{F}_i)$, $i = 1, 2$, be foliated manifolds. A map $f : M_1 \to M_2$ is called *foliation preserving* if $f(L_x) = L_{f(x)}$ for any $x \in M_1$, where L_x is the leaf meeting x. Next, if $(M_1, \mathcal{F}_1) = (M_2, \mathcal{F}_2)$ then f is *leaf preserving* if $f(L_x) = L_x$ for all $x \in M_1$. Throughout, $\mathrm{Diff}^r(M, \mathcal{F})$ will stand for the group of all leaf preserving C^r diffeomorphisms of a foliated manifold $(M, \mathcal{F})$. Define $\mathrm{Diff}^r(M, \mathcal{F})_0$ and $\mathrm{Diff}^r_c(M, \mathcal{F})_0$ analogously. Observe that a perfectness theorem for the compactly supported identity component $\mathrm{Diff}^\infty_c(M, \mathcal{F})_0$, which is a non-transitive counterpart of Thurston's theorem, has been proved by the author [9] and by Tsuboi [19]. Next, the author [10], following Mather [6, II], showed that $\mathrm{Diff}^r_c(M, \mathcal{F})_0$ is perfect provided $1 \le r \le \dim \mathcal{F}$. Observe that, in general, the group $\mathrm{Diff}^r_c(M, \mathcal{F})_0$ is not simple for obvious reasons.

Theorem 1.2. *Let $(M, \mathcal{F})$ be a foliation on a C^∞ smooth manifold M with no leaves of dimension* 0. *Then the commutator subgroup*

$$\mathcal{D} = [\mathrm{Diff}^r_c(M, \mathcal{F})_0, \mathrm{Diff}^r_c(M, \mathcal{F})_0]$$

is simple if and only if $\mathcal{F}_{\mathcal{D}}$ does not have any proper minimal set. In par-

ticular, if $\mathcal{F}$ is regular, and $1 \le r \le \dim \mathcal{F}$ or $r = \infty$, then $\mathrm{Diff}^r_c(M, \mathcal{F})_0$ *is simple if and only if $\mathcal{F}$ has no proper minimal sets.*

In the proof of Theorem 1.1 in Sect. 3 some ideas from Ling [5] are in use.

2. Foliations correspond to a subclass of the class of diffeomorphism groups. Let $1 \le r \le \infty$ and let L be a subset of a C^r manifold M endowed with a C^r differentiable structure which makes it an immersed submanifold. Then L is *weakly imbedded* if for any locally connected topological space N and a continuous map $f : N \to M$ satisfying $f(N) \subset L$, the map $f : N \to L$ is continuous as well. It follows that in this case such a differentiable structure is unique. A *foliation of class C^r* is a partition $\mathcal{F}$ of M into weakly imbedded submanifolds, called leaves, such that the following condition holds. If x belongs to a k-dimensional leaf, then there is a local chart (U, φ) of class C^r with $\varphi(x) = 0$, and $\varphi(U) = V \times W$, where V is open in $\mathbb{R}^k$, and W is open in $\mathbb{R}^{n-k}$, such that if $L \in \mathcal{F}$ then $\varphi(L \cap U) = V \times l$, where $l = \{w \in W : \varphi^{-1}(0, w) \in L\}$. A foliation is called *regular* if all leaves have the same dimension.

Sussmann [17] and Stefan [15], [16] regarded foliations as collections of accessible sets in the following sense.

Definition 2.1. A smooth mapping φ of an open subset of $\mathbb{R} \times M$ into M is said to be a C^r *arrow*, $1 \le r \le \infty$, if

(1) $\varphi(t, \cdot) = \varphi_t$ is a local C^r diffeomorphism for each t, possibly with empty domain,
(2) $\varphi_0 = \mathrm{id}$ on its domain,
(3) $\mathrm{dom}(\varphi_t) \subset \mathrm{dom}(\varphi_s)$ whenever $0 \le s < t$.

Given an arbitrary set $\mathcal{A}$ of arrows, let $\mathcal{A}^*$ be the family of local diffeomorphisms ψ such that $\psi = \varphi(t, \cdot)$ for some $\varphi \in \mathcal{A}$, $t \in \mathbb{R}$. Next, $\hat{\mathcal{A}}$ denotes the set consisting of all local diffeomorphisms which are finite compositions of elements from $\mathcal{A}^*$ or $(\mathcal{A}^*)^{-1} = \{\psi^{-1} : \psi \in \mathcal{A}^*\}$, and of the identity. Then the orbits of $\hat{\mathcal{A}}$ are called *accessible sets* of $\mathcal{A}$.

For $x \in M$ let $\mathcal{A}(x)$, $\bar{\mathcal{A}}(x)$ be the vector subspaces of T_xM generated by

$$\{\dot{\varphi}(t, y) : \varphi \in \mathcal{A}, \varphi_t(y) = x\}, \qquad \{d_y\psi(v) : \psi \in \hat{\mathcal{A}},\ \psi(y) = x, v \in \mathcal{A}(y)\},$$

respectively. Then we have ([15])

Theorem 2.2. *Let $\mathcal{A}$ be an arbitrary set of C^r arrows on M. Then*

(1) *every accessible set of $\mathcal{A}$ admits a (unique) C^r differentiable structure of a connected weakly imbedded submanifold of M;*
(2) *the collection of accessible sets defines a foliation $\mathcal{F}$; and*
(3) *$\mathcal{D}(\mathcal{F}) := \{\bar{\mathcal{A}}(x)\}$ is the tangent distribution of $\mathcal{F}$.*

Let $G \leq \mathrm{Diff}^r(M)$ be an isotopically connected group of diffeomorphisms. Let $\mathcal{A}_G$ be the set of restrictions of isotopies $\mathbb{R} \times M \ni (t,x) \mapsto f_t(x) \in M$ in G to open subsets of $\mathbb{R} \times M$. Then we denote by $\mathcal{F}_G$ the foliation defined by the set $\mathcal{A}_G$ of arrows. Observe that $\hat{\mathcal{A}}_G = \mathcal{A}_G$, and consequently $\bar{\mathcal{A}}_G(x) = \mathcal{A}_G(x)$.

REMARK 2.3. (1) Of course, any subgroup $G \leq \mathrm{Diff}^r(M)$ determines a unique foliation. Namely, G has a unique maximal subgroup G_0 which is isotopically connected.

(2) Denote by G_c the subgroup of all compactly supported elements of G. Then G_c need not be isotopically connected even if G is. In fact, let $G = \mathrm{Diff}^r(\mathbb{R}^n)_0$, $1 \leq r \leq \infty$. Then every $f \in G_c$ is isotopic to the identity but the isotopy need not be in G_c. That is, G_c is not isotopically connected. Observe that the C^0 case is exceptional: due to Alexander's trick for $r = 0$ (see, e.g., [3, p. 70]), G_c is isotopically connected.

Likewise, let $C = \mathbb{R} \times \mathbb{S}^1$ be the annulus and let $G = \mathrm{Diff}^r(C)_0$. Then we have the twisting number epimorphism $T : G_c \to \mathbb{Z}$. It is easily seen that $f \in G_c$ can be joined to id by a compactly supported isotopy iff $T(f) = 0$. Consequently, G_c is not isotopically connected.

Denote by $\mathfrak{G}^r(M)$ (resp. $\mathfrak{G}^r_c(M)$), $1 \leq r \leq \infty$, the collection of isotopically connected (resp. isotopically connected through compactly supported isotopies) groups of C^r diffeomorphisms of M. Next, $\mathfrak{F}^r(M)$ will stand for the set of all foliations of class C^r on M. Then each $G \in \mathfrak{G}^r(M)$ determines a unique foliation from $\mathfrak{F}^r(M)$, denoted by $\mathcal{F}_G$. That is, we have the mapping $\beta_M : \mathfrak{G}^r(M) \ni G \mapsto \mathcal{F}_G \in \mathfrak{F}^r(M)$. Conversely, to any foliation $\mathcal{F} \in \mathfrak{F}^r(M)$ we assign $G_{\mathcal{F}} := \mathrm{Diff}^r_c(M,\mathcal{F})_0$ and we get the mapping $\alpha_M : \mathfrak{F}^r(M) \ni \mathcal{F} \mapsto G_{\mathcal{F}} \in \mathfrak{G}^r_c(M)$. The following is obvious.

PROPOSITION 2.4. *One has* $\beta_M \circ \alpha_M = \mathrm{id}_{\mathfrak{F}^r(M)}$. *In particular*
$$\alpha_M : \mathfrak{F}^r(M) \ni \mathcal{F} \mapsto G_{\mathcal{F}} \in \mathfrak{G}^r_c(M)$$
is an injection identifying the class of C^r *foliations with a subclass of* C^r *diffeomorphism groups.*

Observe that usually $(\alpha_M \circ \beta_M)(G) \in \mathfrak{G}^r_c(M)$ is not a subgroup of G even if $G \in \mathfrak{G}^r_c(M)$. For instance, take the group of Hamiltonian diffeomorphisms of a Poisson manifold (see [20]). See also examples in [11].

REMARK 2.5. Note that we can also define $\alpha'_M : \mathfrak{F}^r(M) \ni \mathcal{F} \mapsto G'_{\mathcal{F}} \in \mathfrak{G}^r(M)$, where $G'_{\mathcal{F}} := \mathrm{Diff}^r(M,\mathcal{F})_0 \in \mathfrak{G}^r(M)$, and we get another identification of the class of C^r foliations with a subclass of C^r diffeomorphism groups. However we prefer α_M to α'_M because of Proposition 2.11 below.

For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M)$ we say that $\mathcal{F}_1$ is a *subfoliation* of $\mathcal{F}_2$ if each leaf of $\mathcal{F}_1$ is contained in a leaf of $\mathcal{F}_2$. We then write $\mathcal{F}_1 \prec \mathcal{F}_2$. By a *flag structure* we mean a finite sequence $\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k$ of foliations of M. Next, by the *intersection* of $\mathcal{F}_1, \mathcal{F}_2$ we mean the partition $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2 := \{L_1 \cap L_2 : L_i \in \mathcal{F}_i,\ i = 1,2\}$ of M. Clearly, if $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ is a foliation then $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2 \prec \mathcal{F}_i$, $i = 1,2$.

It is a rare phenomenon that $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ is a regular foliation if $\mathcal{F}_1, \mathcal{F}_2$ are regular. In the category of (singular) foliations this may happen more often.

PROPOSITION 2.6.

(1) *If the distribution $\mathcal{D}(\mathcal{F}_1 \bar{\cap} \mathcal{F}_2)$ is of class C^r* ([15]) *then $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ is a foliation.*
(2) *If $G_1, G_2 \in \mathfrak{G}^r(M)$ have the intersection $G = G_1 \cap G_2$ isotopically connected then $\mathcal{F}_G = \mathcal{F}_{G_1} \bar{\cap} \mathcal{F}_{G_2}$.*
(3) *For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M)$, if $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ is a foliation then there is $G \in \mathfrak{G}^r(M)$ such that $G \le G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}$ and $\mathcal{F}_G = \mathcal{F}_1 \bar{\cap} \mathcal{F}_2$.*
(4) *For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M)$, if $G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}$ is connected then $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ is a foliation.*

Proof. (1) In fact, the distribution of $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ is then integrable.

(2) Denote by $\mathcal{I}G$ the set of all isotopies in G. Clearly, $\mathcal{I}(G_1 \cap G_2) = \mathcal{I}G_1 \cap \mathcal{I}G_2$ for arbitrary $G_1, G_2 \in \mathfrak{G}^r(M)$. For $x \in M$, set $\mathcal{I}G(x) := \{y \in M : (\exists f \in \mathcal{I}G)(\exists t \in I)\, y = f_t(x)\}$. By definition, $L_x = \mathcal{I}G(x)$, where $L_x \in \mathcal{F}_G$ is a leaf meeting x. Therefore, since G_1, G_2, G are isotopically connected we have $L_x = \mathcal{I}G(x) = \mathcal{I}G_1(x) \cap \mathcal{I}G_2(x) = L^1_x \cap L^2_x$, where $L^i_x \in \mathcal{F}_{G_i}$, $i = 1,2$.

(3) Set $\mathcal{F} = \mathcal{F}_1 \bar{\cap} \mathcal{F}_2$ and $G = G_{\mathcal{F}}$. Use Prop. 2.4.

(4) In view of Prop. 2.4 we have $\mathcal{F}_{G_{\mathcal{F}_0}} = \mathcal{F}_0$ for all $\mathcal{F}_0 \in \mathfrak{F}^r(M)$. Put $G = G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}$. Therefore, in view of (2), $\mathcal{F}_1 \bar{\cap} \mathcal{F}_2 = \mathcal{F}_{G_{\mathcal{F}_1}} \bar{\cap} \mathcal{F}_{G_{\mathcal{F}_2}} = \mathcal{F}_G$ is a foliation. ∎

Let $\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k$ be a flag structure on M and let $x \in M$. If $x \in L_i \in \mathcal{F}_i$ we write $p_i(x) = \dim L_i$, $\bar{p}_i(x) = p_i(x) - p_{i-1}(x)$ $(i = 2, \ldots, k)$ and $q_i(x) = m - p_i(x)$.

DEFINITION 2.7. A chart (U, φ) of M with $\varphi(0) = x$ is called a *distinguished chart* at x with respect to $\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k$ if $U = V_1 \times \cdots \times V_k \times W$ where $V_1 \subset \mathbb{R}^{p_1(x)}$, $V_i \subset \mathbb{R}^{\bar{p}_i(x)}$ $(i \ge 2)$ and $W \subset \mathbb{R}^{q_k(x)}$ are open balls and for any $L_i \in \mathcal{F}_i$ we have

$$\varphi(U) \cap L_i = \varphi(V_1 \times \cdots \times V_i \times l_i),$$

where $l_i = \{w \in V_{i+1} \times \cdots \times V_k \times W : \varphi(0, w) \in L_i\}$ for $i = 1, \ldots, k$.

Observe that actually the above φ is an inverse chart; following [16] we call it a chart for simplicity. Notice as well that in the above definition one need not assume that $\mathcal{F}_i$ is a foliation but only that it is a partition into weakly imbedded submanifolds; that $\mathcal{F}_i$ is a foliation then follows by definition.

THEOREM 2.8. *Let $G_1 \leq \cdots \leq G_k \leq \mathrm{Diff}^r(M)$ be an increasing sequence of diffeomorphism groups of M. Then $\mathcal{F}_{G_1} \prec \cdots \prec \mathcal{F}_{G_k}$ admits a distinguished chart at any $x \in M$.*

In fact, this is a straightforward consequence of Theorem 2 in [11].

COROLLARY 2.9. *Let $G_1 \leq \cdots \leq G_k \leq \mathrm{Diff}^r(M)$ and let (L, σ) be a leaf of $\mathcal{F}_{G_k}$. Then all G_i preserve L, and $\mathcal{F}_{G_1|L} \prec \cdots \prec \mathcal{F}_{G_{k-1}|L}$ is a flag structure on L. Moreover, a distinguished chart at x for $\mathcal{F}_{G_1|L} \prec \cdots \prec \mathcal{F}_{G_{k-1}|L}$ is the restriction to L of a distinguished chart at x for $\mathcal{F}_{G_1} \prec \cdots \prec \mathcal{F}_{G_k}$.*

The following property of paracompact spaces is well-known.

LEMMA 2.10. *If X is a paracompact space and $\mathcal{U}$ is an open cover of X, then there exists an open cover $\mathcal{V}$ starwise finer than $\mathcal{U}$, that is, for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $\mathrm{star}^{\mathcal{V}}(V) \subset U$. Here $\mathrm{star}^{\mathcal{V}}(V) := \bigcup\{V' \in \mathcal{V} : V' \cap V \neq \emptyset\}$. In particular, for all $V_1, V_2 \in \mathcal{V}$ with $V_1 \cap V_2 \neq \emptyset$ there is $U \in \mathcal{U}$ such that $V_1 \cup V_2 \subset U$.*

PROPOSITION 2.11. *If $\mathcal{F} \in \mathfrak{F}^r(M)$ then $G_{\mathcal{F}} = \alpha(\mathcal{F})$ is factorizable.*

Proof. Let $\mathfrak{X}_c(M, \mathcal{F})$ be the Lie algebra of all compactly supported vector fields on M tangent to $\mathcal{F}$. Then there is a one-to-one correspondence between isotopies f_t in $G_{\mathcal{F}}$ and smooth paths X_t in $\mathfrak{X}_c(M, \mathcal{F})$ given by the equation

$$\frac{df_t}{dt} = X_t \circ f_t \quad \text{with} \quad f_0 = \mathrm{id}\,.$$

Let $f = (f_t) \in \mathcal{I}G_{\mathcal{F}}$ and let X_t be the corresponding family in $\mathfrak{X}_c(M, \mathcal{F})$. By considering $f_{(p/m)t} f^{-1}_{(p-1/m)t}$, $p = 1, \ldots, m$, instead of f_t we may assume that f_t is close to the identity.

Let $\mathcal{U}$ be an open cover of M. We choose a family of open sets, $(V_j)_{j=1}^s$, which is starwise finer than $\mathcal{U}$, and satisfies $\mathrm{supp}(f_t) \subset V_1 \cup \cdots \cup V_s$ for each t. Let $(\lambda_j)_{j=1}^s$ be a partition of unity subordinate to (V_j), and let $Y_t^j = \lambda_j X_t$. We set

$$X_t^j = Y_t^1 + \cdots + Y_t^j, \qquad j = 1, \ldots, s,$$

and $X_t^0 = 0$. Each of the smooth families X_t^j integrates to an isotopy g_t^j with support in $V_1 \cup \cdots \cup V_j$. We get the fragmentation

$$f_t = g_t^s = f_t^s \circ \cdots \circ f_t^1,$$

where $f_t^j = g_t^j \circ (g_t^{j-1})^{-1}$, with the required inclusions

$$\mathrm{supp}(f_t^j) = \mathrm{supp}(g_t^j \circ (g_t^{j-1})^{-1}) \subset \mathrm{star}(V_j) \subset U_{i(j)}$$

which hold if f_t is sufficiently small. Thus the group of isotopies of $G_{\mathcal{F}}$ is factorizable. Consequently, $G_{\mathcal{F}}$ itself is factorizable. ∎

REMARK 2.12. The identification α_M enables us to consider several new properties of foliations from $\mathfrak{F}^r(M)$. For instance, one can say that a foliation $\mathcal{F}$ is *perfect* if so is the corresponding diffeomorphism group $G_{\mathcal{F}} = \alpha_M(\mathcal{F})$. As mentioned before, it is known that $G_{\mathcal{F}} = \mathrm{Diff}^r_c(M,\mathcal{F})_0$ is perfect provided $\mathcal{F}$ is regular and $1 \le r \le \dim \mathcal{F}$ or $r = \infty$ ([9], [19], [10]). It is not known whether $G_{\mathcal{F}}$ is perfect for singular foliations and a possible proof seems to be very difficult. In turn, possible perfectness of $G_{\mathcal{F}} = \mathrm{Diff}^r_c(M,\mathcal{F})_0$ with r large is closely related to the simplicity of $\mathrm{Diff}^{n+1}_c(M^n)_0$ (see [4]).

Likewise, one can consider *uniformly perfect* or *bounded* foliations by using the corresponding notions for groups (see [14] and references therein).

Finally consider the following important feature of subclasses of the class $\mathfrak{F}^r(M)$, depending also on M and r. A subclass $\mathfrak{K}$ of $\mathfrak{F}^r(M)$ is called *faithful* if the following holds: For all $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{K}$ and for any group isomorphism $\Phi : \alpha_M(\mathcal{F}_1) \cong \alpha_M(\mathcal{F}_2)$ there is a C^r foliated diffeomorphism $\varphi : (M,\mathcal{F}_1) \cong (M,\mathcal{F}_2)$ such that for all $f \in \alpha_M(\mathcal{F}_1)$, $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$. From the reconstruction results of Rybicki [12] and Rubin [8] it is known that the class of regular foliations of class C^∞, $\mathfrak{F}^\infty_{\mathrm{reg}}(M)$, is faithful.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. First observe that the fact that a foliation $\mathcal{F}$ has no proper minimal set is equivalent to the statement that all leaves of $\mathcal{F}$ are dense.

($\Rightarrow$) Assume that $\emptyset \ne L \subset M$ is a proper closed saturated subset of M. Choose $x \in M \setminus L$. We will prove the following statement:

($*$) there are a ball $U \subset M \setminus L$ with $x \in U$ and $g \in [G_U, G_U]$ such that $g(x) \ne x$.

Then we are done by setting $H := \{g \in [G,G] : g|_L = \mathrm{id}_L\}$. To prove ($*$), choose balls U and V in M such that $x \in V \subset \overline{V} \subset U$. Take $f \in G$ such that $f(x) \ne x$. By assumption, for $\mathcal{U} = \{U, \setminus\overline{V}\}$ we may write $g = g_r \dots g_1$, where all g_i are supported in elements of $\mathcal{U}$. Let $s := \min\{i \in \{1,\dots,r\} : \mathrm{supp}(g_i) \subset U \text{ and } g_i(x) \ne x\}$. Then $g_s \in G_U$ satisfies $g_s(x) \ne x$.

Now take an open W such that $x \in W \subset U$ and $g_s(x) \notin W$. Choose $f \in G_W$ with $f(x) \ne x$ by an argument similar to the above. It follows that $f(g_s(x)) = g_s(x) \ne g_s(f(x))$, and therefore $[f, g_s](x) \ne x$. Thus $g = [f, g_s]$ satisfies the claim.

($\Leftarrow$) First observe the following commutator formulae for all $f, g, h \in G$:

$$[fg, h] = f[g,h]f^{-1}[f,h], \qquad [f, gh] = [f,g]g[f,h]g^{-1}. \tag{3.1}$$

Next, in view of a theorem of Ling [5] we know that $[G,G]$ is a perfect group, that is,

$$[G,G] = [[G,G],[G,G]]. \tag{3.2}$$

Suppose that H is a non-trivial normal subgroup of $[G,G]$. Let $x \in M$ satisfy $h(x) \neq x$ for some $h \in H$. Fix a ball U_0 such that $h(U_0) \cap U_0 = \emptyset$. By the definition of $\mathcal{F}_{[G,G]}$ and the assumption that each leaf $L \in \mathcal{F}_{[G,G]}$ is dense, for every $y \in M$ there are a ball U_y with $y \in U_y$ and $f_y \in [G,G]$ such that $f_y(U_y) \subset U$. Let $\mathcal{U} = \{U_y\}_{y\in M}$.

By Lemma 2.10 we can find an open cover $\mathcal{V}$ starwise finer than $\mathcal{U}$. We denote $\mathcal{U}^G = \{g(U) : U \in \mathcal{U},\ g \in [G,G]\}$ and

$$G^{\mathcal{U}} = \prod_{U\in\mathcal{U}^G} [G_U, G_U].$$

By assumption G is factorizable with respect to $\mathcal{V}$. First we show that $[G,G] \subset G^{\mathcal{U}}$, i.e. any $[g_1, g_2] \in [G,G]$ can be expressed as a product of elements of the form $[h_1, h_2]$, where $h_1, h_2 \in G_U$ for some $U \in \mathcal{U}^G$. In view of (3.1) and (3.2) we may assume that $g_1, g_2 \in [G,G]$. Now the relation $[G,G] \subset G^{\mathcal{U}}$ is an immediate consequence of (3.1) and the fact that $\mathcal{V}$ is starwise finer than $\mathcal{U}$.

Next we have to show that $G^{\mathcal{U}} \subset H$. It suffices to check that for every $f, g \in G_U$ with $U \in \mathcal{U}$ the bracket $[f,g]$ belongs to H. This implies that for every $f, g \in G_U$ with $U \in \mathcal{U}^G$ one has $[f,g] \in H$, since H is a normal subgroup in $[G,G]$.

We have fixed $h \in H$ and U_0 such that $h(U_0) \cap U_0 = \emptyset$. If $U \in \mathcal{U}$ and $f, g \in G_U$, take $k \in [G,G]$ such that $k(U) \subset U_0$, and put $\bar{f} = kfk^{-1}$, $\bar{g} = kgk^{-1}$. It follows that $[h\bar{f}h^{-1}, \bar{g}] = \mathrm{id}$. Therefore, $[\bar{f}, \bar{g}] = [[h, \bar{f}], \bar{g}] \in H$, and we also have $[f,g] \in H$. Thus $G^{\mathcal{U}} \subset H$, and consequently $[G,G] \leq H$, as required. ∎

Proof of Theorem 1.2. By assumption and Prop. 2.11, $\mathrm{Diff}^r(M,\mathcal{F})_0$ is factorizable and non-fixing. Since $\mathrm{Diff}^r(M,\mathcal{F})_0$ is isotopically connected, the first assertion follows from Theorem 1.1. The second assertion is a consequence of $\mathrm{Diff}^r(M,\mathcal{F})_0$ being perfect ([9] and [19] for $r = \infty$, and [6] and [10] for $1 \leq r \leq \dim \mathcal{F}$). ∎

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References

[1] A. Banyaga, *The Structure of Classical Diffeomorphism Groups*, Math. Appl. 400, Kluwer, Dordrecht, 1997.

[2] P. Dazord, A. Lichnerowicz et C. M. Marle, *Structure locale des variétés de Jacobi*, J. Math. Pures Appl. 70 (1991), 101–152.

[3] R. D. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Ann. of Math.

[4] J. Lech and T. Rybicki, *Groups of $C^{r,s}$-diffeomorphisms related to a foliation*, in: Banach Center Publ. 76, Inst. Math., Polish Acad. Sci., 2007, 437–450.

[5] W. Ling, *Factorizable groups of homeomorphisms*, Compos. Math. 51 (1984), 41–50.

[6] J. N. Mather, *Commutators of diffeomorphisms, I–III*, Comment. Math. Helv. 49 (1974), 512–528; 50 (1975), 33–40; 60 (1985), 122–124.

[7] P. Molino, *Riemannian Foliations*, Progr. Math. 73, Birkhäuser, 1988.

[8] M. Rubin, *A reconstruction theorem for smooth foliated manifolds*, Geom. Dedicata 150 (2011), 355–375.

[9] T. Rybicki, *The identity component of the leaf preserving diffeomorphism group is perfect*, Monatsh. Math. 120 (1995), 289–305.

[10] —, *Commutators of C^r-diffeomorphisms acting along leaves*, in: Differential Geometry and Applications (Brno, 1995), Masaryk Univ., 1996, 299–307.

[11] —, *A note on generalized flag structures*, Ann. Polon. Math. 69 (1998), 89–97.

[12] —, *Isomorphisms between leaf preserving diffeomorphism groups*, Soochow J. Math. 22 (1996), 525–542.

[13] —, *Commutators of contactomorphisms*, Adv. Math. 225 (2010), 3291–3326.

[14] —, *Boundedness of certain automorphism groups of an open manifold*, Geom. Dedicata 151 (2011), 175–186.

[15] P. Stefan, *Accesibility and foliations with singularities*, Bull. Amer. Math. Soc. 80 (1974), 1142–1145.

[16] —, *Accessible sets, orbits and foliations with singularities*, Proc. London Math. Soc. 29 (1974), 699–713.

[17] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. 180 (1973), 171–188.

[18] W. Thurston, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. 80 (1974), 304–307.

[19] T. Tsuboi, *On the group of foliation preserving diffeomorphisms*, Foliations 2005, P. Walczak et al. (eds.), World Sci., Singapore, 2006, 411–430.

[20] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progr. Math. 118, Birkhäuser, 1994.

[21] R. Wolak, *Basic forms for transversely integrable singular Riemannian foliations*, Proc. Amer. Math. Soc. 128 (2000), 1542–1545.

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