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Landau's theorem for p-harmonic mappings in several variables

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Abstract. A $2p$-times continuously differentiable complex-valued function $f = u+iv$ in a domain $D \subseteq \mathbb{C}$ is p-harmonic if f satisfies the p-harmonic equation $\Delta^p f = 0$, where p (≥ 1) is a positive integer and Δ represents the complex Laplacian operator. If $\Omega \subset \mathbb{C}^n$ is a domain, then a function $f : \Omega \to \mathbb{C}^m$ is said to be p-harmonic in Ω if each component function f_i $(i \in \{1, \ldots, m\})$ of $f = (f_1, \ldots, f_m)$ is p-harmonic with respect to each variable separately. In this paper, we prove Landau and Bloch's theorem for a class of p-harmonic mappings f from the unit ball $\mathbb{B}^n$ into $\mathbb{C}^n$ with the form

$$f(z) = \sum_{(k_1,\ldots,k_n)=(1,\ldots,1)}^{(p,\ldots,p)} |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} G_{p-k_1+1,\ldots,p-k_n+1}(z),$$

where each $G_{p-k_1+1,\ldots,p-k_n+1}$ is harmonic in $\mathbb{B}^n$ for $k_i \in \{1, \ldots, p\}$ and $i \in \{1, \ldots, n\}$.

1. Introduction and main results. A $2p$ times continuously differentiable complex-valued function $f = u+iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if f satisfies the p-harmonic equation $\Delta^p f = 0$, where

$$\Delta^p f = \Delta(\Delta^{p-1} f) = \underbrace{\Delta \cdots \Delta}_{p\,\text{times}} f,$$

and Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where $z = x + iy \in \mathbb{C}$. If this holds for $p = 1$, then f is (planar) *harmonic*, and if it holds for $p = 2$ then f is (planar) *biharmonic*. If f is harmonic in a simple connected domain D, then $f = h + \overline{g}$, where h and g are analytic in D, and are called the analytic and co-analytic parts of f, respectively. See [AA, AAK1, AAK2, CPW1, CPW2, CPW4, CPW7, CSh, Du, He, Sh] for further discussions on harmonic mappings and biharmonic mappings. More

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generally, every p-harmonic mapping f in a star domain D with center 0 admits the well-known finite Almansi expression

$$f(z)=\sum_{k=1}^{p}|z|^{2(k-1)}f_{p-k+1}(z), \tag{1.1}$$

where f_{p-k+1} is harmonic in D for each $k\in\{1,\dots,p\}$ (see [ACL, p. 4, Proposition 1.3] or [CPW3, CPW5]).

Let $C(X,Y)$ denote the set of all continuous functions $f:\ X\to Y$, where X and Y are topological spaces. If $Y=\mathbb{C}$, we simply write $C(X)=C(X,Y)$.

Definition 1.1. Let $\mathbb{C}^n=\{z=(z_1,\dots,z_n):\ z_1,\dots,z_n\in\mathbb{C}\}$ denote the complex vector space of dimension n. Suppose Ω is a domain in $\mathbb{C}^n$. A vector-valued function $f=(f_1,\dots,f_m):\ \Omega\to\mathbb{C}^m$ is said to be *p-harmonic* in Ω if

(a) $f_i\in C(\Omega)$ for each $i\in\{1,\dots,m\}$, and
(b) each component f_i of f is p-harmonic with respect to each variable separately.

For $a=(a_1,\dots,a_n),\ z\in\mathbb{C}^n$, we define the Euclidean inner product $\langle\cdot,\cdot\rangle$ by

$$\langle z,a\rangle=z\cdot\overline{a}=z_1\overline{a}_1+\cdots+z_n\overline{a}_n$$

so that the Euclidean length of z in $\mathbb{C}^n$ is defined by

$$|z|=\langle z,z\rangle^{1/2}=(|z_1|^2+\cdots+|z_n|^2)^{1/2}.$$

Denote the ball in $\mathbb{C}^n$ with center z' and radius r by

$$\mathbb{B}^n(z',r)=\{z\in\mathbb{C}^n:|z-z'|<r\}.$$

In particular, $\mathbb{B}^n$ denotes the unit ball $\mathbb{B}^n(0,1)$. Set $\mathbb{B}^1=\mathbb{D}$, the open unit disk in $\mathbb{C}$.

We use $\mathcal{H}_m^p(\mathbb{B}^n)$ to denote the set of all p-harmonic mappings f from $\mathbb{B}^n$ into $\mathbb{C}^m$. As in the one-dimensional case, we say that f is *separately harmonic* (resp. *separately biharmonic*) when $p=1$ (resp. $p=2$). By the representation (1.1) and Definition 1.1, we easily have the following basic result, and so we omit its proof.

Proposition 1.2. *Every $f\in\mathcal{H}_m^p(\mathbb{B}^n)$ has the representation*

$$f(z)=\sum_{(k_1,\dots,k_n)=(1,\dots,1)}^{(p,\dots,p)}|z_1|^{2(k_1-1)}\cdots|z_n|^{2(k_n-1)}G_{p-k_1+1,\dots,p-k_n+1}(z),$$

where each $G_{p-k_1+1,\dots,p-k_n+1}$ is separately harmonic in $\mathbb{B}^n$ for $k_1,\dots,k_n\in\{1,\dots,p\}$.

Let $\overline{z}$ denote the conjugate of z, that is, $\overline{z}=(\overline{z}_1,\dots,\overline{z}_n)$. Sometimes it is convenient to identify the point $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ with an $n\times 1$ column matrix so that

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

For a vector-valued function $f = (f_1, \ldots, f_m)$ defined on a domain in $\mathbb{C}^n$, we denote by $\partial f/\partial z_j$ the column vector formed by the partial derivatives of the component functions, namely, $\partial f_1/\partial z_j, \ldots, \partial f_m/\partial z_j$, so that

$$f_z = \left(\frac{\partial f}{\partial z_1} \ \cdots \ \frac{\partial f}{\partial z_n}\right) := \left(\frac{\partial f_i}{\partial z_j}\right)_{m\times n},$$

the matrix formed by these column vectors. Similarly, we use

$$f_{\overline{z}} = \left(\frac{\partial f}{\partial \overline{z}_1} \ \cdots \ \frac{\partial f}{\partial \overline{z}_n}\right) := \left(\frac{\partial f_i}{\partial \overline{z}_j}\right)_{m\times n}$$

to denote the matrix formed by the column vectors $\partial f/\partial \overline{z}_j$, where $j \in \{1, \ldots, n\}$. For an $n \times n$ matrix $A = (a_{ij})_{n\times n}$, the operator norm of A is defined by

$$|A| = \sup_{z\neq 0} \frac{|Az|}{|z|} = \max\{|A\theta| : \theta \in \partial\mathbb{B}^n\}.$$

One of the long-standing open problems in function theory is to determine the precise value of the schlicht Landau–Bloch constant for analytic functions of $\mathbb{D}$. It has attracted much attention (see [LiMi, Mi1, Mi2, Mi3] and references therein). For general holomorphic mappings of more than one complex variable, no Landau–Bloch constant exists (cf. [Wu]). In order to obtain some analogs of Landau–Bloch's theorem for mappings with several complex variables, it is necessary to restrict the class of mappings considered (see [CG1, CPW6, FG, Li, Ta, Wu]).

Recently, many authors studied the class of p-harmonic mappings (see [Ad, AdH, Ar, ArL, CPW3, CPW5, Ma]). For instance, in [CPW3], the authors discussed the p-harmonic Bloch mappings and proved a Bloch and Landau's theorem for a class of p-harmonic mappings. The main aim of the present paper is to establish Landau and Bloch's theorems for p-harmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$. Our main result follows.

Theorem 1.3. *Let $f \in \mathcal{H}_n^p(\mathbb{B}^n)$ and*

$$f(z) = \sum_{(k_1,\ldots,k_n)=(1,\ldots,1)}^{(p,\ldots,p)} |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} G_{p-k_1+1,\ldots,p-k_n+1}(z),$$

where all $G_{p-k_1+1,\ldots,p-k_n+1}$ are harmonic for $k_1, \ldots, k_n \in \{1, \ldots, p\}$. Suppose that $f(0) = 0$, $|\det f_z(0)| - \alpha = |f_{\overline{z}}(0)| = 0$, and for any $z \in \mathbb{B}^n$ and

$k_1, \ldots, k_n \in \{1, \ldots, p\}$,

$$|G_{p-k_1+1,\ldots,p-k_n+1}(z)| \leq M,$$

where α and M are positive constants. Then there is a constant $\rho_0 \in (0,1)$ such that f is univalent in $|z| < \rho_0$, where ρ_0 satisfies

$$\frac{\alpha}{(nM)^{n-1}} - \frac{4M(2n-1)[5n+2\sqrt{2}(n+1)]\rho}{\pi\sqrt{1/2-\rho^2}} - 2\sum_{(k_1,\ldots,k_n)\neq(1,\ldots,1)}^{(p,\ldots,p)} \left[M\rho^{2(k_1+\cdots+k_n)-2n-1}\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2} + \frac{[n+(n+1)\rho]M\rho^{2(k_1+\cdots+k_n)-2n}}{(1-\rho^2)} \right] = 0$$

and $f(\mathbb{B}^n)$ contains a univalent ball of radius at least R_0, where

$$R_0 = \frac{\alpha\rho_0}{(nM)^{n-1}} - \frac{4M(2n-1)[5n+2\sqrt{2}(n+1)]}{\pi}\left[\frac{\sqrt{2}}{2} - \Big(\frac{1}{2}-\rho_0^2\Big)^{1/2}\right] - \sum_{(k_1,\ldots,k_n)\neq(1,\ldots,1)}^{(p,\ldots,p)} \left[\frac{M\rho_0^{2(k_1+\cdots+k_n)-2n}}{k_1+\cdots+k_n-n}\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2} + \frac{2[n+(n+1)\rho_0]M\rho_0^{2(k_1+\cdots+k_n)-2n+1}}{(1-\rho_0^2)[2(k_1+\cdots+k_n)-2n+1]} \right].$$

We use $\mathcal{H}_q(\mathbb{B}^n)$ to denote the harmonic Hardy class consisting of all harmonic mappings $f \in \mathcal{H}_n^1(\mathbb{B}^n)$ such that

$$\|f\|_q = \sup_{0<r<1}\Big(\int_{\partial\mathbb{B}^n} |f(r\zeta)|^q\, d\sigma(\zeta)\Big)^{1/q} < \infty,$$

where $q \in (0,\infty)$ and $d\sigma$ denotes the normalized surface measure on $\partial\mathbb{B}^n$. By applying Theorem 1.3, we have

Corollary 1.4. *Suppose that $f \in \mathcal{H}_q(\mathbb{B}^n)$ satisfies $f(0)=0$, $|\det f_z(0)| - 1 = |f_{\overline{z}}(0)| = 0$, and $\|f\|_q \leq K_0$ for some constant $K_0 > 0$ and $q \geq 1$. Then $f(\mathbb{B}^n)$ contains a univalent ball of radius*

$$R \geq \max_{0<r<1} \varphi(r),$$

where

$$\varphi(r) = r\left[\frac{\rho(r)}{(nK(r))^{n-1}} - \frac{4K(r)[5n+2\sqrt{2}(n+1)]}{\pi}\Big(\frac{1}{\sqrt{2}} - \sqrt{1/2-\rho^2(r)}\Big)\right]$$

with

$$\rho(r) = \frac{1}{\sqrt{2(1+t^2)}}, \qquad t = \frac{4n^{n-1}K^n(r)(2n-1)[5n+2\sqrt{2}\,(n+1)]}{\pi}$$

and

$$K(r)=\frac{2^{1/q}K_0}{r(1-r)^{(2n-1)/q}}.$$

We remark that, as $\lim_{r\to 0+}\varphi(r)=\lim_{r\to 1-}\varphi(r)=0$, the maximum of $\varphi(r)$ in Corollary 1.4 does exist.

DEFINITION 1.5. A continuous complex-valued function f defined on a domain $\Omega\subset\mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z\in\Omega$ and $\theta\in\partial\mathbb{B}^n$, the function $f(z+\theta\zeta)$ is harmonic in $\{\zeta:|\zeta|<d(z)\}$, where $d(z)$ denotes the distance from z to the boundary $\partial\Omega$ of Ω (cf. [Ru]). Let $\mathcal{PH}_n(\mathbb{B}^n)$ denote the set of all pluriharmonic mappings of $\mathbb{B}^n$ into $\mathbb{C}^n$.

It follows from [Ru, Theorem 4.4.9] that a real-valued function u defined on a domain $\Omega\subset\mathbb{C}^n$ is pluriharmonic if and only if u is the real part of a holomorphic function on Ω. We remark that a function f defined from $\mathbb{B}^n$ into $\mathbb{C}^n$ is pluriharmonic if and only if f has a representation $f=h+\overline{g}$, where g and h are holomorphic mappings (cf. [CG2]). It is not difficult to show that functions $f\in\mathcal{PH}_n(\mathbb{B}^n)$ are harmonic. This fact follows from Lelong's well-known result that a separately harmonic function is indeed harmonic or, using the continuity assumption, from Avanissian's well-known result. Clearly, $\mathcal{PH}_1(\mathbb{D})$ is the class of planar harmonic mappings in $\mathbb{D}$ (see [CSh, Du]).

THEOREM 1.6. *Let* $f\in\mathcal{H}_n^p(\mathbb{B}^n)$ *and*

$$f(z)=\sum_{(k_1,\dots,k_n)=(1,\dots,1)}^{(p,\dots,p)}|z_1|^{2(k_1-1)}\cdots|z_n|^{2(k_n-1)}G_{p-k_1+1,\dots,p-k_n+1}(z),$$

where $G_{p-k_1+1,\dots,p-k_n+1}\in\mathcal{PH}_n(\mathbb{B}^n)$ *for all* $k_1,\dots,k_n\in\{1,\dots,p\}$. *Suppose* $f(0)=0$, $|\det f_z(0)|-\alpha=|f_{\overline{z}}(0)|=0$ *and for any* $z\in\mathbb{B}^n$, $k_1,\dots,k_n\in\{1,\dots,p\}$,

$$|G_{p-k_1+1,\dots,p-k_n+1}(z)|\le M,$$

where α *and* M *are positive constants. Then there is a constant* $\rho_0\in(0,1)$ *such that* f *is univalent in* $|z|<\rho_0$, *where* ρ_0 *satisfies*

$$\frac{\alpha\pi^{n-1}}{(4M)^{n-1}}-\frac{4(m_3+m_4)M\rho}{\pi}-\frac{\alpha\pi^{n-1}}{(4M)^{n-1}}-\frac{4(m_3+m_4)M\rho}{\pi}$$
$$-2\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)}\bigg[\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2}\rho^{2(k_1+\cdots+k_n)-2n-1}M$$
$$+\frac{4M\rho^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho^2)}\bigg]=0$$

and $f(\mathbb{B}^n)$ contains a univalent ball of radius at least R_0, where

$$m_3 = 2\sqrt{2}\left(\frac{3+\sqrt{17}}{(1+\sqrt{17})\sqrt{5-\sqrt{17}}}\right) \approx 4.199595,$$

$m_4 \approx 2.598076$ *is a constant and*

$$R_0 = \rho_0\left\{\frac{\alpha\pi^{n-1}}{(4M)^{n-1}} - \frac{2(m_1+m_2)M\rho_0}{\pi} - \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)}\left[\frac{(\sum_{i=1}^{n}(k_i-1)^2)^{1/2}\rho_0^{2(k_1+\cdots+k_n)-2n-1}M}{k_1+\cdots+k_n-n} + \frac{8M\rho_0^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho_0^2)[2(k_1+\cdots+k_n)-2n+1]}\right]\right\}.$$

We remark that Theorems 1.3 and 1.6 are generalizations of [CPW3, Theorem 2] to the case of p-harmonic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$.

In Section 2, we will prove several necessary lemmas. The proofs of Theorem 1.3, Corollary 1.4 and Theorem 1.6 will be given in Section 3.

2. Several lemmas

LEMMA 2.1. *Let $f : \mathbb{D} \to \mathbb{B}^n \subset \mathbb{C}^n$ be a harmonic mapping with $f(0)=0$. Then*

$$|f(z)| \le \frac{4}{\pi}\arctan|z| \le \frac{4}{\pi}|z|$$

and this inequality is sharp for each point $z \in \mathbb{D}$.

Proof. For any fixed point $z_0 \in \mathbb{D}$, let $F(z) = \langle f(z_0), f(z)\rangle/|f(z_0)|$ in $\mathbb{D}$, where $f(z_0) \neq 0$. It is not difficult to see that F is a planar harmonic mapping and $|F(z)| < 1$ in $\mathbb{D}$. Then, by [He, Lemma], we have

$$\frac{|\langle f(z_0), f(z)\rangle|}{|f(z_0)|} = |F(z)| \le \frac{4}{\pi}\arctan|z|,$$

which implies that

$$|f(z_0)| \le \frac{4}{\pi}\arctan|z_0|.$$

The desired result follows from the arbitrariness of z_0. ∎

A matrix-valued function $A(z) = (a_{i,j}(z))_{n\times n}$ is called *harmonic* if each entry $a_{i,j}(z)$ is a harmonic mapping from an open subset $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}$.

LEMMA 2.2. *Let $A(z) = (a_{i,j}(z))_{n\times n}$ be a matrix-valued harmonic mapping of $\mathbb{B}^n(0,r)$. If $A(0) = 0$ and $|A(z)| \le M$ in $\mathbb{B}^n(0,r)$, then*

$$|A(z)| \le \frac{4M}{\pi}\frac{|z|}{r}\left(1+\frac{2(n-1)r}{\sqrt{r^2-|z|^2}}\right) \le \frac{4M(2n-1)}{\pi}\frac{|z|}{\sqrt{r^2-|z|^2}}.$$

Proof. For an arbitrary $\theta = (\theta_1, \ldots, \theta_n) \in \partial\mathbb{B}^n$, we let

$$P_\theta(z) = A(z)\theta = (p_1(z), \ldots, p_n(z)).$$

Fix $z = (z_1, \ldots, z_n) \in \mathbb{B}^n(0, r)$. Then we let

$$r_0 = \sqrt{r^2 - (|z_2|^2 + \cdots + |z_n|^2)}$$

and we define

$$F(w) = P_\theta(wr_0, z_2, \ldots, z_n) - P_\theta(0, z_2, \ldots, z_n)$$

in $\mathbb{D}$. Then $|F(w)| \leq 2M$ in $\mathbb{D}$ and $F(0) = 0$. By Lemma 2.1, we have

$$|F(w)| \leq \frac{8M}{\pi}|w| = \frac{8M}{\pi} \frac{\sqrt{|\zeta|^2 - (|z_2|^2 + \cdots + |z_n|^2)}}{r_0} \leq \frac{8M}{\pi} \frac{|\zeta|}{\sqrt{r^2 - |\zeta|^2}},$$

which implies

$$|P_\theta(z)| \leq \frac{8M}{\pi} \frac{|z|}{\sqrt{r^2 - |z|^2}} + |P_\theta(0, z_2, \ldots, z_n)|,$$

where $\zeta = (r_0 w, z_2, \ldots, z_n)$. Repeating this process, we get

$$\begin{aligned}
|P_\theta(0, z_2, \ldots, z_n)| &\leq |P_\theta(0, 0, z_3, \ldots, z_n)| + \frac{8M}{\pi} \frac{|z|}{\sqrt{r^2 - |z|^2}} \\
&\leq |P_\theta(0, 0, 0, z_4, \ldots, z_n)| + \frac{16M}{\pi} \frac{|z|}{\sqrt{r^2 - |z|^2}} \\
&\leq \cdots \\
&\leq |P_\theta(0, \ldots, 0, z_n)| + \frac{8(n-2)M}{\pi} \frac{|z|}{\sqrt{r^2 - |z|^2}} \\
&\leq \frac{4M}{\pi} \frac{|z|}{r} + \frac{8(n-2)M}{\pi} \frac{|z|}{\sqrt{r^2 - |z|^2}},
\end{aligned}$$

which gives

$$|P_\theta(z)| \leq \frac{4M}{\pi} \frac{|z|}{r} \left(1 + \frac{2(n-1)r}{\sqrt{r^2 - |z|^2}}\right) \leq \frac{4M(2n-1)}{\pi} \frac{|z|}{\sqrt{r^2 - |z|^2}}.$$

The arbitrariness of θ yields the desired inequality. ∎

LEMMA 2.3. *Let $f \in \mathcal{H}_n^1(\mathbb{B}^n)$ with $|f(z)| \leq M$ in $\mathbb{B}^n$, where M is a positive constant. Then*

$$\max\{|f_z(z)|, |f_{\overline{z}}(z)|\} \leq M \frac{n + (n+1)|z|}{1 - |z|^2}.$$

Proof. Let $f = (f_1, \ldots, f_n)$ and $\theta = (\theta_1, \ldots, \theta_n) \in \partial\mathbb{B}^n$. Without loss of generality, we assume that f is also harmonic on $\partial\mathbb{B}^n$. By the Poisson integral formula, we have

$$f(z) = \int_{\partial\mathbb{B}^n} \frac{1-|z|^2}{|z-\zeta|^{2n}} f(\zeta)\, d\sigma(\zeta),$$

where $d\sigma$ denotes the normalized surface measure on $\partial\mathbb{B}^n$. In particular,

$$\int_{\partial\mathbb{B}^n} \frac{d\sigma(\zeta)}{|z-\zeta|^{2n}} = \frac{1}{1-|z|^2}.$$

For any $j, k \in \{1, \dots, n\}$, we have

$$(f_j(z))_{z_k} = \int_{\partial\mathbb{B}^n} \frac{-\overline{z}_k|\zeta - z|^2 - n(1-|z|^2)(\overline{z}_k - \overline{\zeta}_k)}{|z-\zeta|^{2n+2}} f_j(\zeta)\, d\sigma(\zeta),$$

which gives

$$\begin{aligned}
&\Big|\sum_{k=1}^n (f_j(z))_{z_k} \cdot \theta_k\Big|^2 \\
&\quad = \Big|\sum_{k=1}^n \int_{\partial\mathbb{B}^n} \frac{[\overline{z}_k|\zeta - z|^2 + n(1-|z|^2)(\overline{z}_k - \overline{\zeta}_k)]\theta_k}{|z-\zeta|^{2n+2}} f_j(\zeta)\, d\sigma(\zeta)\Big|^2 \\
&\quad = \Big|\int_{\partial\mathbb{B}^n} \frac{\sum_{k=1}^n [\overline{z}_k|\zeta - z|^2 + n(1-|z|^2)(\overline{z}_k - \overline{\zeta}_k)]\theta_k}{|z-\zeta|^{2n+2}} f_j(\zeta)\, d\sigma(\zeta)\Big|^2 \\
&\quad \le \Big[\int_{\partial\mathbb{B}^n} \frac{[|z|\,|\zeta - z|^2 + n(1-|z|^2)|\zeta - z|]|f_j(\zeta)|}{|z-\zeta|^{2n+2}}\, d\sigma(\zeta)\Big]^2 \\
&\quad \le \Big[\int_{\partial\mathbb{B}^n} \frac{[|z|\,|\zeta - z| + n(1-|z|^2)]^2}{|z-\zeta|^{2n+2}}\, d\sigma(\zeta)\Big] \cdot \Big[\int_{\partial\mathbb{B}^n} \frac{|f_j(\zeta)|^2}{|z-\zeta|^{2n}}\, d\sigma(\zeta)\Big].
\end{aligned}$$

In the second inequality above, we have used the classical Cauchy–Schwarz inequality. Now we have

$$\begin{aligned}
\sum_{j=1}^n\Big|\sum_{k=1}^n (f_j(z))_{z_k} \cdot \theta_k\Big|^2 &\le \Big[\int_{\partial\mathbb{B}^n} \frac{[|z|\,|\zeta - z| + n(1-|z|^2)]^2}{|z-\zeta|^{2n+2}}\, d\sigma(\zeta)\Big] \\
&\quad \cdot \Big[\int_{\partial\mathbb{B}^n} \frac{\sum_{j=1}^n |f_j(\zeta)|^2}{|z-\zeta|^{2n}}\, d\sigma(\zeta)\Big] \\
&\le \frac{M^2}{1-|z|^2}\Big[\int_{\partial\mathbb{B}^n} \frac{[|z|\,|\zeta - z| + n(1-|z|^2)]^2}{|z-\zeta|^{2n+2}}\, d\sigma(\zeta)\Big] \\
&\le \frac{M^2}{1-|z|^2}\Big[\int_{\partial\mathbb{B}^n} \frac{[|z| + n(1+|z|)]^2}{|z-\zeta|^{2n}}\, d\sigma(\zeta)\Big] \\
&\le M^2 \frac{[|z| + n(1+|z|)]^2}{(1-|z|^2)^2},
\end{aligned}$$

which implies

$$|f_z(z)| \le M\frac{n+(n+1)|z|}{1-|z|^2}.$$

A similar argument shows that

$$|f_{\overline{z}}(z)| \le M\frac{n+(n+1)|z|}{1-|z|^2}.$$

The proof of the lemma is finished. ∎

In the proof of the next lemma, the following result is used.

LEMMA A ([CPW3, Lemma 1] or [CPW4, Theorem 1.1]). *Let f be a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}$ such that $|f(z)| \le M$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$. Then $|a_0| \le M$ and for any $n \ge 1$,*

$$|a_n| + |b_n| \le 4M/\pi. \tag{2.1}$$

In particular,

$$|f_z(0)| + |f_{\overline{z}}(0)| \le 4M/\pi. \tag{2.2}$$

The estimate (2.1) *is sharp. The extremal functions are $f(z) \equiv M$ or*

$$f_n(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right),$$

where $|\alpha| = |\beta| = 1$.

LEMMA 2.4. *Let φ be a harmonic mapping of $\mathbb{D}$ into $\mathbb{C}^m$ and suppose $|\varphi(z)| \le M$ in $\mathbb{D}$. Then*

$$\max\{|\varphi_z(0)|, |\varphi_{\overline{z}}(0)|\} \le 4M/\pi. \tag{2.3}$$

Proof. Let $\alpha = |\varphi_z(0)|$ and $\beta = |\varphi_{\overline{z}}(0)|$. We first prove $\alpha \le 4M/\pi$. Without loss of generality, we may assume that $\alpha > 0$.

Let $\varphi(z) = (\mu_1(z), \dots, \mu_m(z))$. Since each component function μ_k of φ is harmonic in $\mathbb{D}$, φ has the representation

$$\varphi = (\varphi_1 + \overline{\psi}_1, \dots, \varphi_m + \overline{\psi}_m)$$

where φ_k and $\overline{\psi}_k$ are the analytic and co-analytic parts of μ_k in $\mathbb{D}$. Let

$$F(z) = \frac{1}{\alpha}\big[\big(\varphi_1(z) + \overline{\psi_1(z)}\,\big)\overline{\varphi_1'(0)} + \dots + \big(\varphi_m(z) + \overline{\psi_m(z)}\,\big)\overline{\varphi_m'(0)}\,\big].$$

Clearly, $F_z(0) = \alpha$. It follows from the classical Cauchy–Schwarz inequality that

$$|F(z)| \le |\varphi(z)| \le M$$

in $\mathbb{D}$. Applying (2.2) to F shows that

$$\alpha = F_z(0) \le 4M/\pi. \tag{2.4}$$

If $\beta > 0$, then we consider the function

$$P(z) = \frac{1}{\beta}\big[\big(\varphi_1(z) + \overline{\psi_1(z)}\,\big)\psi_1'(0) + \cdots + \big(\varphi_m(z) + \overline{\psi_m(z)}\,\big)\psi_m'(0)\big].$$

Now, applying (2.2) to P, we have

$$\beta = P_{\overline{z}}(0) \le 4M/\pi. \tag{2.5}$$

The desired inequality (2.3) follows from (2.4) and (2.5). ∎

We now recall the following lemma from [CG1, GK, Li].

Lemma B ([CG1, Lemma 2] or [GK, Lemma 9.2.2] or [Li, Lemma 4]). *Let A be an $n \times n$ complex matrix. Then for any unit vector $\theta \in \partial\mathbb{B}^n$,*

$$|A\theta| \ge \frac{|\det A|}{|A|^{n-1}}.$$

In the proof of the next lemma, we shall make use of the automorphism group $\mathrm{Aut}(\mathbb{B}^n)$ consisting of all biholomorphic self-mappings of the unit ball $\mathbb{B}^n$. We recall the following facts from [Ru]:

(a) For $a \in \mathbb{B}^n$, let

$$\phi_a(z) = \frac{a - P_a z - (1-|a|^2)^{1/2} Q_a z}{1 - \langle z, a\rangle},$$

where

$$P_a z = \frac{a\langle z, a\rangle}{\langle a, a\rangle} \quad \text{and} \quad Q_a z = z - P_a z.$$

Then $\phi_a \in \mathrm{Aut}(\mathbb{B}^n)$.

(b) For $z \in \mathbb{B}^n$ and $\phi \in \mathrm{Aut}(\mathbb{B}^n)$,

$$|\phi'(z)\theta| \ge \frac{1 - |\phi(z)|^2}{(1-|z|^2)^{1/2}} \tag{2.6}$$

and

$$|\det \phi'(z)| = \left(\frac{1 - |\phi(z)|^2}{1 - |z|^2}\right)^{(n+1)/2}, \tag{2.7}$$

where $\theta \in \partial\mathbb{B}^n$.

Lemma 2.5. *Let $f \in \mathcal{PH}_n(\mathbb{B}^n)$ and $|f(z)| \le M$ in $\mathbb{B}^n$. Then*

$$\max\{|f_z(z)|, |f_{\overline{z}}(z)|\} \le \frac{4M}{\pi(1-|z|^2)} \tag{2.8}$$

and

$$\max\{|\det f_z(z)|, |\det f_{\overline{z}}(z)|\} \le \frac{(4M)^n}{\pi^n (1-|z|^2)^{(n+1)/2}}. \tag{2.9}$$

Proof. For any $\zeta \in \mathbb{D}$ and a fixed $\theta \in \partial\mathbb{B}^n$, define $\varphi : \mathbb{D} \to \mathbb{C}^n$ by

$$\varphi(\zeta) = f(\zeta\theta).$$

Obviously, $|\varphi(\zeta)| \le M$. By the chain rule, we have

$$\varphi_\zeta(0) = \sum_{k=1}^{n} \theta_k \cdot \frac{\partial f}{\partial z_k}(0) = f_z(0) \cdot \theta, \qquad \varphi_{\overline{\zeta}}(0) = \sum_{k=1}^{n} \overline{\theta}_k \cdot \frac{\partial f}{\partial \overline{z_k}}(0) = f_{\overline{z}}(0) \cdot \overline{\theta},$$

where $\theta = (\theta_1, \dots, \theta_n)$. By Lemma 2.4,

$$|\varphi_\zeta(0)| = |f_z(0) \cdot \theta| \le 4M/\pi, \tag{2.10}$$

$$|\varphi_{\overline{\zeta}}(0)| = |f_{\overline{z}}(0) \cdot \overline{\theta}| \le 4M/\pi. \tag{2.11}$$

The arbitrariness of θ shows that (2.8) holds when $z = 0$.

Next, we fix $z_0 \in \mathbb{B}^n$ with $z_0 \ne 0$. Let $\phi \in \mathrm{Aut}(\mathbb{B}^n)$ be such that ϕ maps 0 to z_0, $T = f \circ \phi$ and $w = \phi(z)$ for $z \in \mathbb{B}^n$. By calculations, we have

$$|T_z| = |f_w\phi'| = \max_{\theta\in\partial\mathbb{B}^n} |f_w\phi'\theta| = \max_{\theta\in\partial\mathbb{B}^n} \left(\left| f_w \frac{\phi'\theta}{|\phi'\theta|} \right| |\phi'\theta| \right),$$

$$|T_{\overline{z}}| = |f_{\overline{w}}\overline{\phi'}| = \max_{\theta\in\partial\mathbb{B}^n} |f_{\overline{w}}\overline{\phi'}\theta| = \max_{\theta\in\partial\mathbb{B}^n} \left(\left| f_{\overline{w}} \frac{\overline{\phi'}\theta}{|\overline{\phi'}\theta|} \right| |\overline{\phi'}\theta| \right).$$

By (2.6),

$$|T_z(0)| \ge (1 - |z_0|^2)|f_w(z_0)|, \qquad |T_{\overline{z}}(0)| \ge (1 - |z_0|^2)|f_{\overline{w}}(z_0)|.$$

Similar arguments to those in the proofs of (2.10) and (2.11) yield

$$\max\left\{|f_w(z_0)|, |f_{\overline{w}}(z_0)|\right\} \le \frac{4M}{\pi(1 - |z_0|^2)}. \tag{2.12}$$

Hence (2.8) follows from (2.12) and the arbitrariness of $z_0 \in \mathbb{B}^n \backslash \{0\}$.

Next we prove inequality (2.9). Inequality (2.8) and Lemma B imply that (2.9) holds when $z = 0$. So, we fix an arbitrary $\xi \in \mathbb{B}^n$ with $\xi \ne 0$. Let $\psi \in \mathrm{Aut}(\mathbb{B}^n)$ be such that ψ maps 0 to ξ, $S = f \circ \psi$ and $u = \psi(z)$ for $z \in \mathbb{B}^n$. By (2.7), we have

$$|\det \psi'(0)| = (1 - |\xi|^2)^{(n+1)/2}.$$

Hence

$$|\det S_z(0)| = |\det f_u(\xi)|\,|\det(\psi'(0))| = |\det f_u(\xi)|(1 - |\xi|^2)^{(n+1)/2}. \tag{2.13}$$

Since $|S(z)| \le M$, we see that

$$|\det S_z(0)| \le \frac{(4M)^n}{\pi^n}. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$|\det f_u(\xi)| \le \frac{(4M)^n}{\pi^n(1 - |\xi|^2)^{(n+1)/2}}. \tag{2.15}$$

Similarly, we have

$$\text{(2.16)} \qquad |\det f_{\overline{u}}(\xi)| \leq \frac{(4M)^n}{\pi^n(1-|\xi|^2)^{(n+1)/2}}.$$

Therefore (2.9) follows from (2.15), (2.16) and the arbitrariness of $\xi \in \mathbb{B}^n\backslash\{0\}$. ∎

LEMMA C ([CG1, Lemma 4]). *Let $A = (a_{i,j}(z))_{n\times n}$ be a holomorphic mapping of $\mathbb{B}^n(0,r)$ into the space of $n\times n$ complex matrices; that is, each $a_{i,j}(z)$ is a holomorphic mapping of $\mathbb{B}^n(0,r)$ into $\mathbb{C}$. If $A(0) = 0$ and $|A(z)| \leq M$ for $z \in \mathbb{B}^n(0,r)$, then*

$$|A(z)| \leq \frac{M}{r}|z|.$$

3. Proofs of Theorem 1.3, Corollary 1.4 and Theorem 1.6

Proof of Theorem 1.3. For each $z \in \mathbb{B}^n(0,\sqrt{2}/2)$, using Lemma 2.3, we have

$$\begin{aligned}|(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)| &\leq |(G_{p,\dots,p})_z(0)| + |(G_{p,\dots,p})_z(z)| \\ &\leq nM + \frac{M[n+(n+1)|z|]}{1-|z|^2} \\ &\leq M[3n+\sqrt{2}(n+1)].\end{aligned}$$

By Lemma 2.2, for each $z \in \mathbb{B}^n(0,\sqrt{2}/2)$, we have

$$|(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)| \leq \frac{m_1|z|}{\sqrt{1/2-|z|^2}},$$

where

$$m_1 = 4M(2n-1)[3n+\sqrt{2}\,(n+1)]/\pi.$$

By Lemmas B and 2.3, we deduce that for each $\theta \in \partial\mathbb{B}^n$,

$$|(G_{p,\dots,p})_z(0)\theta| \geq \frac{\alpha}{|(G_{p,\dots,p})_z(0)|^{n-1}} \geq \frac{\alpha}{(nM)^{n-1}}.$$

From the assumption of Theorem 1.3, we obtain

$$|f_{\overline{z}}(0)| = |(G_{p,\dots,p})_{\overline{z}}(0)| = 0.$$

A similar argument shows that for each $z \in \mathbb{B}^n(0,\sqrt{2}/2)$,

$$\begin{aligned}|(G_{p,\dots,p})_{\overline{z}}(z) - (G_{p,\dots,p})_{\overline{z}}(0)| &\leq |(G_{p,\dots,p})_{\overline{z}}(z)| + |(G_{p,\dots,p})_{\overline{z}}(0)| \\ &= |(G_{p,\dots,p})_{\overline{z}}(z)| + |f_{\overline{z}}(0)| = |(G_{p,\dots,p})_{\overline{z}}(z)| \\ &\leq \frac{m_2|z|}{\sqrt{1/2-|z|^2}},\end{aligned}$$

where

$$m_2 = 4M(2n-1)[2n+\sqrt{2}\,(n+1)]/\pi.$$

Let ξ_1 and ξ_2 be two distinct points in $\mathbb{B}^n(0,\rho)$ with $\rho \le \sqrt{2}/2$, let $[\xi_1,\xi_2]$ denote the segment from ξ_1 to ξ_2, and let

$$dz = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix}, \quad d\overline{z} = \begin{pmatrix} d\overline{z}_1 \\ \vdots \\ d\overline{z}_n \end{pmatrix}, \tag{3.1}$$

which may be conveniently written as

$$dz = (dz_1, \dots, dz_n)^T, \quad d\overline{z} = (d\overline{z}_1, \dots, d\overline{z}_n)^T,$$

where T means the matrix transpose. First we have

$$f_z(z) = (G_{p,\dots,p})_z(z) + \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [(G_{p-k_1+1,\dots,p-k_n+1}(z))^T P_{k_1,\dots,k_n} + |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_z(z)].$$

Similarly,

$$f_{\overline{z}}(z) = (G_{p,\dots,p})_{\overline{z}}(z) + \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [(G_{p-k_1+1,\dots,p-k_n+1}(z))^T \overline{P}_{k_1,\dots,k_n} + |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_{\overline{z}}(z)].$$

Then

$$\begin{aligned} |f(\xi_1) - f(\xi_2)| &= \Big| \int_{[\xi_1,\xi_2]} f_z(z)dz + f_{\overline{z}}(z)\, d\overline{z} \Big| \\ &\ge \Big| \int_{[\xi_1,\xi_2]} f_z(0)\, dz + f_{\overline{z}}(0)\, d\overline{z} \Big| \\ &\quad - \Big| \int_{[\xi_1,\xi_2]} (f_z(z) - f_z(0))\, dz + (f_{\overline{z}}(z) - f_{\overline{z}}(0))\, d\overline{z} \Big| \\ &\ge J_1 - J_2 - J_3 - J_4, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \Big| \int_{[\xi_1,\xi_2]} (G_{p,\dots,p})_z(0)\, dz + (G_{p,\dots,p})_{\overline{z}}(0)\, d\overline{z} \Big|, \\ J_2 &= \Big| \int_{[\xi_1,\xi_2]} [(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)]\, dz \\ &\qquad + [(G_{p,\dots,p})_{\overline{z}}(z) - (G_{p,\dots,p})_{\overline{z}}(0)]\, d\overline{z} \Big|, \end{aligned}$$

$$J_3 = \Big| \int_{[\xi_1,\xi_2]} \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [(G_{p-k_1+1,\dots,p-k_n+1}(z))^T P_{k_1,\dots,k_n} + |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_z(z)]\, dz \Big|,$$

$$J_4 = \Big| \int_{[\xi_1,\xi_2]} \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [(G_{p-k_1+1,\dots,p-k_n+1}(z))^T \overline{P}_{k_1,\dots,k_n} + |z_1|^{2(k_1-1)} \dots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_{\overline{z}}(z)]\, d\overline{z} \Big|,$$

with

$$P_{k_1,\dots,k_n} = ((k_1-1) z_1^{k_1-2} \overline{z}_1^{k_1-1} |z_2|^{2(k_2-1)} \cdots |z_n|^{2(k_n-1)}, \dots \\ \dots, (k_n-1) z_n^{k_n-2} \overline{z}_n^{k_n-1} |z_1|^{2(k_1-1)} \cdots |z_{n-1}|^{2(k_{n-1}-1)})$$

and

$$\overline{P}_{k_1,\dots,k_n} = ((k_1-1) z_1^{k_1-1} \overline{z}_1^{k_1-2} |z_2|^{2(k_2-1)} \cdots |z_n|^{2(k_n-1)}, \dots \\ \dots, (k_n-1) z_n^{k_n-1} \overline{z}_n^{k_n-2} |z_1|^{2(k_1-1)} \cdots |z_{n-1}|^{2(k_{n-1}-1)}).$$

Now, as $f_{\overline{z}}(0) = (G_{p,\dots,p})_{\overline{z}}(0) = 0$, we have

$$J_1 = \left| \int_{[\xi_1,\xi_2]} (G_{p,\dots,p})_z(0) \frac{dz}{|dz|} |dz| \right| \geq |\xi_1 - \xi_2| \frac{\alpha}{(nM)^{n-1}}.$$

Next,

$$\begin{aligned} J_2 &\leq \int_{[\xi_1,\xi_2]} |(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)|\, |dz| \\ &\quad + \int_{[\xi_1,\xi_2]} |(G_{p,\dots,p})_{\overline{z}}(z) - (G_{p,\dots,p})_{\overline{z}}(0)|\, |d\overline{z}| \\ &\leq |\xi_1 - \xi_2| \frac{(m_1+m_2)\rho}{\sqrt{1/2 - \rho^2}}. \end{aligned}$$

Finally,

$$J_3 \leq \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} \Big\{ \int_{[\xi_1,\xi_2]} \big(|(G_{p-k_1+1,\dots,p-k_n+1}(z))^T P_{k_1,\dots,k_n}| + \big| |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_z(z) \big| \big) |\, dz| \Big\}$$

$$\leq |\xi_1 - \xi_2| \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} \left[\Big(\sum_{i=1}^{n} (k_i-1)^2 \Big)^{1/2} \rho^{2(k_1+\cdots+k_n)-2n-1} M + \frac{[n+(n+1)\rho] M \rho^{2(k_1+\cdots+k_n)-2n}}{(1-\rho^2)} \right],$$

because

$$|(G_{p-k_1+1,\dots,p-k_n+1}(z))^T P_{k_1,\dots,k_n}| \leq \Big(\sum_{i=1}^{n} (k_i-1)^2 \Big)^{1/2} \rho^{2(k_1+\cdots+k_n)-2n-1} M$$

and

$$\big| |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_z(z) \big| \leq \frac{[n+(n+1)\rho] M \rho^{2(k_1+\cdots+k_n)-2n}}{(1-\rho^2)}.$$

A similar estimate holds for J_4. Using these estimates, we deduce that

$$|f(\xi_1) - f(\xi_2)| \geq J_1 - J_2 - J_3 - J_4 \geq |\xi_1 - \xi_2| \psi(\rho),$$

where

$$\psi(\rho) = \frac{\alpha}{(nM)^{n-1}} - \frac{(m_1+m_2)\rho}{\sqrt{(1/2)-\rho^2}} - 2 \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} \left[\Big(\sum_{i=1}^{n} (k_i-1)^2 \Big)^{1/2} \rho^{2(k_1+\cdots+k_n)-2n-1} M + \frac{[n+(n+1)\rho] M \rho^{2(k_1+\cdots+k_n)-2n}}{(1-\rho^2)} \right].$$

Then it is easy to see that the function $\psi(\rho)$ is strictly decreasing in $(0, \sqrt{2}/2)$,

$$\lim_{\rho\to 0+} \psi(\rho) = \frac{\alpha}{(nM)^{n-1}} \quad \text{and} \quad \lim_{\rho\to\sqrt{2}/2} \psi(\rho) = -\infty.$$

Hence there exists a unique $\rho_0 \in (0, \sqrt{2}/2)$ satisfying $\psi(\rho_0) = 0$. This implies that $f(z)$ is univalent in $\mathbb{B}^n(0, \rho_0)$.

Furthermore, for any z' in $\{z' : |z'| = \rho_0\}$,

$$|f(z') - f(0)| \geq \Big| \int_{[0,z']} (G_{p,\dots,p})_z(0)\, dz + (G_{p,\dots,p})_{\overline{z}}(0)\, d\overline{z} \Big| - \Big| \int_{[0,z']} [(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)]\, dz + [(G_{p,\dots,p})_{\overline{z}}(z) - (G_{p,\dots,p})_{\overline{z}}(0)] d\overline{z} \Big|$$

$$
\begin{aligned}
&-\Big|\int\limits_{[0,z']}\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)}[(G_{p-k_1+1,\dots,p-k_n+1}(z))^T P_{k_1,\dots,k_n}\\
&\qquad\qquad + |z_1|^{2(k_1-1)}\dots|z_n|^{2(k_n-1)}(G_{p-k_1+1,\dots,p-k_n+1})_z(z)]\,dz\Big|\\
&-\Big|\int\limits_{[0,z']}\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)}[(G_{p-k_1+1,\dots,p-k_n+1}(z))^T \overline{P}_{k_1,\dots,k_n}\\
&\qquad\qquad + |z_1|^{2(k_1-1)}\dots|z_n|^{2(k_n-1)}(G_{p-k_1+1,\dots,p-k_n+1})_{\overline{z}}(z)]\,d\overline{z}\Big|\\
&\geq \frac{\alpha\rho_0}{(nM)^{n-1}}-(m_1+m_2)[\sqrt{2}/2-(1/2-\rho_0^2)^{1/2}]\\
&\quad -\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)}\bigg[\frac{(\sum_{i=1}^n(k_i-1)^2)^{1/2}\rho_0^{2(k_1+\dots+k_n)-2n}M}{k_1+\dots+k_n-n}\\
&\qquad\qquad +\frac{2[n+(n+1)\rho_0]M\rho_0^{2(k_1+\dots+k_n)-2n+1}}{(1-\rho_0^2)[2(k_1+\dots+k_n)-2n+1]}\bigg]\\
&> \rho_0\psi(\rho_0)=0.
\end{aligned}
$$

The proof of the theorem is complete. ∎

Proof of Corollary 1.4. Without loss of generality, we may assume that f is also harmonic on $\partial\mathbb{B}^n$. By the Poisson integral representation, we have

$$
f(z)=\int\limits_{\partial\mathbb{B}^n}\frac{1-|z|^2}{|z-\zeta|^{2n}}f(\zeta)\,d\sigma(\zeta)
$$

in $\mathbb{B}^n$. By Jensen's inequality, we get

$$
|f(z)|^q\leq\int\limits_{\partial\mathbb{B}^n}\frac{1-|z|^2}{|z-\zeta|^{2n}}|f(\zeta)|^q\,d\sigma(\zeta)\leq\frac{2\|f\|_q^q}{(1-|z|)^{2n-1}},
$$

which gives

$$
|f(z)|\leq\frac{2^{1/q}K_0}{(1-|z|)^{(2n-1)/q}}.
$$

For $r\in(0,1)$, let $F(\zeta)=f(r\zeta)/r$ in $\mathbb{B}^n$. Then

$$
|F(\zeta)|\leq\frac{2^{1/q}K_0}{r(1-r)^{(2n-1)/q}}=K(r).
$$

Replacing M in Theorem 1.3 by $K(r)$ and applying Theorem 1.3 to F, we deduce that $F(\mathbb{B}^n)$ contains a univalent ball of radius $R_0\geq\varphi(r)/r$. Then $f(\mathbb{B}^n)$ contains a univalent ball of radius $R\geq\max_{0<r<1}\varphi(r)$. ∎

Proof of Theorem 1.6. By Lemma 2.5, we see that for any $z \in \mathbb{B}^n$,

$$|(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)| \le \frac{4M}{\pi}\left(1 + \frac{1}{1-|z|^2}\right) = \frac{4M}{\pi}\,\frac{2-|z|^2}{1-|z|^2}.$$

Let $W_1(r) = (2-r^2)/[r(1-r^2)]$ for $r \in (0,1)$. It is easy to see that

$$W_1(r_1) = \min_{r\in(0,1)} W_1(r),$$

where $r_1 = \sqrt{(5-\sqrt{17})/2} \approx 0.662153$. We denote $W_1(r_1)$ by m_3. Then

$$m_3 = 2\sqrt{2}\left(\frac{3+\sqrt{17}}{(1+\sqrt{17})\sqrt{5-\sqrt{17}}}\right) \approx 4.199595.$$

By Lemma A, we see that for z in the disk $\{z : |z| \le r_1\}$,

$$|(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)| \le \frac{4m_3M}{\pi}|z|. \tag{3.2}$$

On the other hand, by Lemmas B and 2.5, we conclude that for any $\theta \in \partial\mathbb{B}^n$,

$$|(G_{p,\dots,p})_z(0)\theta| \ge \frac{\alpha}{|(G_{p,\dots,p})_z(0)|^{n-1}} \ge \frac{\alpha\pi^{n-1}}{(4M)^{n-1}}. \tag{3.3}$$

A similar argument gives the inequality

$$|(G_{p,\dots,p})_{\overline{z}}(z) - (G_{p,\dots,p})_{\overline{z}}(0)| \le \frac{4M}{\pi}\,\frac{1}{1-|z|^2}$$

in $\mathbb{B}^n$.

Let $W_2(r) = 1/[r(1-r^2)]$ in $(0,1)$. Then

$$W_2(r_2) = \min_{r\in(0,1)} \{W_2(r)\},$$

where $r_2 = \sqrt{3}/3 \approx 0.577350$. We denote $W_2(r_2)$ by m_4. Then $m_4 \approx 2.598076$.

By Lemma A, we have

$$|(G_{p,\dots,p})_{\overline{z}}(z)| \le \frac{4m_4M}{\pi}|z| \tag{3.4}$$

for all z in the disk $\{z : |z| \le r_2\}$.

Let ξ_1 and ξ_2 be two distinct points in $\mathbb{B}^n(0,\rho)$ with $\rho \le r_2$. Following the proof of Theorem 1.3, we deduce from (3.2)–(3.4) (together with the notations for dz and $d\overline{z}$ given in (3.1)) that

$$
\begin{aligned}
|f(\xi_1)-f(\xi_2)| \geq & \Big|\int_{[\xi_1,\xi_2]} (G_{p,\dots,p})_z(0)\,dz + (G_{p,\dots,p})_{\overline{z}}(0)\,d\overline{z}\Big| \\
& - \Big|\int_{[\xi_1,\xi_2]} [(G_{p,\dots,p})_z(z)-(G_{p,\dots,p})_z(0)]\,dz \\
& \qquad + [(G_{p,\dots,p})_{\overline{z}}(z)-(G_{p,\dots,p})_{\overline{z}}(0)]d\overline{z}\Big| \\
& - \Big|\int_{[\xi_1,\xi_2]} \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [(G_{p-k_1+1,\dots,p-k_n+1}(z))^T P_{k_1,\dots,k_n} \\
& \qquad + |z_1|^{2(k_1-1)}\cdots|z_n|^{2(k_n-1)}(G_{p-k_1+1,\dots,p-k_n+1})_z(z)]\,dz\Big| \\
& - \Big|\int_{[\xi_1,\xi_2]} \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [(G_{p-k_1+1,\dots,p-k_n+1}(z))^T \overline{P}_{k_1,\dots,k_n} \\
& \qquad + |z_1|^{2(k_1-1)}\cdots|z_n|^{2(k_n-1)}(G_{p-k_1+1,\dots,p-k_n+1})_{\overline{z}}(z)]\,d\overline{z}\Big| \\
\geq & |\xi_1-\xi_2|\Big\{\frac{\alpha\pi^{n-1}}{(4M)^{n-1}} - \frac{4(m_3+m_4)M\rho}{\pi} \\
& - 2\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} \Big[\rho^{2(k_1+\cdots+k_n)-2n-1}M\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2} \\
& \qquad + \frac{4M\rho^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho^2)}\Big]\Big\},
\end{aligned}
$$

where $P_{k_1,\dots,k_n}$ and $\overline{P}_{k_1,\dots,k_n}$ are as in the proof of Theorem 1.3.

Finally, we let

$$
\begin{aligned}
\phi(\rho) = & \frac{\alpha\pi^{n-1}}{(4M)^{n-1}} - \frac{4(m_3+m_4)M\rho}{\pi} \\
& - 2\sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} \Big[\rho^{2(k_1+\cdots+k_n)-2n-1}M\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2} \\
& \qquad + \frac{4M\rho^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho^2)}\Big].
\end{aligned}
$$

Then it is easy to see that $\phi(\rho)$ is a strictly decreasing function in $(0,1)$,

$$
\lim_{\rho\to 0+}\phi(\rho) = \frac{\alpha\pi^{n-1}}{(4M)^{n-1}} \quad \text{and} \quad \lim_{\rho\to 1-}\phi(\rho) = -\infty.
$$

Hence there exists a unique $\rho_0 \in (0, \sqrt{3}/3)$ satisfying $\phi(\rho_0)=0$, which shows that f is univalent in $\mathbb{B}^n(0,\rho_0)$.

Furthermore, by inequalities (3.2)–(3.4) and Lemma 2.5, we deduce that for any z' in $\{z' : |z'| = \rho_0\}$,

$$
\begin{aligned}
|f(z') - f(0)| \geq & \Big| \int_{[0,z']} (G_{p,\dots,p})_z(0)\, dz + (G_{p,\dots,p})_{\overline{z}}(0)\, d\overline{z} \Big| \\
& - \Big| \int_{[0,z']} [(G_{p,\dots,p})_z(z) - (G_{p,\dots,p})_z(0)]\, dz \\
& \qquad + [(G_{p,\dots,p})_{\overline{z}}(z) - (G_{p,\dots,p})_{\overline{z}}(0)]\, d\overline{z} \Big| \\
& - \Big| \int_{[0,z']} \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [G_{p-k_1+1,\dots,p-k_n+1}(z) P_{k_1,\dots,k_n} \\
& \qquad + |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_z(z)]\, dz \Big| \\
& - \Big| \int_{[0,z']} \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} [G_{p-k_1+1,\dots,p-k_n+1}(z) \overline{P}_{k_1,\dots,k_n} \\
& \qquad + |z_1|^{2(k_1-1)} \cdots |z_n|^{2(k_n-1)} (G_{p-k_1+1,\dots,p-k_n+1})_{\overline{z}}(z)]\, d\overline{z} \Big| \\
\geq & \rho_0 \bigg\{ \frac{\alpha\pi^{n-1}}{(4M)^{n-1}} - \frac{2(m_3+m_4)M\rho_0}{\pi} \\
& - \sum_{(k_1,\dots,k_n)\neq(1,\dots,1)}^{(p,\dots,p)} \bigg[\frac{\rho_0^{2(k_1+\cdots+k_n)-2n-1} M (\sum_{i=1}^n (k_i-1)^2)^{1/2}}{k_1+\cdots+k_n-n} \\
& \qquad + \frac{8M\rho_0^{2(k_1+\cdots+k_n)-2n}}{\pi(1-\rho_0^2)[2(k_1+\cdots+k_n)-2n+1]} \bigg] \bigg\} \\
> & \rho_0 \psi(\rho_0) = 0.
\end{aligned}
$$

The proof of the theorem is complete. ∎

We remark that the univalent disk of radius ρ_0 in Theorem 1.6 is larger than the one obtained in Theorem 1.3. From the definition of ψ (resp. ϕ), we see that the function ψ (resp. ϕ) is strictly decreasing in $(0, \sqrt{2}/2)$ (resp. $(0, \sqrt{3}/3)$), where ψ (resp. ϕ) is as in the proof of Theorem 1.3 (resp. Theorem 1.6). Hence there is a unique solution $x \in (0, \sqrt{2}/2)$ (resp. $x \in (0, \sqrt{3}/3)$) such that $\psi(x) = 0$ (resp. $\phi(x) = 0$). Without loss of generality, let $\rho_1 \in (0, \sqrt{2}/2)$ be such that $\psi(\rho_1) = 0$, and let $\rho_2 \in (0, \sqrt{3}/3)$ be such that $\phi(\rho_2) = 0$. By calculations, we see that

$$
\frac{\alpha\pi^{n-1}}{(4M)^{n-1}} > \frac{\alpha}{(nM)^{n-1}}, \qquad \frac{4(m_3+m_4)Mx}{\pi} < \frac{(m_1+m_2)x}{\sqrt{1/2 - x^2}}
$$

and

$$Mx^{2(k_1+\cdots+k_n)-2n-1}\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2}+\frac{[n+(n+1)x]Mx^{2(k_1+\cdots+k_n)-2n}}{(1-x^2)}$$
$$\geq Mx^{2(k_1+\cdots+k_n)-2n-1}\Big(\sum_{i=1}^{n}(k_i-1)^2\Big)^{1/2}+\frac{4Mx^{2(k_1+\cdots+k_n)-2n}}{\pi(1-x^2)},$$

where $x\in(0,\sqrt{2}/2)$. This implies that $\rho_1<\rho_2\leq\sqrt{3}/3$.

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