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On locally biholomorphic surjective mappings

by Ewa Ligocka (Warszawa)

Abstract. We prove that each open Riemann surface can be locally biholomorphically (locally univalently) mapped onto the whole complex plane. We also study finite-to-one locally biholomorphic mappings onto the unit disc. Finally, we investigate surjective biholomorphic mappings from Cartesian products of domains.

0. Introduction. The motivation for writing this note came from the book [6]. We find there the Gunning–Narasimhan theorem which says that each open Riemann surface can be expressed as a Riemann domain over $\mathbb{C}$. Moreover, the whole book [6] is devoted to the study of holomorphic functions on Riemann domains over $\mathbb{C}^n$.

If X is a complex connected manifold and f is a locally biholomorphic map from X onto $\mathbb{C}^n$, then the pair (X, f) forms a *Riemann domain* over $\mathbb{C}^n$.

The natural question arises: What can we say about the image of X under a locally biholomorphic map $f : X \to \mathbb{C}^n$? In particular, for which X can we find such an f with $f(X) = \mathbb{C}^n$?

The main result of the present note is that for every open Riemann surface there exists a locally biholomorphic mapping from X *onto* $\mathbb{C}$. This implies that every Cartesian product of n such surfaces can be thought of as a Riemann domain over the whole $\mathbb{C}^n$. We do not know if this is true for every Riemann–Stein domain over $\mathbb{C}^n$.

In the rest of this paper we study locally biholomorphic mappings with finite fibers. We give geometric conditions on a domain $D \subset \mathbb{C}$ sufficient for the existence of a locally biholomorphic mapping with finite fibers from D *onto* the unit disc. The Fornæss–Stout theorem [1] implies that such a domain can be mapped locally biholomorphically onto each Riemann surface by a mapping with finite fibers.

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We shall use standard notations. The symbol $\langle a,b\rangle$ stands for the closed interval with ends a and b. We deal with domains in $\mathbb{C}^n$ or n-dimensional complex manifolds.

1. The case of $n=1$. We start with the following

THEOREM 1. *For every entire function* $g\neq$ const *and every domain* $D\subset\mathbb{C}$ *there exists a homography* h *such that the function*

$$F(z)=(\exp(g\circ h)-1)\exp(\exp(g\circ h))$$

maps D *onto* $\mathbb{C}$. *If* $g'(z)\neq 0$ *on* $\mathbb{C}$ *then* $F'(z)\neq 0$ *on* D *and thus* F *is a local biholomorphism.*

Proof. The function $\varphi(z)=(z-1)e^z$ maps $\mathbb{C}\setminus\{0\}$ *onto* $\mathbb{C}$ by the Picard theorem. The derivative $\varphi'(z)=ze^z$ vanishes only at zero. The function e^g has an essential singularity at ∞. Hence $e^{g(1/z)}$ has an essential singularity at zero. The Julia theorem ([7], see also [3, Ch. 2, §7, Th. 5]) implies that there exists a halfline l with end at zero such that for every open neighborhood U of zero and every open angle A with bisectrix l the function $e^{g(1/x)}$ takes every value from $\mathbb{C}\setminus\{0\}$ on $U\cap A$. Let $l^{\perp}$ denote the line through zero perpendicular to l, and let H be the half-plane with $\partial H=l^{\perp}$ and $l\subset H$.

Now let D be a domain in $\mathbb{C}$. If $D=\mathbb{C}$ then it suffices to put $h(z)=z$. The function

$$F=\varphi\circ e^g=(\exp g-1)\exp(\exp g)$$

is as required.

If $D\neq\mathbb{C}$ then we can take $a\in D$ and $b\in\partial D$ such that

$$\varrho=\operatorname{dist}(a,\partial D)=|a-b|.$$

Let Δ denote the open disc with center a and radius ϱ. We can find a homography h_1 which maps Δ onto H such that $h_1(b)=0$ (note that each conformal map from a disc onto a half-plane must be a homography). Let $h=1/h_1$. Then $e^{g\circ h}$ maps D onto $\mathbb{C}\setminus\{0\}$ and hence

$$F=\varphi\circ e^{g\circ h}=(\exp(g\circ h)-1)\exp(\exp(g\circ h))$$

maps D onto $\mathbb{C}$. We have

$$F'(z)=\varphi'(\exp(g\circ h)(z))\cdot\exp(g\circ h)(z)\cdot g'(h(z))\cdot h'(z)$$

and $F'(z)=0$ if $g'(h(z))=0$.

COROLLARY 1. *For each domain* $D\subset\mathbb{C}$ *there exists a homography* h *such that*

$$F_h(z)=(\exp h(z)-1)\exp(\exp h(z))$$

maps D *onto* $\mathbb{C}$ *and* $F'(z)\neq 0$ *for* $z\in D$.

Proof. Put $g(z)=z$ in Theorem 1.

Corollary 2. *Let X be a connected complex manifold and let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function. There exists a homography h such that the mapping $F_h \circ f$ maps X onto $\mathbb{C}$.*

Proof. Apply Corollary 1 to the domain $D = f(X)$.

The term "open Riemann surface" will stand for "one-dimensional connected, noncompact complex manifold".

Our Corollary 2 yields in particular the following

Theorem 2. *For each open Riemann surface X there exists a locally biholomorphic map p from X onto $\mathbb{C}$.*

In other words, X can be represented as a Riemann domain *over the whole plane* $\mathbb{C}$.

Proof. By the Gunning–Narasimhan theorem ([4], see also [6, Ch. 1, §1.11]) there exists a locally biholomorphic mapping φ from X into $\mathbb{C}$. We can now apply Corollary 2.

The mappings constructed in Theorems 1 and 2 were all infinite-to-one. Thus a natural question arises: Does there exist a finite-to-one locally biholomorphic mapping from X onto $\mathbb{C}$?

We have the following

Proposition 1. *Let $D = \mathbb{C} \setminus \{0\}$ and let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a locally biholomorphic map onto $\mathbb{C}$. Then for all $w \in \mathbb{C}$ except at most one, the set $f^{-1}(w)$ is infinite.*

To prove this we shall need the following

Lemma 1. *If R is a rational function which maps $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C}$, then there exists $z_0 \in \mathbb{C} \setminus \{0\}$ such that $R'(z_0) = 0$.*

Proof of Lemma 1. The derivative $R'(z)$ is rational and holomorphic on $\mathbb{C} \setminus \{0\}$. The Laurent series of $R'(z)$ at zero has the form

$$R'(z) = \sum_{k=0}^{N} a_k z^k + \sum_{k=1}^{M} a_{-k} z^{-k}.$$

If $R'(z) \neq 0$ on $\mathbb{C} \setminus \{0\}$ then R' must be zero either at 0 or at ∞. In the first case

$$R'(z) = a_N z^N, \qquad R(z) = \frac{a_N z^{N+1}}{N+1} + c$$

and therefore $R(z) \neq c$ for $z \in \mathbb{C} \setminus \{0\}$. In the second case

$$R'(z) = a_{-M} z^{-M}, \qquad M \neq 1, \qquad R(z) = \frac{a_{-M} z^{-M+1}}{-M+1} + c$$

and again $R(z) \neq c$ on $\mathbb{C} \setminus \{0\}$. In both cases we have obtained a contradiction.

Proof of Proposition 1. If there exist two different values w_1 and w_2 such that the sets $\{f^{-1}(w_1)\}$ and $\{f^{-1}(w_2)\}$ are finite then f has an essential singularity neither at 0 nor at ∞ (Picard theorem). Thus f must be rational, which contradicts Lemma 1.

Definition 1. Let $f : X \xrightarrow{\text{onto}} Y$. We shall say that f is *m-valent*, $m \in \mathbb{N}$, if for each $y \in Y$ the set $\{f^{-1}(g)\}$ has no more than m elements.

It turns out that the case of $D = \mathbb{C} \setminus \{0\}$ is in some sense exceptional:

Theorem 3. *Let D be a finitely connected domain in $\mathbb{C}$, not biholomorphic to $\mathbb{C} \setminus \{0\}$. Then there exist $m \in \mathbb{N}$ and an m-valent locally biholomorphic mapping f from D onto $\mathbb{C}$.*

Proof. We shall consider several cases:

(a) $D = \mathbb{C} \setminus \{a_1, \ldots, a_N\}$, $a_i \neq a_j$, $i \neq j$, $N \geq 2$. Let $M \in \mathbb{N}$, $M \geq N$, be chosen such that

$$\frac{M+1}{M+2} \cdot \frac{a_i - a_1}{a_2 - a_1} \neq 1 \quad \text{for } 2 \leq i \leq N.$$

Let $\varphi(z) = z^{M+1}(z-1)$ and let h be a linear mapping sending a_1 to 0 and a_2 to $\frac{M+1}{M+2}$. Put $f = \varphi \circ h$. We have $f'(z) \neq 0$ on D. Moreover $1 \in h(D)$ and $f(h^{-1}(1)) = 0$. Any other value $w \neq 0$ in $\mathbb{C}$ is taken by φ at $\geq M+1$ different points. Hence f maps D *onto* $\mathbb{C}$. The mapping f is $(M+2)$-valent.

(b) $D = B(0,1)$, the unit disc. We can construct f as the superposition $f = f_4 \circ f_3 \circ f_2 \circ f_1$, where $f_1 = h^3$, h is a biholomorphic map of $B(0,1)$ onto the upper halfdisc and

$$f_2(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad f_3(z) = \frac{z}{\sqrt{3}}, \qquad f_4(z) = z^3 - z.$$

Note that f_1 maps $B(0,1)$ onto $B(0,1) \setminus \{0\}$, $f_3 \circ f_2 \circ f_1$ maps $B(0,1)$ onto $\mathbb{C} \setminus \langle -1/\sqrt{3}, 1/\sqrt{3} \rangle$ and $f_4(z) = z^3 - z$ maps $\mathbb{C} \setminus \langle -1/\sqrt{3}, 1/\sqrt{3} \rangle$ onto $\mathbb{C}$. It is easy to check that $f'(z) \neq 0$ on D. Thus by the Riemann theorem we have already proved the assertion of Theorem 3 for all simply connected domains.

(c) If D is a k-connected domain, $k \geq 2$, and D is not biholomorphic to any of the domains from (a) then by the Koebe theorem (see [3, Ch. 5, §6, Th. 2 and remarks at the end of §6]), D is biholomorphic to the annulus $A = \{z : 0 \leq r < |z| < 1\}$ with $k-2$ closed disc or points removed. There exists an open angle α, $|\alpha| = 2\pi k/q$, $k/q \in \mathbb{Q}$, such that $\alpha \cap A = \alpha \cap \{\text{biholomorphic image of } D\}$. Hence $g(z) = z^{q+1}$ maps the biholomorphic image of D onto the whole annulus A. Thus we have obtained a locally biholomorphic $(q+1)$-valent mapping D onto A.

It now suffices to find $m \in \mathbb{N}$ and an m-valent locally biholomorphic mapping A onto $\mathbb{C}$. If $r = 0$ then $A = B(0,1) \setminus \{0\}$ and we can take the superposition $f_4 \circ f_3 \circ f_2$, where f_i, $i = 2, 3, 4$, are the mappings from item (b). If $r > 0$ we can take the mapping $\varphi_r(z) = \frac{z-r}{1-rz}$, use the function φ_r^3 which maps A onto $B(0,1) \setminus \{0\}$, and proceed as before.

If our domain D is biholomorphic to a bounded domain we can ask whether there exists an m-valent locally biholomorphic mapping f from D onto the unit disc $B(0,1)$.

We start with the following

PROPOSITION 2. *Let D be a domain contained in $B(0,1)$. Assume that there exist $z_0 \in \partial B(0,1)$ and $r > 0$ such that*

$$D \cap B(z_0, r) = B(0,1) \cap B(z_0, r).$$

Then there exists a locally biholomorphic map f from D onto $B(0,1)$ which is m-valent with $m \le 24$.

Proof. By the Riemann theorem we can find a biholomorphic mapping h from $B(0,1)$ onto the strip

$$\left\{z : \operatorname{Re} z < 0,\ |\operatorname{Im} z| < \tfrac{3}{2}\pi\right\}$$

which extends to $\overline{B(0,1)}$ in such a way that the arc

$$B(z_0, r) \cap \partial B(0,1)$$

is mapped onto

$$\left\langle -a - \tfrac{3}{2}\pi i, -\tfrac{3}{2}\pi i\right\rangle \cup \left\langle -\tfrac{3}{2}\pi i, \tfrac{3}{2}\pi i\right\rangle \cup \left\langle \tfrac{3}{2}\pi i, \tfrac{3}{2}\pi i - a\right\rangle, \qquad a > 0.$$

Thus the set $h(B(0,1) \cap B(z_0, r))$ contains a rectangle R with vertices $\left(-b - \frac{3}{2}\pi i, -\frac{3}{2}\pi i, \frac{3}{2}\pi i, \frac{3}{2}\pi i - b\right)$ for some b with $0 < b < a$. By assumption we have $R \subset h(D)$. Hence $g = e^h$ is a 2-valent locally biholomorphic mapping from D into $B(0,1)$ such that $B(0,1) \setminus g(D) \subset B(0, e^{-b})$. Let $r = \max(|z| : z \in B(0,1) \setminus g(D))$ and let $z_0 \in B(0,1) \setminus g(D)$ be such that $|z_0| = r$. Put

$$\varphi = \frac{z - z_0}{1 - \overline{z}_0 z}.$$

Then φ^3 is a 3-valent locally biholomorphic mapping from $g(D)$ onto $B(0,1) \setminus \{0\}$ (cf. (c) of the proof of Theorem 3).

Let ψ be a biholomorphic (conformal) map from $B(0,1) \setminus \{0\}$ onto $\Omega_l \setminus \{q\}$, where Ω_l is the rectangle with vertices

$$\left(-l + \tfrac{5}{2}\pi i, \tfrac{5}{2}\pi i, -l - \tfrac{1}{2}\pi i, -\tfrac{1}{2}\pi i\right), \qquad l > 0,\ q \in \mathbb{R} \cap \Omega_l.$$

The map e^ψ sends $B(0,1) \setminus \{0\}$ onto the annulus

$$A_l = \{z : 1/e^l < |z| < 1\}.$$

It is a 2-valent locally biholomorphic map.

Let B_c, $0 < c < 1$, denote the domain

$$B(0,1) \setminus \langle 0, c^{-1}(1-\sqrt{1-c^2})\rangle.$$

We can find l and c such that the domains A_l and B_c are biholomorphic (conformally equivalent).

Let

$$\Phi_c(z) = z \cdot \frac{z-c}{1-cz}.$$

It maps $B(0,1)$ onto $B(0,1)$ because it is a Blaschke product. The derivative Φ_c' vanishes only at

$$z = (1-\sqrt{1-c^2})c^{-1}.$$

We also have $\Phi_c(1) = \Phi_c(-1) = 1$. Hence Φ_c is a 2-valent locally biholomorphic map from B_c *onto* the unit disc.

Let D be a domain in $\mathbb{C}$. We shall say that a closed connected set $K \subset \widehat{\mathbb{C}} \setminus D$ is an *isolated component* of $\widehat{\mathbb{C}} \setminus D$ if there exists an open set U such that $K \subset U$ and $U \setminus K \subset D$.

As a consequence of Proposition 2 we get

Theorem 4. *Let $D \subset \mathbb{C}$ be a domain such that $\widehat{\mathbb{C}} \setminus D$ contains an isolated component K not equal to a single point. Then there exists an m-valent locally biholomorphic mapping f from D onto the unit disc such that $m \le 24$.*

Proof. By the Riemann theorem there exists a biholomorphic map h from $\widehat{\mathbb{C}} \setminus K$ onto $B(0,1)$. Since K is a component of $\widehat{\mathbb{C}} \setminus K$, the set $h((\widehat{\mathbb{C}} \setminus D) \setminus K)$ is a compact subset of $B(0,1)$. Hence $h(D)$ satisfies the assumptions of Proposition 2.

Corollary 3. *For every finitely connected domain D not equal to*

$$\mathbb{C} \setminus \{a_1, \ldots, a_M\}, \qquad M \in \mathbb{N},$$

there exists an m-valent locally biholomorphic map $f : D \xrightarrow{\text{onto}} B(0,1)$ such that $m \le 24$.

Corollary 4. *Theorem 4 is valid for every "Swiss cheese" domain.*

We can also prove the following

Theorem 5. *Let $D \subset \mathbb{C}$ be a domain such that $\widehat{\mathbb{C}} \setminus D$ contains a continuum K not equal to a single point, for which the boundary of $\widehat{\mathbb{C}} \setminus K$ is locally connected, and $\widehat{\mathbb{C}} \setminus K$ is connected. Put $K_1 = \partial(\widehat{\mathbb{C}} \setminus K)$ and assume that $K_1 \cap \overline{(\widehat{\mathbb{C}} \setminus D) \setminus K} \ne K_1$. Then there exists an m-valent locally biholomorphic mapping f from D onto the unit disc such that $m \le 24$.*

Proof. Let g be a biholomorphic map from $B(0,1)$ onto $\widehat{\mathbb{C}} \setminus K$. By the Carathéodory theorem, g extends to a continuous map $\widetilde{g}$ from $\overline{B(0,1)}$ onto $\overline{\widehat{\mathbb{C}} \setminus K} = (\widehat{\mathbb{C}} \setminus K) \cup K_1$. By assumption there exists $a \in K_1$ and an open neighborhood U of a (in $\widehat{\mathbb{C}}$) such that

$$U \cap [(\widehat{\mathbb{C}} \setminus D) \setminus K] = \emptyset.$$

Hence $\widetilde{g}^{-1}(U)$ is an open set in $\overline{B(0,1)}$ containing boundary points of $B(0,1)$. We have $g^{-1}(U) \subset g^{-1}(D)$. Thus the assumptions of Proposition 2 are satisfied for the domain $g^{-1}(D)$.

Theorems 4 and 5 are very interesting in conjuction with the deep result due to Fornæss and Stout [1]: For every n-dimensional paracompact complex manifold X there exists an m-valent locally biholomorphic mapping from an n-dimensional polydisc onto X such that $m \le 4^n(2n+1)+2$. This implies the following

THEOREM 6. *If a domain D satisfies the assumptions of either Theorem 4 or Theorem 5 then for each connected Riemann surface X (compact or open) there exist $m \in \mathbb{N}$ and an m-valent locally biholomorphic mapping f from D onto X for which $m \le 12 \cdot 2 \cdot 14 = 336$.*

The following problem seems to be difficult.

PROBLEM 1. Are the conclusions of Theorems 4, 5 and 6 valid for every bounded domain D?

2. The case of $n > 1$. In this part we state some theorems concerning Cartesian products of one-dimensional domains or manifolds.

Our Theorem 2 yields immediately

THEOREM 7. *Let X be an n-dimensional complex manifold biholomorphic to a Cartesian product of n open Riemann surfaces. Then there exists a locally biholomorphic mapping from X onto $\mathbb{C}^n$. Hence X can be represented as a Riemann domain over the whole $\mathbb{C}^n$.*

A natural question arises: Which Riemann domains (X,p) admit locally biholomorphic mappings from X *onto* $\mathbb{C}^n$?

It is easy to see that if $f : X \to \mathbb{C}^n$ is a locally biholomorphic map onto $\mathbb{C}^n$ then also its extension $\widetilde{f} : \widetilde{X} \to \mathbb{C}^n$ to the envelope of holomorphy $(\widetilde{X}, \widetilde{p})$ of (X,p) is a locally biholomorphic mapping onto $\mathbb{C}^n$.

Hence a correct statement of our problem is the following:

PROBLEM 2. For which Riemann–Stein domains (X,p) over $\mathbb{C}^n$ does there exist a locally biholomorphic mapping $f : X \to \mathbb{C}^n$?

We conjecture that the answer may be positive for every such (X,p). Our conjecture is motivated by:

PROPOSITION 3. *Let (X, p) be a Riemann–Stein domain over $\mathbb{C}^n$. Then for each compact subset $K \subset X$ there exists an almost proper mapping $f : X \to \mathbb{C}^n$ such that f is locally biholomorphic on an open neighborhood of K.*

Note that an almost proper mapping must be surjective.

Proof. This follows from Theorem VIIC 2 of [5], which says that the almost proper mappings from X onto $\mathbb{C}^n$ are dense in $(H(X))^n$ in the compact-open topology because X is Stein. Since X is a Riemann domain there exists a locally biholomorphic mapping from X into $\mathbb{C}^n$.

There exists a sequence of almost proper mappings f_k which tends to p almost uniformly (uniformly on compact sets). Hence each point $x \in X$ has a neighborhood U_x such that the Jacobian of $f_k \circ p^{-1}$ tends to the Jacobian of $p \circ p^{-1}$ (which is equal to one) uniformly on $p(U_x)$. Thus for every compact $K \subset X$ there exists k_0 and an open neighborhood U of K such that $\operatorname{rank} f_k = n$ on U for every $k > k_0$.

It should be mentioned here that Fornæss and Stout proved in [2] that the unit ball in $\mathbb{C}^n$ can be mapped by an $m(n)$-valent locally biholomorphic mapping onto each n-dimensional connected, paracompact complex manifold.

In particular there exists an $m(n)$-valent locally biholomorphic mapping from the unit ball onto $\mathbb{C}^n$.

Our Theorems 4, 5 and 6 imply

THEOREM 8. *Let $X = D_1 \times \ldots \times D_n$ where D_i, $i = 1, \ldots, n$, satisfy the assumptions of either Theorem 4 or Theorem 5. Then for each connected, paracompact n-dimensional complex manifold Y there exists an m-valent and locally biholomorphic mapping f from X onto Y such that $m \leq (24)^n[(2n+1)4^n + 2]$.*

Proof. Theorems 4 and 5 imply that there exists such an f if Y is the unit polydisc in $\mathbb{C}^n$. Now the result of Fornæss and Stout [1] implies the assertion.

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Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: elig@mimuw.edu.pl

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