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Asymptotic behaviour of semigroups of nonnegative operators on a Banach lattice

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Abstract. Asymptotic convergence theorems for semigroups of nonnegative operators on a Banach lattice, on $C(X)$ and on $L^p(X)$ $(1 \le p \le \infty)$ are proved. The general results are applied to a class of semigroups generated by some differential equations.

Introduction. In ergodic theory an important role is played by some noncompact Markov operators. For example, they map the unit ball in L^1 into itself. A. Lasota and J. A. Yorke [LY1] proved the convergence of the iterates of such operators under the assumption of the existence of a lower function. Some conditions for asymptotic stability and sweeping of Markov operators in $L^1(X)$ were given by R. Rudnicki [R]. A. Lasota and J. A. Yorke [LY2] discuss the case where the operator is a Markov operator on the family of finite Borel measure sets. Positive operators and semigroups of positive operators on $C(X)$ have been investigated by R. Rudnicki [R], A. Lasota and R. Rudnicki [LR], and A. Lasota and J. A. Yorke [LY1]. J. Socała [S] discusses the case of nonnegative operators on a Banach lattice, on $C(X)$ and on L^p $(1 \le p \le \infty)$.

The purpose of this paper is to give a necessary and sufficient condition for the convergence of a semigroup of nonnegative linear operators on a Banach lattice. Our results are a straightforward extension of the Lasota–Rudnicki theorem [LR] and they are based on the idea of the lower function.

The organization of the paper is as follows. Section 1 contains the main convergence theorems. In Section 2 we show an application to semigroups generated by some differential equations. Our result is different from the theorems of T. Dłotko and A. Lasota [DL], K. Łoskot and R. Rudnicki [ŁR],

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A. Lasota and R. Rudnicki [LR], D. Jama [J], A. Lasota and J. A. Yorke [LY1], H. Inaba [I], P. Pichór and R. Rudnicki [PR], [PR1], R. Rudnicki and M. C. Mackey [RM], and A. Gmira and L. Veron [GV].

1. Convergence theorems. Let $(V, \leq, \|\cdot\|)$ be a Banach lattice. For $f \in V$ we define $f^+ = \sup(0, f)$, $f^- = \sup(0, -f)$.

By V_+ we shall denote the set of all nonzero nonnegative elements of V, i.e.

$$V_+ = \{f \in V : f \geq 0,\ \|f\| \neq 0\}.$$

A linear operator $P : V \to V$ will be called *nonnegative* if $Pf \geq 0$ for $f \in V_+$.

A nonnegative operator $P : V \to V$ is said to be *exponentially stationary* if there exist $\lambda > 0$, $f_0 \in V_+$ and a continuous linear functional $L : V \to \mathbb{R}$ such that

$$Pf_0 = \lambda f_0, \tag{1}$$

$$\lim_{n\to\infty} \|\lambda^{-n} P^n f - f_0 Lf\| = 0 \quad \text{for } f \in V. \tag{2}$$

Remark 1. Let P be an exponentially stationary operator. Then the operator Q given by $Q = \lambda^{-1} P$ satisfies

$$\lim_{n\to\infty} \|Q^n f - f_0\| = 0 \tag{3}$$

for all f with $Lf = 1$. Note that (3) does not imply stability in the sense of Lyapunov.

A family $\{T(t)\}$ $(t \geq 0)$ of continuous linear operators on V is called a *strongly continuous semigroup* if $T(t+s) = T(t)T(s)$ for $t, s \geq 0$ and

$$\lim_{t\to 0^+} \|T(t)f - f\| = 0 \quad \text{for } f \in V.$$

A strongly continuous semigroup $\{T(t)\}$ of nonnegative operators on V is called *exponentially stationary* if there exist $\lambda > 0$, $f_0 \in V_+$ and a continuous linear functional $L : V \to \mathbb{R}$ such that $T(t)f_0 = \lambda^t f_0$ for $t \geq 0$ and

$$\lim_{t\to\infty} \|\lambda^{-t} T(t)f - f_0 Lf\| = 0 \quad \text{for } f \in V.$$

Lemma 1. *A strongly continuous semigroup $\{T(t)\}$ of nonnegative operators on V is exponentially stationary if and only if there exists $s > 0$ such that the operator $T(s)$ is exponentially stationary.*

Proof. Evidently if the semigroup $\{T(t)\}$ is exponentially stationary, then for every $s > 0$ the operator $T(s)$ is exponentially stationary. Now assume there exists $s > 0$ such that $T(s)$ is exponentially stationary. We will show that $\{T(t)\}$ is exponentially stationary. We will follow the idea of A. Lasota and R. Rudnicki (see [LR, proof of Theorem 1.1]). There exist

$\lambda > 0$, $f_0 \in V_+$ and a continuous linear functional $L : V \to \mathbb{R}$ such that $T(s)f_0 = \lambda^s f_0$ and

$$\lim_{n\to\infty} \|\lambda^{-ns}T(ns)f - f_0 Lf\| = 0 \quad \text{for } f \in V. \tag{4}$$

We can assume $Lf_0 = 1$. Substituting $f = T(t)f_0$ into (4) we obtain $T(t)f_0 = f_0 LT(t)f_0$. Write $\beta(t) = LT(t)f_0$. The function β is continuous and satisfies the Cauchy equation $\beta(t+r) = \beta(t)\beta(r)$ for $t, r \geq 0$. This implies that β is an exponential function. Since $\beta(s) = \lambda^s$, we have $\beta(t) = \lambda^t$, $t \geq 0$, and consequently

$$T(t)f_0 = \lambda^t f_0. \tag{5}$$

Let α be a constant such that $\|T(t)\| \leq \alpha$ for $t \in [0,s]$. Then $\|T(t)f\| \leq \alpha\|f\|$ for $t \in [0,s]$ and $f \in V$. Consequently,

$$\|\lambda^{ns}T(ns+t)f - T(t)f_0 Lf\| \leq \alpha\|\lambda^{ns}T(ns)f - f_0 Lf\|$$

for $t \in [0,s]$ and $f \in V$. From this inequality and (4) it follows that the semigroup $\{T(t)\}$ is exponentially stationary. ∎

From this lemma and [S, Theorems 1–3] we have the following theorems.

THEOREM 1. *A strongly continuous semigroup $\{T(t)\}$ of nonnegative operators on V is exponentially stationary if and only if there exists $s > 0$, a set $D \subseteq V_+$ dense in V_+ and $h \in V_+$ such that $\|T(ns)f\| \neq 0$ for $f \in D \cup \{h\}$ and $n \in \mathbb{N}$ and the following three conditions hold*:

(I) $\displaystyle \lim_{n\to\infty} \left\| \left(\frac{T(ns)f}{\|T(ns)f\|} - h \right)^{-} \right\| = 0$ *for* $f \in D$.

(II) $\displaystyle \limsup_{n\to\infty} \frac{\|T(ns)f\|}{\|T(ns)h\|} < \infty$ *for* $f \in V_+$.

(III) *The sequence $\{T(ns)h/\|T(ns)h\|\}$ has a convergent subsequence.*

We shall use the following conditions:

(III′) For some $g_0 \in D$ the sequence $\{T(ns)g_0/\|T(ns)g_0\|\}$ has a weakly convergent subsequence.

(IV) There exists $\gamma > 0$ such that

$$\lim_{n\to\infty} \left\| \left(\gamma h - \frac{T(ns)h}{\|T(ns)h\|} \right)^{-} \right\| = 0.$$

(IV′) There exists $\gamma > 0$ such that for $f \in D$ and $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \left\| \left(\frac{T(ns)f}{\|T(ns)f\|} - \frac{T(ns+ms)h}{\gamma\|T(ns+ms)h\|} \right)^{-} \right\| = 0.$$

THEOREM 2. *A strongly continuous semigroup $\{T(t)\}$ of nonnegative operators on V is exponentially stationary if and only if there exists $s > 0$,* *a set* $D \subseteq V+$ *dense in* V_+, $h \in V_+$ *and* $\gamma > 0$ *such that* $\|T(ns)f\| \neq 0$ *for* $f \in D \cup \{h\}$, $n \in \mathbb{N}$ *and they satisfy conditions* I, II, III′, IV (*or* I, II, III′, IV′).

Now let V be a Banach lattice. In Theorem 3 below we shall assume that the unit ball of V contains a largest element 1_X, that is, $\|1_X\| = 1$ and $f \le 1_X$ for $f \in V$, $\|f\| = 1$ (for example $C(X)$, $L^\infty(X)$). We say that $f \in V_{++}$ if $f \in V_+$ and $f \ge \alpha 1_X$ for some $\alpha > 0$.

Theorem 3 (see [R], [ŁR], [LY1]). *Let* $\{T(t)\}$ *be a strongly continuous semigroup of nonnegative operators on* V, D *a dense subset of* V_+, *and* $\alpha > 0$, $s > 0$. *Assume that for every* $f \in D$ *there is an integer* $n_0(f)$ *such that*

$$\|T(ns)f\| \neq 0, \quad T(ns)f/\|T(ns)f\| \ge \alpha \cdot 1_X \quad \text{for } n \ge n_0(f)$$

and that, *for some* $g \in V_{++}$, *the sequence* $\{T(ns)g/\|T(ns)g\|\}$ *has a weakly convergent subsequence. Then* $\{T(t)\}$ *is exponentially stationary.*

Now let $V = L^p(X, \Sigma, \mu)$ $(1 \le p < \infty)$ and $\|\cdot\| = \|\cdot\|_{L^p}$, where (X, Σ, μ) is a σ-finite measure space.

Theorem 4. *A strongly continuous semigroup* $\{T(t)\}$ *of nonnegative operators on* V *is exponentially stationary if and only if there exists* $s > 0$, *a set* $D \subseteq V_+$ *dense in* V_+ *and* $h \in V_+$ *such that* $\|T(ns)f\| \neq 0$ *for* $f \in D \cup \{h\}$, $n \in \mathbb{N}$ *and they satisfy conditions* I, II.

2. Asymptotic behaviour of semigroups generated by differential equations. Let V be a Banach lattice. A strongly continuous semigroup $\{T(t)\}$ $(t \in [0,\infty))$ will be called *analytic* if for every $f \in V$ the function

$$(0,\infty) \ni t \mapsto T(t)f \in V$$

is analytic.

Let $\{T(t)\}$ be a strongly continuous semigroup of nonnegative operators on V. We denote by $D(A)$ the set of all $f \in V$ such that the limit

$$Af = \lim_{t \to 0^+} \frac{T(t)f - f}{t}$$

exists. The operator $A : D(A) \to V$ is called the *infinitesimal generator* of the semigroup $\{T(t)\}$ (see [LM]).

Now let

$$V = \{f \in C([0,\pi]) : f(0) = f(\pi) = 0\},$$
$$\|f\| = \sup\{|f(X)| : x \in [0,\pi]\} \quad \text{for } f \in V.$$

Define $A_1 : D(A_1) \to V$ by

$$A_1 f = \alpha_1 f'' + P_0 f + \alpha_2 f \quad \text{for } f \in D(A_1) = \{f \in V : f'' \in V\},$$

where P_0 is a nonnegative operator on V and $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 > 0$.

THEOREM 5. *The operator A_1 generates an exponentially stationary semigroup.*

Proof. It is sufficient to show that the operator

$$A_2 = \alpha f'' + Pf \quad \text{for } f \in D(A_2) = D(A_1),$$

where $P = P_0/\|P_0\|$, $\alpha = \alpha_1/\|P_0\|$, generates an exponentially stationary semigroup $\{T(t)\}$. The operator

$$Af = \alpha f'' \quad \text{for } f \in D(A) = D(A_1)$$

generates an analytic semigroup $\{T_0(t)\}$ of nonnegative operators on V (see [St, Theorem 5]). It is well known that

$$p_n(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \tag{6}$$

$$T_0(t)f(x) = \sum_{n=1}^{\infty} p_n(f)e^{-\alpha t n^2}\sin nx \tag{7}$$

for $f \in V$ and $t \in (0,\infty)$. Hence the operator A_2 generates an analytic semigroup $\{T(t)\}$ (see [H]) and

$$T_{n+1}(t)f = \int_0^t T_0(t-s)PT_n(s)f\,ds, \tag{8}$$

$$T(t)f = \sum_{n=0}^{\infty} T_n(t)f \tag{9}$$

for $f \in V$ and $n \ge 0$ (see [LM]). It is easy to show that

$$|\sin nx| \le n\sin x \quad \text{for } x \in [0,\pi], \tag{10}$$

$$|p_n(f)| \le 2\|f\| \quad \text{for } f \in V. \tag{11}$$

If $f \in V$ and $f \ge 0$ then from (10) it follows that

$$|p_n(f)| \le np_1(f). \tag{12}$$

LEMMA 2. *There exist $t_1, \beta_1 > 0$ such that for $f \in V^+$ and $n \ge n_0(f)$ we have*

$$T(nt_1)f(x)/\|T(nt_1)f\| \ge h(x), \quad \textit{where} \quad h(x) = \beta_1 \sin x,\ x \in [0,\pi].$$

Proof. There exists a constant δ such that

$$\|T_n(t)\| \le e^{-\alpha t}\delta^{n+1}t^n/n! \quad \text{for } n \ge 0,\ t \in [0,\infty), \tag{13}$$

$$\|T(t)\| \le \delta e^{\delta t} \quad \text{for } t \in [0,\infty). \tag{14}$$

There exists $t_1 > 1$ such that

$$2e^{-3\alpha t}\sum_{n=2}^{\infty} n^2 e^{-\alpha t(n^2-4)} \le 1 \quad \text{for } t \ge t_1.$$

Let $f \in V_+$. According to (7), (10) and (12) we obtain

$$|T_0(t)f(x) - p_1(f)e^{-\alpha t}\sin x| \le \sum_{n=2}^{\infty} n^2 p_1(f) e^{-\alpha t n^2}\sin x \le 2^{-1}p_1(f)e^{-\alpha t}\sin x$$

for $t \ge t_1$ and $x \in [0,\pi]$. Since $T_n(t)$ is nonnegative we have

$$T(t)\, f \ge T_0(t)f \ge 2^{-1}p_1(f)e^{-\alpha t}\sin \tag{15}$$

for $t \ge t_1$. From (6) we obtain

$$p_1(T(t)f) \ge 2^{-1}p_1(f)e^{-\alpha t} \quad \text{for } t \ge t_1. \tag{16}$$

There exists a constant t_2 such that $0 < t_2 < t_1$ and

$$\mu := 2\delta e^{\alpha t_1 + \delta t_1}(1 - e^{-\delta t_2}) < 1. \tag{17}$$

Put $\beta_2 = \sum_{n=1}^{\infty} n e^{-\alpha t_2 n}$. From (7) and (12) we have

$$\|T_0(t)f\| \le \sum_{n=1}^{\infty} n p_1(f) e^{-\alpha t n} \le p_1(f)\beta_2 \quad \text{for } t \ge t_2.$$

According to (8) and (13) we obtain

$$\|T_{n+1}(t)f\| \le \|f\|\delta^{n+2}\, t_2^{n+1}((n+1)!)^{-1} + \int_{t_2}^{t} \delta\|T_n(s)f\|\, ds \quad \text{for } t \ge t_2,\ n \ge 1.$$

By an induction argument it is easy to verify that

$$\|T_n(t)f\| \le \|f\|\delta^{n+1}(t^n - (t-t_2)^n)(n!)^{-1} + p_1(f)\beta_2\delta^n t^n (n!)^{-1} \quad \text{for } t \ge t_2.$$

Hence from (9),

$$\|T(t)f\| \le \|f\|\delta e^{\delta t}(1 - e^{-\delta t_2}) + p_1(f)\beta_2 e^{\delta t}$$

for $t \ge t_2$ and according to (16) and (17) we have

$$\frac{\|T(t_1)f\|}{p_1(T(t_1)f)} \le \mu\,\frac{\|f\|}{p_1(f)} + \eta,$$

where $\eta = 2\beta_2 e^{\alpha t_1 + \delta t_1}$. It follows by induction that there exists a number $n_1(f)$ such that

$$\|T(nt_1)f\|/p_1(T(nt_1)f) \le 2\eta/(1-\mu) \quad \text{for } n \ge n_1(f).$$

From (15) and (14) we have

$$\frac{T(nt_1+t_1)f(x)}{\|T(nt_1+t_1)f\|} \ge \frac{2^{-1}p_1(T(nt_1)f)e^{-\alpha t_1}\sin x}{\delta e^{\delta t_1}\|T(nt_1)f\|}$$

for $n \ge 1$. The last two inequalities finish the proof. ■

LEMMA 3. *If* $g \in D(A_2)$, *then*

(a) $\|g'\| \le \alpha^{-1}\pi(\|A_2 g\| + \|g\|)$,

(b) $g \le \beta_4(\|A_2 g\| + \|g\|)h$,

where

$$\beta_4 = \max\{\pi^2/3^{1/2}\alpha\beta_1,\ 2\pi/\alpha\beta_1\}$$

and $h(x) = \beta_1 \sin x$, $x \in [0,\pi]$.

Proof. (a) There exists $x_0 \in [0,\pi]$ such that $g'(x_0) = 0$. From the Lagrange theorem it follows that

$$|g'(x) - g'(x_0)| \le \|g''\|\,|x - x_0| \le \pi\|g''\|$$

for $x \in [0,\pi]$ and consequently $\|g'\| \le \pi\|g''\|$. By the definition of A_2 and P this completes the proof.

(b) For $y \in [0,\pi/3]$ we have

$$\begin{aligned} g(y) &= \int_0^y g'(x)\,dx \le 2\,\|g'\| \int_0^y \cos x\,dx \\ &\le 2\|g'\|\sin y \le 2\alpha^{-1}\pi(\|A_2 g\| + \|g\|)\sin y. \end{aligned}$$

The last formula is also true for $y \in [2\pi/3,\pi]$. Let $y \in [\pi/3, 2\pi/3]$. Since $1 \le 2\cdot 3^{-1/2}\sin y$, from the Lagrange theorem and (a) we have

$$g(y) \le \frac{\pi}{2}\,\|g'\| \le 3^{-1/2}\alpha^{-1}\pi^2(\|A_2 g\| + \|g\|)\sin y,$$

and this finishes the proof. ■

Proof of Theorem 5. Let $f \in V_+$. For $t > 0$ we have $T(t)f \in D_{A_2}$ (see [H]) and according to Lemma 3(b),

$$T(nt_1)f \le \beta_4(\|A_2 T(nt_1)f\| + \|T(nt_1)f\|)h \quad \text{for } n \ge 1. \tag{18}$$

From Lemma 1 there exists a number m such that

$$h \le T(mt_1)h/\|T(mt_1)h\|.$$

Hence there exists a constant $\beta_5(f)$ such that

$$T(mt_1)f \le \beta_5(f)T(mt_1)h$$

and consequently

$$\|T(nt_1)f\| \le \beta_5(f)\|T(nt_1)h\| \quad \text{for } n \ge m. \tag{19}$$

Since $A_2T(nt_1)h = T(nt_1)A_2h$ (see [H]),

$$\|A_2T(nt_1)h\| \le \beta_5(A_2h)\|T(nt_1)h\| \quad \text{for } n \ge m. \tag{20}$$

From (18),

$$T(nt_1)h \le \beta_4(\beta_5(A_2h)+1)\|T(nt_1)h\|h. \tag{21}$$

By the Ascoli theorem, (20) and Lemma 2(a) it follows that the sequence $\{T(nt_1)h/\|T(nt_1)h\|\}$ contains a convergent subsequence. Consequently, from Theorem 1, Lemma 1 and (19) the semigroup $\{T(t)\}$ is exponentially stationary. ∎

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