ANNALES
POLONICI MATHEMATICI
107.1 (2013)

# On Noether and strict stability, Hilbert exponent, and relative Nullstellensatz

by Chia-chi Tung (Mankato, MN)

**Abstract.** Conditions characterizing the membership of the ideal of a subvariety $\mathfrak{S}$ arising from (effective) divisors in a product complex space $Y \times X$ are given. For the algebra $\mathcal{O}_Y[V]$ of relative regular functions on an algebraic variety $V$, the strict stability is proved, in the case where $Y$ is a normal space, and the Noether stability is established under a weakened condition. As a consequence (for both general and complete intersections) a global Nullstellensatz is derived for divisors in $Y \times \mathbb{C}^N$, respectively, $Y \times \mathbb{P}^N(\mathbb{C})$. Also obtained are a principal ideal theorem for relative divisors, a generalization of the Gauss decomposition rule, and characterizations of solid pseudospherical harmonics on a semi-Riemann domain. A result towards a more general case is as follows: Let $\mathfrak{D}_j$, $1 \le j \le p$, be principal divisors in $X$ associated to the components of a $q$-weakly normal map $g = (g_1, \dots, g_p) : X \to \mathbb{C}^p$, and $S := \bigcap \mathfrak{S}_{|\mathfrak{D}_j|}$. Then for any proper slicing $(\varphi, g, D)$ of $\{\mathfrak{D}_j\}_{1 \le j \le p}$ (where $D \subset X$ is a relatively compact open subset), there exists an explicitly determined Hilbert exponent $\mathfrak{h}_{\mathfrak{D}_1 \cdots \mathfrak{D}_p, D}$ for the ideal of the subvariety $\mathfrak{S} = Y \times (S \cap D)$.

**1. Introduction.** Determination of the membership of an ideal generated by polynomials over an algebraically closed field is a basic problem in polynomial ideal theory. A result of O. Forster [Fo] in this direction can be stated as follows: if $\mathfrak{A}$ is a closed primary ideal in a Stein algebra $\Gamma(X, \mathcal{O}_X)$ (as a Fréchet space), then the ideal $\mathcal{I}(S)$ of all holomorphic functions vanishing on the subvariety $S = \mathcal{V}(\mathfrak{A})$ is precisely the radical ideal $\sqrt{\mathfrak{A}}$; moreover, there exists an integer $\mathfrak{h} \ge 0$ such that $(\sqrt{\mathfrak{A}})^{\mathfrak{h}} \subseteq \mathfrak{A}$. F. Lorenz [Lor, p. 281] remarked that "actually finding such an $\mathfrak{h}$ for a specified $\mathfrak{A}$, described say by a set of generators, is a different matter". The well-known Hilbert Nullstellensatz ([W, p. 59]), Forster's results, and Max Noether's criterion for the individual membership of a polynomial ideal are seemingly loosely related. And all these results share some common ground with the (local) Rückert Nullstellensatz ([GR$_3$, p. 82]). The present work is motivated in part by a wish to bring out (to some extent) the possible connections between them.

---

2010 *Mathematics Subject Classification*: Primary 32C15; Secondary 14C17, 32C25.
*Key words and phrases*: strict stability, Noether stability, Hilbert number, Hilbert exponent, mapping multiplicity, intersection number.

DOI: 10.4064/ap107-1-1  

As it gains in usefulness to allow subvarieties (arising from different situations) to depend on a parameter, an attempt is made in this work to find conditions under which the overall and individual membership of the ideal $\mathcal{I}(\mathfrak{S})$ can be determined for a subvariety $\mathfrak{S}$ lying in a product space $Y \times X$, and a corresponding Hilbert exponent $\mathfrak{h}$ explicitly described or estimated.

Let $V$ be a positive-dimensional affine algebraic variety in $\mathbb{C}^N$ and $Y$ a complex space. There are two useful properties concerning the complex algebra $\mathcal{O}_Y[V]$ of all $Y$-regular functions on $V$ (see §2). The first (Theorem 2.1), valid in the case where $Y$ is a normal space, concerns the *strict stability* of the subalgebra $\mathcal{B} = \mathcal{O}_Y[V]$ in $\mathcal{O}(Y \times \mathbb{C}^N)$, meaning that, if $f \in \mathcal{B} \setminus \{0\}$ and $\psi \in \mathcal{O}(Y \times \mathbb{C}^N) \setminus \{0\}$, then both $f/\psi$ (defined in terms of local extensions of $f$) and $\psi$ are equivalent to some elements of $\mathcal{B}$ whenever $f/\psi \in \mathcal{O}(Y \times \mathbb{C}^N)$. The second property (Theorem 2.2), a generalization of the well-known Noether criterion, is the assertion that for any reduced complex space $Y$, the subalgebra $\mathcal{O}_Y[V]$ of $\mathcal{O}(Y \times V)$ is (relatively) *Noether-stable*, meaning that, if $f$ and $g_j$, $1 \leq j \leq q$, are elements of $\mathcal{O}_Y[V] \setminus \{0\}$ such that the Noether condition

$$f_w \equiv 0 \ (\langle g_{1,w}, \ldots, g_{q,w} \rangle) \tag{1.1}$$

holds in $\mathcal{O}_{Y \times V, w}$ at every point $w$ of the subvariety $\mathfrak{S} := \mathcal{V}(g_1, \ldots, g_q)$ with $\pi(w)$ lying off some thin subset of codimension $\geq 2$ in $\mathbb{C}^N$ (where $\pi : Y \times \mathbb{C}^N \to \mathbb{C}^N$ denotes the projection), then the Noether relation

$$f \equiv 0 \ (\langle g_1, \ldots, g_q \rangle_Y) \tag{1.2}$$

is valid in the ring $\mathcal{O}_Y[V]$. This result, which extends the Płoski–Tworzewski Theorem [PT, Proposition 2.1] to the relative case, is proved by refining the argument of Tworzewski [Tw, p. 2] for the Max Noether Theorem. In consequence a global Nullstellensatz for divisors in a relative affine, respectively, projective, variety (Theorems 2.3 and 4.2) can be deduced. The proof naturally involves the local notion of Hilbert number (to be defined below).

Moreover, deeper study of subvarieties in affine or projective spaces necessitates the consideration of a relative semiglobal Nullstellensatz; this is treated in §3 (and summarized below). Especially, it will be shown (in Theorem 3.3) that, for a complete intersection $S$ of divisors on an affine algebraic variety $V$, there is an intrinsically determined Hilbert exponent $\mathfrak{h}_S$ ((3.9)) for the ideal of all $Y$-regular functions $f : Y \times V \to \mathbb{C}$ vanishing on the product subvariety $\mathfrak{S} = Y \times S$ (compare the results of Płoski–Tworzewski [PT, Theorem 3.1 and Proposition 2.2]). If $\mathfrak{F}$ is a complete intersection of divisors in a projective space $\mathbb{P}^N(\mathbb{C})$ with defining system $\mathcal{F}$ (consisting of homogeneous polynomials $f_j$), the following result is obtained (Theorem 4.3): Assume that the set $\mathfrak{F} \cap \{z = [z_0, \ldots, z_N] \mid z_k = 0\}$ is thin in $\mathfrak{F}$ for each $k \in \mathbb{Z}[0, N]$. Then there exists a Hilbert exponent $\mathfrak{h}_{\{\mathcal{F}\}}$ (defined by (4.3)–(4.4)) for all elements in $\mathcal{P}ol_{Y,N+1}(d) \cap \mathfrak{I}_Y(Y \times \mathfrak{F})$ (in the sense of (4.5)); moreover, if either

$\langle f_1, \ldots, f_q \rangle_Y$ is a prime ideal or the system $\mathcal{F}$ defines $\mathfrak{F}$ minimally, then $\mathfrak{I}_Y(Y \times \mathfrak{F}) = \langle f_1, \ldots, f_q \rangle_Y$. As applications, a generalization of the Gauss decomposition rule and characterizations of solid pseudospherical harmonics on a semi-Riemann domain are given in §5.

Turning next to local and semiglobal situations, let $X$, $Y$ denote (unless otherwise mentioned) reduced, pure-dimensional complex spaces with a countable base of open sets, and $D \Subset X$ a (relatively compact) open subset. Let $S$ be a subvariety of $X$ and set $S_D := S \cap D$. If $S = \mathcal{V}(g_1, \ldots, g_p) := \{g_1 = \ldots = g_p = 0\}$ with $g_j \in \mathcal{O}(X) := \Gamma(X, \mathcal{O}_X)$, the (complex) algebra of holomorphic functions on $X$, then denote by $\mathcal{I}_a(S_a)$ the ideal of germs of holomorphic functions vanishing on the germ of $S$ at a point $a$, and $\langle g_{1,a}, \ldots, g_{p,a} \rangle$ the ideal generated by the $g_{j,a}$'s in $\mathcal{O}_{X,a}$. If $f_a \in \mathcal{I}_a(S_a)$, then by the Rückert Nullstellensatz the Hilbert relation

$$f_a^r \equiv 0 \ (\langle g_{1,a}, \ldots, g_{p,a} \rangle) \tag{1.3}$$

holds in $\mathcal{O}_{X,a}$ for some integer $r \geq 0$. The integer $r$ can actually be chosen to depend only on the ideal generated by the $g_{j,a}$, $1 \leq j \leq p$. Therefore, following Forster [Fo, p. 325], define the *Hilbert number*

$$\mathfrak{h}(g, a) := \min\{r \in \mathbb{Z}[0, \infty) \mid f_a^r \equiv 0 \ (\langle g_{1,a}, \ldots, g_{p,a} \rangle), \ \forall f_a \in \mathcal{I}_a(S_a)\}, \tag{1.4}$$

where $g = (g_1, \ldots, g_p) : X \to \mathbb{C}^p$. A continuous map $\pi : X \to N$ (between topological spaces) is said to be *light at* $a \in X$, if for some neighborhood $U$ of $a$, the fibers of $\pi\rfloor U$ consist of isolated points. If $g : U \to \mathbb{C}^m$ is holomorphic in an open set $U \subseteq \mathbb{C}^m$ and is light at $a \in U$, then (according to [GH, p. 669] or [Ts, p. 110]) the germ relation (1.3) holds with $r$ equal to the multiplicity of $g$ at $a$ (see (3.1)). Thus

$$\mathfrak{h}(g, a) \leq \nu_g^0(a). \tag{1.5}$$

A system $\mathfrak{F} = \{g_j\}_{1 \leq j \leq p}$ defining a subvariety $S$ is said to be *minimal* if $\mathfrak{h}(g, a) = 1$ for all $a \in S$ ([Ts, pp. 118–119]; cf. [Tu$_2$, p. 127]). For the convenience of the reader, the definition of the *multiplicity* "$\nu_g(a)$" at $a \in X$ of a light, respectively, pure fibering, holomorphic map $g$ is recalled at the beginning of §3 (with some basic properties summarized in the Appendix, §6).

The inequality (1.5) for the Hilbert number bespeaks the relevance of considering intersecting divisors arising from the component functions $g_j$. In the following every divisor $\mathfrak{D}$ mentioned is assumed to be effective, and its support is denoted by $\mathfrak{S}_{|\mathfrak{D}|}$. To give a result towards the general case, let $\mathfrak{D}_j$, $1 \leq j \leq p$, be principal divisors in $X$ and $S := \bigcap \mathfrak{S}_{|\mathfrak{D}_j|}$.

Definition 1.1. A *proper slicing of* $\{\mathfrak{D}_j\}$ is a triple $(\varphi, g, D)$ such that:

(i) $\varphi : D \to \mathbb{C}^q$ is holomorphic with $\varphi^{-1}(0) \cap S_D \neq \emptyset$;

(ii) $g = (g_1, \ldots, g_p)$, with each component giving a defining equation $g_j = 0$ for $\mathfrak{D}_j$;

(iii) there exist an open, connected neighborhood $Q$ of the origin $0 \in \mathbb{C}^q$ and a connected, pure $(m-q)$-dimensional locally irreducible analytic subset $N$ of an open subset in $\mathbb{C}^p$ such that the junction $(g, \varphi) : \tilde{D} := D \cap g^{-1}(N) \cap \varphi^{-1}(Q) \to N \times Q$ is proper and light.

Note that if $\pi = (\pi_1, \ldots, \pi_p) : X \to \mathbb{C}^p$ is a holomorphic map of pure rank $n = m - q > 0$ (see §3) with a locally irreducible (local) image at $a \in X$, then there exists a holomorphic map $\varphi : D \to \mathbb{C}^q$ defined in an open neighborhood $D \subseteq \mathbb{C}^m$ of $a$ such that $(\varphi, \pi, D)$ is a proper slicing of $\{(\pi_j)\}_{1 \le j \le p}$, $(\pi_j)$ being the divisor associated to $\pi_j$; this is shown in [Tu$_1$, Lemma 2.1.4]. To each proper slicing $(\varphi, g, D)$ of a system $\{\mathfrak{D}_j\}$ of principal divisors in $X$ is associated a positive integer

$$\mathfrak{h}_{\mathfrak{D}_1 \cdots \mathfrak{D}_p, D} := \sum_{z \in \tilde{D}} \nu_{g,\varphi}(z; w, t), \qquad \forall (w,t) \in N \times Q, \tag{1.6}$$

called the *slicing degree*, which is independent of $(w,t) \in N \times Q$ (see Property 6.1). Here $\nu_{g,\varphi}(z; w, t) := \nu_{(g,\varphi)}(z)$ denotes the multiplicity of the light mapping $(g, \varphi)$ at a point $z$ of the fiber over $(w,t)$ of $(g,\varphi)\rfloor\tilde{D}$, and is set to be zero at all other points. The slicing degree actually depends only on the divisors $\mathfrak{D}_j$ and the map $\varphi$ (as can be shown by invoking an extended version of the invariance of the multiplicity under an invertible holomorphic matrix transformation [St$_2$, Theorem 6.1]).

As a generalization of the notion of an analytic covering space, a holomorphic map $\pi : X \to \mathbb{C}^p$ is said to be *$q$-weakly normal* if the following conditions are met: $\pi$ has (i) pure rank $m - q > 0$ where $0 \le q < m := \dim X$, (ii) locally irreducible, weakly normal (local) image ([Tu$_1$, p. 104]) at each point of $Z := \pi^{-1}(0)$. A result of Gunning [Gu, Theorem 3, p. 32] can be generalized (Proposition 3.1): If $\pi = (\pi_1, \ldots, \pi_p) : X \to \mathbb{C}^p$ is $q$-weakly normal, then for any proper slicing $(\varphi, \pi, D)$ of $\{(\pi_j)\}_{1 \le j \le p}$, the algebra $\mathcal{O}(Y \times \tilde{D})$ is an integral algebraic extension, of degree at most $\mathfrak{h} = \mathfrak{h}_{(\pi_1)\cdots(\pi_p),D}$, of the lifted algebra

$$\tilde{\mathcal{O}}_{[Y, N \times Q]} := (\mathrm{id}_Y, (\pi, \varphi)\rfloor\tilde{D})^* \mathcal{O}(Y \times N \times Q). \tag{1.7}$$

Essential to the proof (and that of the Nullstellensatz 3.1) is the construction, for an element $f \in C^0(Y \times \tilde{D})$, of a monic *characteristic pseudopolynomial* (in a sense similar to that of [GR$_3$, p. 138]) arising from a Riemannian fiber product of $f$ (see (3.4)). The fact that the holomorphy of such a pseudopolynomial follows from the Hartogs theorem (on the equivalence of separate and joint holomorphy) seems to provide a lacking application of the latter, in view of the remark of Grauert and Remmert [GR$_3$, p. 2]. Further, if $S := \bigcap \mathfrak{S}_{|(g_j)|}$ is determined by the component divisors of a $q$-weakly normal map $g = (g_1, \ldots, g_p) : X \to \mathbb{C}^p$, then the slicing degree $\mathfrak{h} = \mathfrak{h}_{(g_1)\cdots(g_p),D}$ of any proper slicing $(\varphi, g, D)$ of $\{(g_j)\}_{1\le j\le p}$ serves as a *Hilbert exponent* (independent of the parameter $y \in Y$) for the ideal of $\mathfrak{S}_{D,Y} := Y \times S_D$ (Theorem 3.1):

$$\mathcal{I}(\mathfrak{S}_{D,Y}) = \sqrt[\mathfrak{h}]{\langle \tilde{g}_1, \ldots, \tilde{g}_p\rangle_{\{Y\}}}, \tag{1.8}$$

where $\langle \tilde{g}_1, \ldots, \tilde{g}_p\rangle_{\{Y\}}$ denotes the ideal in $\mathcal{O}(Y \times \tilde{D})$ generated by the functions $\tilde{g}_j := g_j\rfloor\tilde{D}$, $1 \le j \le p$. The question whether the ideal $\mathcal{I}(\mathfrak{S})$ of a general subvariety $\mathfrak{S}$ in $Y \times X$ admits a Hilbert exponent (as in (1.8)) remains open. By the use of suitable slicing maps of $g$ and properties of the multiplicity, it can be shown that at each point $a_0 \in S \setminus (S_{\mathrm{sing}} \cap X_{\mathrm{sing}})$ there is a neighborhood $U$ such that the multiplicity "$\nu_g^0(a_0)$" gives a Hilbert exponent for $\mathfrak{S}_{U,Y}$. If $\mathcal{D}$ is a principal divisor in $X$ with support $S$, then the Hilbert number of a local defining function $g$ of $\mathcal{D}$ at a normal point $a$ of $X$ is given by

$$\mathfrak{h}(g, a) = \max\{\nu_g^0(c_\mu) \mid c_\mu \in S_{\mathrm{reg}} \cap \mathfrak{B}_\mu\}, \tag{1.9}$$

the maximum being taken over all branches $\mathfrak{B}_\mu$ of $S_U$, for a suitable neighborhood $U$ of $a$; moreover, $\mathfrak{h}(g, a)$ gives the smallest Hilbert exponent for $\mathfrak{S}_{U,Y}$ (Theorem 3.2). The related question as to when the ideal associated to a (general) divisor in $Y \times \mathbb{P}^N(\mathbb{C})$ admits a principal generator is partially answered in Theorem 4.1 (see also Propositions 3.3 and 4.1).

**2. Strict and Noether stability.** For $f : Y \times X \to \mathbb{C}$, set $f^{(y)}(z) := f(y, z)$ for $(y, z) \in Y \times X$, and define $\Delta(f) := \{y \in Y \mid f^{(y)} \equiv 0\}$. Let $\mathcal{P}ol_{Y,N}(d)$ denote the set of all (*holomorphic*) *homogeneous pseudopolynomials* (*of degree* $d > 0$) *over* $Y$ in $N$ indeterminates, that is,

$$G(X_1, \ldots, X_N) = \sum_{\|\mu\|=d} a_\mu X_1^{\mu_1} \cdots X_N^{\mu_N}, \tag{2.1}$$

where $a_\mu \in \mathcal{O}(Y)$ and at least one $a_\mu \not\equiv 0$. Set $\mathcal{P}ol_{Y,N}(0) := \mathcal{O}(Y)$. Then the graded ring

$$\mathcal{O}_Y[X_1, \ldots, X_N] := \bigoplus_{d\ge 0} \mathcal{P}ol_{Y,N}(d)$$

is a left module over the ring $\mathcal{O}(Y)$. Denote by $\mathcal{P}ol_{Y,N}$ the union of all its summands $\mathcal{P}ol_{Y,N}(d)$ (omitting the reference to $Y$ if $Y$ is a single point). An element $Q \in \mathcal{O}_Y[X_1, \ldots, X_N]$ is naturally identified with a function $Q : Y \times \mathbb{C}^N \to \mathbb{C}$ (by evaluation at each $(y, z) \in Y \times \mathbb{C}^N$). A mapping $g = (g_1, \ldots, g_p)$ with $g_j \in \mathcal{O}_Y[X_1, \ldots, X_N]$ (or $g_j \in \mathcal{P}ol_{Y,N}$) of (generic) positive degree is said to be *primitive over* $Y$ if the set $\Delta(g) := \bigcup_{j=1}^p \Delta(g_j)$ is almost thin of codimension 2 (thus possibly empty) ([AS, p. 14]).

In the following let $V$ denote a positive-dimensional affine algebraic variety in $\mathbb{C}^N$. An equivalence relation, "$F \equiv G$" (with respect to $V$), is defined in $\mathcal{O}_Y[X_1, \dots, X_N]$ by setting $F \equiv G$ if and only if at each point $y_0 \in Y$ there is a neighborhood $Y_0$ such that $F = G$ on $Y_0 \times V$. Every such equivalence class is called a *$Y$-regular function on $V$*. The set $\mathcal{O}_Y[V]$ of all $Y$-regular functions on $V$ forms a (complex) algebra under the usual operations. Denote by $\mathfrak{A}_Y = \langle g_1, \dots, g_p \rangle_Y$ the ideal in $\mathcal{O}_Y[V]$ generated by elements $g_j \in \mathcal{O}_Y[V]$, $1 \leq j \leq p$,. Similarly, let $\mathcal{V}(\mathfrak{A}) = \mathcal{V}(g_1, \dots, g_p)$ be the subvariety in $Y \times V$ defined by the equations $g_j = 0$, $1 \leq j \leq p$ (with the same notation in case $V = \mathbb{C}^N$ and the $g_j$ belong to $\mathcal{P}ol_{Y,N}$). Denote by $\mathcal{J}_Y(\mathfrak{S})$ the ideal of all $Y$-regular functions on $V$ vanishing on a subvariety $\mathfrak{S} \subseteq Y \times V$.

In analogy with the notion of stable subalgebra of entire functions in $\mathbb{C}^N$ ([BD, p. 268]), a property of recurring use concerning the relative algebra $\mathcal{O}_Y[V]$ is stated below. This is an immediate consequence of the generalized Ronkin theorem of [Tu$_2$, 4.1]:

THEOREM 2.1. *If $Y$ is a normal complex space, then the subalgebra $\mathcal{O}_Y[V]$ is strictly stable in $\mathcal{O}(Y \times \mathbb{C}^N)$.*

In Płoski–Tworzewski [PT, Proposition 2.1], the Max Noether theorem is generalized to an affine algebraic variety in $\mathbb{C}^N$. This result can be further extended to the relative case (by refining the proof of [PT, p. 33]):

THEOREM 2.2 (Relative Noether Theorem). *For any reduced complex space $Y$, the subalgebra $\mathcal{O}_Y[V]$ is Noether-stable in $\mathcal{O}(Y \times V)$.*

*Proof.* At first a relative version of the Max Noether Theorem in $\mathbb{C}^N$ will be proved by modifying the argument of Tworzewski [Tw, p. 2] (with some of the original steps included for completeness): "If $G_j$, $1 \leq j \leq p$, and $F$ are $Y$-regular functions on $\mathbb{C}^N$ such that $F_{\mathfrak{w}} \equiv 0$ $(\langle (G_1)_{\mathfrak{w}}, \dots, (G_p)_{\mathfrak{w}} \rangle)$ in $\mathcal{O}_{Y \times \mathbb{C}^N, \mathfrak{w}}$ at every point $\mathfrak{w} \in \mathfrak{S} := \mathcal{V}(G_1, \dots, G_p)$ with $\pi(\mathfrak{w})$ lying off some thin subset of codimension $\geq 2$ in $\mathbb{C}^N$ ([AS, p. 14]), then $F \equiv 0$ $(\langle G_1, \dots, G_p \rangle_Y)$ in $\mathcal{O}_Y[\mathbb{C}^N]$".

Let $\Gamma \subset Y \times \mathbb{C}^N \times \mathbb{C}^p$ be the graph of $G = (G_1, \dots, G_p)$. Then for each $y \in Y$, the set $\mathcal{Z}^{(y)} := \{(z, w) \mid (y, z, w) \in (\{y\} \times \mathbb{C}^N \times \{0\}) \cup \Gamma\}$ is an $N$-dimensional algebraic set in $\mathbb{C}^N \times \mathbb{C}^p$. Take $\mathfrak{w} = (y_0, a_0) \in Y \times (\mathbb{C}^N \setminus S)$, where $S$ is a thin analytic subset of codimension $\geq 2$ in $\mathbb{C}^N$ relative to which the above mentioned Noether condition holds. Then on some neighborhood $Y_0 \times \Delta$ of $(y_0, a_0)$ in $Y \times (\mathbb{C}^N \setminus S)$, holomorphic functions $\lambda_j$, $1 \leq j \leq p$, can be chosen so that the function $\Psi : Y_0 \times \Delta \times \mathbb{C}^p \to \mathbb{C}$,

$$\Psi(y, z, w) := F^{(y)}(z) - \sum_{j=1}^{p} \lambda_j^{(y)}(z) w_j, \quad \forall (y, z, w) \in Y_0 \times \Delta \times \mathbb{C}^p,$$

vanishes on $\Gamma \cap (Y_0 \times \Delta \times \mathbb{C}^p)$. It suffices to consider the case $F \not\equiv 0$ in

$Y_0 \times \mathbb{C}^N$. Let $\psi^{(y)} : \mathcal{Z}^{(y)} \to \mathbb{C}$ be defined by

$$\psi^{(y)}(z,w) = \begin{cases} F(y,z) & \text{if } w = 0, \\ 0 & \text{if } (y,z,w) \in \Gamma. \end{cases}$$

The assumption on the germs of $F$ implies that $\psi^{(y)}$ is well-defined for each $y \in Y_0$. Clearly

$$\psi^{(y)} \rfloor (\Delta \times \mathbb{C}^p) \cap \mathcal{Z}^{(y)} = \Psi^{(y)} \rfloor (\Delta \times \mathbb{C}^p) \cap \mathcal{Z}^{(y)},$$

where $\Psi^{(y)}(z,w) := \Psi(y,z,w)$. Therefore $\psi^{(y)}$ extends holomorphically to a function $\tilde{\psi}^{(y)}$ on $(\mathbb{C}^N \setminus S) \times \mathbb{C}^p$, hence also to $\mathbb{C}^N \times \mathbb{C}^p$. Moreover, it is easy to check that the graph of $\tilde{\psi}^{(y)}$ is an algebraic subset of $\mathbb{C}^N \times \mathbb{C}^p \times \mathbb{C}$. Then by Serre's Algebraic Graph Theorem ([Łoj, p. 464]), there exists a polynomial $Q^{(y)} : \mathbb{C}^N \times \mathbb{C}^p \to \mathbb{C}$ such that $Q^{(y)} \rfloor \mathcal{Z}^{(y)} = \tilde{\psi}^{(y)}$. Thus

$$Q^{(y)} \rfloor (\mathbb{C}^N \times \mathbb{C}^p) \cap \mathcal{Z}^{(y)} = \Psi^{(y)} \rfloor (\mathbb{C}^N \times \mathbb{C}^p) \cap \mathcal{Z}^{(y)}.$$

Write

$$Q^{(y)}(z,w) = Q_0^{(y)}(z) + \sum_{j=1}^{p} w_j \sum_{\mu^{(j)}} a^{(y)}_{\mu^{(j)}} w_1^{\mu_1^{(j)}} \cdots w_p^{\mu_p^{(j)}} z_1^{\mu_{p+1}^{(j)}} \cdots z_N^{\mu_{p+N}^{(j)}}.$$

By the definition of $\psi^{(y)}$ one has $F^{(y)} = Q_0^{(y)}$ and

$$F^{(y)}(z) = \sum_{j=1}^{p} \lambda_j(y,z) G_j^{(y)}(z), \tag{2.2}$$

where

$$\lambda_j(y,z) := -\sum_{\mu^{(j)}} a^{(y)}_{\mu^{(j)}} (G_1^{(y)})^{\mu_1^{(j)}} \cdots (G_p^{(y)})^{\mu_p^{(j)}} z_1^{\mu_{p+1}^{(j)}} \cdots z_N^{\mu_{p+N}^{(j)}}$$

for $(y,z) \in Y_0 \times \mathbb{C}^N$. Consequently, there exist functions $c_{\rho(j,\mu^{(j)})} \in \mathcal{O}(Y_0)$, $1 \le j \le p$, such that

$$F^{(y)}(z) = -\sum_{j=1}^{p} \sum_{\mu^{(j)}} \sum_{\rho(j,\mu^{(j)})} a^{(y)}_{\mu^{(j)}} c^{(y)}_{\rho(j,\mu^{(j)})} z_1^{\rho(j,\mu^{(j)})_1} \cdots z_N^{\rho(j,\mu^{(j)})_N}.$$

Then for each $(\mu, j)$ with some $c^{(y)}_{\rho(j,\mu^{(j)})} \not\equiv 0$ in $Y_0$, one has

$$a_{\mu^{(j)}} = \frac{\partial^{\rho(j,\mu^{(j)})} F}{\partial z^{\rho(j,\mu^{(j)})}} \in \mathcal{O}((Y_0 \setminus A) \times \mathbb{C}^N), \tag{2.3}$$

where $A$ is a thin analytic subset of $Y_0$. Since the function $F$ agrees with the one defined by the right-hand side of (2.2) where any term containing $a^{(y)}_{\mu^{(j)}}$ is dropped if every ensuing coefficient $c_{\rho(j,\mu^{(j)})} \equiv 0$, it follows from the formula (2.3) that $F \equiv 0$ $(\langle G_1, \ldots, G_p \rangle_Y)$ in $\mathcal{O}_Y[\mathbb{C}^N]$.

The general case of an affine algebraic variety can now be deduced from the preceding. Let $f$ and $g_j$, $1 \le j \le q$, be elements of $\mathcal{O}_Y[V] \setminus \{0\}$ such that the Noether condition (1.1) holds at every point $\mathfrak{w}$ of $\mathfrak{S} := \mathcal{V}(g_1, \dots, g_q)$ with $\pi(\mathfrak{w})$ lying off some thin subset of codimension $\ge 2$ in $\mathbb{C}^N$. Locally at a given point $y_0 \in Y$, the space $Y$ is embeddable as a subvariety in an open subset of $\mathbb{C}^p$. There is a thin analytic subset $S$ of codimension $\ge 2$ in $\mathbb{C}^N$ such that if $\mathfrak{w} = (y_0, a_0) \in Y \times (\mathbb{C}^N \setminus S)$, then a local equation

$$(2.4) \qquad F^{(y)}\ (\zeta) - \sum_{j=1}^{q} \eta_j(y,\zeta) G_j^{(y)}(\zeta) = R^{(y)}\ (\zeta), \quad \forall (y,\zeta) \in \Omega_1 \times \varDelta_1,$$

holds with $\eta_j \in \mathcal{O}(\Omega_1 \times \varDelta_1)$, $1 \le j \le q$, where $\Omega_1 \times \varDelta_1$ is a product neighborhood of $(y_0, a_0)$ in $\mathbb{C}^p \times (\mathbb{C}^N \setminus S)$, $F$, respectively, $G_j$, $1 \le j \le q$, an element of $\mathcal{O}_{\Omega_1}[X_1, \dots, X_N]$ inducing $f$, respectively, $g_j$, $1 \le j \le q$, on $\Omega_1 \times V$, and $R$ a function in $\mathcal{O}(\Omega_1 \times \varDelta_1)$ vanishing on $Y_1 \times U_1 := (Y \times V) \cap (\Omega_1 \times \varDelta_1)$.

Regarding the set $\mathbb{C}^p \times V$ as an affine algebraic susbset in $\mathbb{C}^{p+N}$ there exists a set $\{G_j\}_{q+1 \le j \le q+s}$ of generators of the ideal $\mathcal{J}(\mathbb{C}^p \times V)$ in the ring $\mathbb{C}[\zeta_1, \dots, \zeta_p, X_1, \dots, X_N]$. It follows from Serre's Lemma ([Łoj, p. 458]) that at every point $(w_0, a_0) \in \mathbb{C}^p \times V$ the germs of such $G_j$'s generate the ideal $\mathcal{I}_{(w_0,a_0)}(\mathbb{C}^p \times V)$ in $\mathcal{O}_{\mathbb{C}^{p+N},(w_0,a_0)}$. Therefore, given $(y_0, a_0) \in \mathfrak{S}$ with $a_0 \in \varDelta_1 \setminus S$, there is a product neighborhood $\Omega_0 \times \varDelta_{a_0} \subseteq \Omega_1 \times (\varDelta_1 \setminus S)$ of $(y_0, a_0)$ such that, for all $(y, \zeta) \in \Omega_0 \times \varDelta_{a_0}$,

$$F^{(y)}(\zeta) - \sum_{j=1}^{q} \eta_j^{(y)}(\zeta') G_j^{(y)}(\zeta) = \sum_{j=q+1}^{q+s} \xi_j^{(y)}(\zeta') G_j(\zeta)$$

for some $\xi_j \in \mathcal{O}(\Omega_0 \times \varDelta_{a_0})$, $q+1 \le j \le q+s$. Thus at every point $(y_0, a_0) \in \tilde{\mathfrak{S}}$ with $a_0 \in \mathbb{C}^N \setminus S$, where $\tilde{\mathfrak{S}} := \mathcal{V}(G_1, \dots, G_{q+s})$, the germs of the elements $G_j$, $1 \le j \le q+s$, generate the germ of $F$ in $\mathcal{O}_{Y_0 \times \mathbb{C}^N,(y_0,a_0)}$, where $Y_0 := Y \cap \Omega_0$. By the preceding relative Noether Theorem, there exist elements $\hat{\lambda}_j \in \mathcal{O}_{Y_0}[X_1, \dots, X_N]$, $1 \le j \le q+s$, such that

$$F = \sum_{j=1}^{q} \hat{\lambda}_j G_j + \sum_{j=q+1}^{q+s} \hat{\lambda}_j G_j \quad \text{in } Y_0 \times \mathbb{C}^N.$$

Here the second sum $H := \sum_{j=q+1}^{q+s} \hat{\lambda}_j G_j$ belongs to $\mathcal{O}_{Y_0}[X_1, \dots, X_N]$ and vanishes on $Y_0 \times V$, thus completing the proof of the general case. ∎

A (local) relative version of the Hilbert Nullstellensatz for polynomial ideals is given by Łojasiewicz–Płoski [Łoj, p. 407]. Theorem 2.2 allows for an extension of this result to relative regular functions on an algebraic variety. The proof of [Łoj, p. 407]) will need to be modified and the existence of the Hilbert number (1.4) established. The latter is stated in the next proposition, which is an easy consequence of the Rückert Nullstellensatz ([Łoj, p. 284] and the Noetherian character of the local rings of $X$; its proof is omitted.

Proposition 2.1. *Let $g = (g_1, \ldots, g_p) : X \to \mathbb{C}^p$ be a holomorphic map.*

(1) *The Hilbert number $\mathfrak{h}(g, a)$ is a well-defined positive integer for every point $a \in S := \mathcal{V}(g_1, \ldots, g_p)$.*
(2) *If $b \in X \setminus S$, then $\mathcal{O}_{X,b} = \langle g_{1,b}, \ldots, g_{p,b} \rangle$.*

Theorem 2.3 (Nullstellensatz for an algebraic variety). *Assume that $Y$ is a connected complex space and $V$ an affine algebraic variety. Let $\mathfrak{A}_Y$ be the ideal in $\mathcal{O}_Y[V]$ generated by $Y$-regular functions $g_j$, $j = 1, \ldots, p$, on $V$.*

(1) *If the subvariety $\mathfrak{S} := \mathcal{V}(\mathfrak{A}_Y)$ is empty, then $\mathfrak{A}_Y = \mathcal{O}_Y[V]$.*
(2) *If $\mathfrak{S}$ is not empty, then $\mathcal{J}_Y(\mathfrak{S})$ admits a Hilbert exponent $\mathfrak{h} > 0$ over $Y$, namely,*

$$\mathcal{J}_Y(\mathfrak{S}) = \sqrt[\mathfrak{h}]{\mathfrak{A}_Y}. \tag{2.5}$$

(3) *If $Y$ is a Stein space and $\{g_j\}_{1 \le j \le p}$ generates an ideal in $\mathcal{O}(Y \times V)$ that is an intersection of prime ideals, then $\mathcal{J}_Y(\mathfrak{S}) = \mathfrak{A}_Y$.*

*Proof.* Let $\{G_{p+1}, \ldots, G_{p+s}\}$ be a polynomial system defining $V$ in $\mathbb{C}^N$ and let $Q \in \mathcal{J}_Y(\mathfrak{S})$. Given a point $y_0 \in Y$, let $Y_1$ be a neighborhood such that $Q$, respectively, each $g_j$, $1 \le j \le p$, is induced by an element $\tilde{Q}$, respectively, $G_j$, in $\mathcal{O}_{Y_1}[X_1, \ldots, X_N]$. Let ${}^hQ$, respectively, ${}^hG_j$, $1 \le j \le p+s$, be the homogenization of $\tilde{Q}$, respectively, $G_j$, $1 \le j \le p+s$ (in the indeterminates $X_0, X_1, \ldots, X_N$, so that when restricted to $Y_1 \times \{1\} \times \mathbb{C}^N$, ${}^hQ$ reduces to $\tilde{Q}$, respectively, ${}^hG_j$ to $G_j$). Suppose that for a point $(t^0, z^0)$ with $t^0 \neq 0$, $({}^hG_j)^{(y)}(t^0, z^0) = 0$ for all $j = 1, \ldots, p+s$. Then $G_j(y, z^0/t^0) = 0$ for $1 \le j \le p$ and $G_j(z^0/t^0) = 0$ for $p+1 \le j \le p+s$. Hence $(y, z^0/t^0) \in \mathfrak{S}$, which implies that $\tilde{Q}(y, z^0/t^0) = 0$ and hence $({}^hQ)^{(y)}(t^0, z^0) = 0$. One may also require that $({}^hQ)^{(y)}(0, z^0) = 0$. Thus the function ${}^hQ$ vanishes on the subvariety ${}^h\mathfrak{S} := \{{}^hG_1 = 0, \ldots, {}^h G_{p+s} = 0\} \subset Y_1 \times \mathbb{C}^{N+1}$. By Proposition 2.1 (applied to the point $w_0 = (y_0, 0) \in Y \times \mathbb{C}^{N+1}$), there exist a positive integer $\mathfrak{h} = \mathfrak{h}(y_0)$ (independent of ${}^hQ$) and a connected neighborhood $Y_0 \times \Delta_0 \subseteq Y_1 \times \mathbb{C}^{N+1}$ of $w_0$ such that

$$({}^hQ)^{\mathfrak{h}}\,(y, \zeta) = \sum_{j=1}^{p+s} \lambda_j(y, \zeta)({}^hG_j)(y, \zeta), \qquad (y, \zeta) \in Y_0 \times \Delta_0, \tag{2.6}$$

for suitable $\lambda_j \in \mathcal{O}(Y_0 \times \Delta_0)$, $1 \le j \le p+s$. Note that if ${}^h\mathfrak{S} = \emptyset$, then the above relation holds for $\mathfrak{h} = 0$. The subalgebra $\mathcal{O}_{Y_0}[X_1, \ldots, X_{N+1}]$ being Noether-stable in $\mathcal{O}(Y \times \mathbb{C}^{N+1})$, there exist $\tilde{\lambda}_j \in \mathcal{O}_{Y_0}[X_1, \ldots, X_{N+1}]$, $1 \le j \le p+s$, such that the relation (2.6), with each $\lambda_j$ replaced by $\tilde{\lambda}_j$, remains valid for all $(y, \zeta) \in Y_0 \times \mathbb{C}^{N+1}$. This shows that $Q \in \sqrt[\mathfrak{h}]{\mathfrak{A}_Y}$. The function

$y_0 \mapsto \mathfrak{h}(y_0)$, being locally constant on the connected space $Y$, is a constant on $Y$. Consequently, the global Hilbert relation (2.5) is proved.

Assume now that $Y$ is Stein and the ideal $\langle g_1, \ldots, g_q \rangle$ in $\mathcal{O}(Y \times V)$ generated by $g_j$, $1 \leq j \leq q$, is an intersection of prime ideals. Since the product space $Y \times V$ is Stein, according to Forster [Fo, Satz 4, p. 315] (or Siu [S, p. 297]) the set of all holomorphic functions vanishing on $\mathfrak{S}$ is precisely given by the ideal $\langle g_1, \ldots, g_q \rangle$. Consequently the Noether stability of $\mathcal{O}_Y[V]$ in $\mathcal{O}(Y \times V)$ implies that the ideal $\mathcal{J}_Y(\mathfrak{S})$ is globally generated by the $g_j$'s over $Y$. ∎

**3. The slicing degree and a semiglobal relative Nullstellensatz.** Let $M$ and $N$ be complex spaces of pure dimension $m > 0$ and $n$ respectively with $q := m - n \geq 0$, $f : M \to N$ a holomorphic map, and $a \in M$. A holomorphic map $\varphi : U \to \mathbb{C}^q$ is called a *slicing map of $f$ at $a$*, if the junction $(f, \varphi) : U \to N \times \mathbb{C}^q$ is light. Denote by $\Phi_a^0(f)$ the set of all slicing maps of $f$ at $a$. The map $f$ is said to be a *$q$-fibering* if $F_z := f^{-1}(f(z))$ has pure dimension $q$ for all $z \in M$. The *rank of $f$ at $a$* is defined by

$$\operatorname{rank}_a f := \dim_a M - \dim_a F_a.$$

The map $f$ is said to be of *pure rank* if $\operatorname{rank}_z f = \text{const}$ for all $z \in M$. Assume that $N$ is locally irreducible. Let $f : M \to N$ be a holomorphic $q$-fibering. If $q = 0$, choose an open neighborhood $U \Subset M$ of $a$ with $F_a \cap \overline{U} = \{a\}$. Then

$$1 \leq \nu_f(a) := \limsup_{z \to a} \# F_z \cap U \tag{3.1}$$

is an integer independent of the choice of $U$, called the *multiplicity of $f$ at $a$*.

Consider now the case where $f : M \to N$ is a holomorphic $q$-fibering at $a$ with $q > 0$. Given $\varphi \in \Phi_a^0(f)$ the restriction $\psi = \varphi \rfloor F_a \cap U$ is light at $a$, hence the *covering index* $\nu_\varphi^f(a) := \nu_\psi(a)$ is well-defined. An element $\varphi \in \Phi_a^0(f)$ is called *regular* if there exists a local embedding $\alpha : U \to U' \subseteq G$ into an open set $G \subseteq \mathbb{C}^{e_a}$, where $e_a$ is the embedding dimension of $M$ at $a$, and a regular holomorphic map $\tilde{\varphi} : G \to \mathbb{C}^q$ such that $\tilde{\varphi} \circ \alpha = \varphi \rfloor U$.

Property 3.1 ([Tu$_1$, (2.1.5)]). *A holomorphic map $f : M \to N$ is a $q$-fibering at $a$ if and only if the set $\Phi_a^1(f)$ of all regular slicing maps of $f$ at $a$ is not empty.*

Define

$$d_a(f) := \operatorname{Min} \{ \nu_\varphi^f(a) \mid \varphi \in \Phi_a^1(f) \}.$$

Let $\Phi_a^2(f)$ be the set of all $\varphi \in \Phi_a^1(f)$ with *minimal covering index*, that is,

$$\nu_\varphi^f(a) = d_a(f). \tag{3.2}$$

Definition 3.1. The *multiplicity of $f$ at $a$* is defined by

$$\nu_f(a) := \operatorname{Min}\{\nu_{(f,\varphi)}(a) \mid \varphi \in \Phi_a^2(f)\}. \tag{3.3}$$

Note that in Stoll [$\mathrm{St}_1$] different definitions are given for the *multiplicity* "$\nu_f(a)$" and *order* "$\tilde{\nu}_f(a)$" of $f$ at $a$ (see [$\mathrm{St}_1$, p. 48]), and these two definitions agree at each point where $M$ is locally irreducible. Stoll's "multiplicity" is not needed in this paper (and an example shows that it does not agree with the one defined above at singular points of $M$; see [$\mathrm{Tu}_1$, p. 125]). For the multiplicity (3.3) one sees that, in general, $\nu_f(a) \geq \tilde{\nu}_f(a)$ ([$\mathrm{Tu}_1$, Lemma 2.2.3]). Also it is shown in [$\mathrm{Tu}_1$, Proposition 2.2.5] that: *if $M$, $N$ are pure-dimensional, $N$ is locally irreducible, and $f : M \to N$ is $q$-fibering, then at every simple point $a$ of $M$, $\nu_f(a) = \tilde{\nu}_f(a) = \nu_{(f,\varphi)}(a)$ for all $\varphi \in \Phi_a^2(f)$.* Furthermore, according to Draper [D, Proposition 5.1], if $a$ is a simple point of $M$ and $N$ is a normal space, then "$\tilde{\nu}_f(a)$" agrees with the (classical) "intersection multiplicity" (see [D, §4]):

$$\tilde{\nu}_f(a) = \mathfrak{i}(\Gamma \cdot (M \times \{b\}) \cdot (L \times \{N\}, (a,b)),$$

where $\Gamma$ denotes the graph of $f$, $b = f(a)$, and $L \subset M$ is an analytic subset of pure dimension $m - q$ at $a$ and has compact closure which meets $F_a$ in an isolated point at $a$. If further $M$, $N$ are complex manifolds, then the latter (intersection multiplicity) agrees with that defined by Borel–Haefliger [BH].

If $\pi : X \to U \subseteq \mathbb{C}^m$ is a finite branched analytic covering, then a function $f \in C^0(X)$ is holomorphic if and only if $f$ is integral over the subalgebra $\pi^*(\mathcal{O}_U)$ ([GuRo, pp. 104–105] and [Gu, Theorem 3, p. 32]). This characterization carries over to the more general case of a $q$-weakly normal mapping:

Proposition 3.1. *If $\pi = (\pi_1, \ldots, \pi_p) : X \to \mathbb{C}^p$ is $q$-weakly normal, then for any proper slicing $(\varphi, \pi, D)$ of $\{(\pi_j)\}_{1 \leq j \leq p}$:*

(i) *The algebra $\mathcal{O}(Y \times \tilde{D})$ is an integral algebraic extension, of degree at most $\mathfrak{h} = \mathfrak{h}_{(\pi_1)\cdots(\pi_p),D}$ (see (1.6)), of the lifted algebra $\tilde{\mathcal{O}}_{[Y,N\times Q]}$ (see (1.7)).*

(ii) *If an element $f \in C^0(Y \times \tilde{D})$ is holomorphic, then there exists a monic element $P \in \tilde{\mathcal{O}}_{[Y,N\times Q]}[X_1]$ such that $P(f(y,z), y, \pi(z), \varphi(z)) \equiv 0$ in $Y \times \tilde{D}$; the converse assertion holds if, in addition, $X$ and $Y$ are normal spaces.*

*Proof.* Since $\pi : X \to \mathbb{C}^p$ is $q$-weakly normal, the slicing degree $\mathfrak{h} = \mathfrak{h}_{(\pi_1)\cdots(\pi_p),D}$ given by the expression (1.6) is a positive integer independent of $(w,t) \in N \times Q$ (Property 6.1 below). Given an element $f \in C^0(Y \times \tilde{D})$, consider the associated *Riemannian fiber product*

$$\omega(y, w, t, \zeta) := \prod_{z^* \in \tilde{D}} (\zeta - f^{(y)}(z^*))^{\nu_{\pi,\varphi}(z^*;w,t)} \tag{3.4}$$

defined in $Y \times N \times Q \times \mathbb{C}$. The continuity of the function $\omega$ is shown in [Tu$_1$, (1.2.20)]. Restricting $\omega$ to $\{y\} \times N \times Q \times \mathbb{C}$ (for fixed $y \in Y$) gives rise to a polynomial function

$$\omega^{(y)}(w,t,\zeta) = \sum_{k=0}^{\mathfrak{h}} (-1)^k \mathfrak{p}_k^{(y)}(w,t)\,\zeta^{\mathfrak{h}-k} \tag{3.5}$$

on $N \times Q \times \mathbb{C}$, where the integer $\mathfrak{h}$ is the slicing degree of $\{(\pi_j)\}_{1\le j\le p}$. Here the coefficient $\mathfrak{p}_k^{(y)}(w,t)$ arises from the $k$th elementary symmetric function of the values of the restriction $f^{(y)}$ on $\pi^{-1}(w) \cap \varphi^{-1}(t) \cap D$, and each $\mathfrak{p}_k \in C^0(Y \times N \times Q)$. The mapping $\omega : Y \times N \times Q \times \mathbb{C} \to \mathbb{C}$ is shown to be separately holomorphic in each variable [Tu$_1$, (1.2.19)]. Hence by the Hartogs theorem ([GR$_1$, Satz 29]), $\omega$ is jointly holomorphic; moreover, as a function of $(y,w,t)$, $\omega$ is holomorphic in $Y \times N \times Q$. Clearly, for fixed $y \in Y$, $\omega^{(y)}(\pi(z), \varphi(z), f^{(y)}(z)) = 0$ for all $z \in \tilde{D}$. Hence

$$f^{\mathfrak{h}}\,(y,z) = \sum_{k=0}^{\mathfrak{h}-1} (-1)^{k+1} \mathfrak{p}_k^{(y)}(g(z),\varphi(z))(f^{(y)}(z))^{\mathfrak{h}-k}, \qquad z \in \tilde{D}, \tag{3.6}$$

thus $f$ is integral of degree $\mathfrak{h}$ over $\tilde{\mathcal{O}}_{[Y,N\times Q]}$.

Conversely, assume that $f \in C^0(Y \times \tilde{D})$ is integral over $\tilde{\mathcal{O}}_{[Y,N\times Q]}$. The map $\hat{\pi} := (\mathrm{id}_Y, (\pi,\varphi)\rfloor \tilde{D}) : Y \times \tilde{D} \to Y \times N \times Q$ being light, proper and holomorphic, the Andreotti–Stoll theorem [AS, Lemma 2.2, pp. 45–46] (on finite branched analytic coverings) and the same argument as in [GuRo, Lemma, pp. 104–105] imply that $f$ is holomorphic in $Y \times \tilde{D}$ off a thin analytic subset. Hence so is $f$ in $Y \times \tilde{D}$, by the normality of the latter. ∎

THEOREM 3.1 (Nullstellensatz for a $q$-weakly normal mapping). *Assume that $g = (g_1, \ldots, g_p) : X \to \mathbb{C}^p$ is $q$-weakly normal. Then:*

(i) *For any proper slicing $(\varphi, g, D)$ of $\{(g_j)\}_{1\le j\le p}$, the slicing degree $\mathfrak{h} = \mathfrak{h}_{(g_1)\cdots(g_p),D}$ gives a Hilbert exponent for $\mathfrak{S}_{D,Y} := Y \times S_D$, where $S := \bigcap \mathfrak{S}_{|(g_j)|}$.*

(ii) *At every point $a \in S \setminus (S_{\mathrm{sing}} \cap X_{\mathrm{sing}})$, the multiplicity $\nu_g^0(a)$ gives a Hilbert exponent for $\mathfrak{S}_{U,Y}$, for some neighborhood $U$ of $a$, and consequently*

$$\mathfrak{h}(g,a) \le \nu_g^0(a).$$

*Proof.* (i) Assume that the function $Q^{(y)}$ belongs to $\mathcal{I}(S_D)$ for every $y \in Y$. Since each $\mathfrak{p}_k^{(y)}(w,t)$ is expressible as a polynomial (without constant term) in the push-forwards

$$\sum \{\nu_{g,\varphi}(z^*; w,t)(Q^{(y)}(z^*))^s \mid z^* \in g^{-1}(w) \cap \varphi^{-1}(t) \cap D\},$$

where $s = 1, 2, \ldots$, one has $\mathfrak{p}_k^{(y)}(0,t) = 0$. Thus for each $k = 1, \ldots, \mathfrak{h}$, the function $\mathfrak{p}_k^{(y)}(g(z),\varphi(z))$ can be written as

$$\mathfrak{p}_k^{(y)}(g(z),\varphi(z)) = \sum_{\|\mu\|>0} h_\mu(y,\varphi(z))\, g_1(z)^{\mu_1}\cdots g_p(z)^{\mu_p}, \qquad z \in \tilde{D},$$

with holomorphic coefficients $h_\mu(y,\varphi(z))$ in $Y\times\tilde{D}$. Consequently, for each $k=1,\dots,\mathfrak{h}-1$, the function $\mathfrak{p}_k^{(y)}(g(z),\varphi(z))$ belongs to the ideal generated by the restrictions $g_j\rfloor\tilde{D}$. By virtue of the identity (3.6), this implies that there exist $\lambda_j\in\mathcal{O}(Y\times\tilde{D})$, $j=1,\dots,p$, such that

$$(Q(y,z))^{\mathfrak{h}} = \lambda_1(y,z)\,g_1(z)+\cdots+\lambda_p(y,z)\,g_p(z), \qquad (y,z)\in Y\times\tilde{D}. \tag{3.7}$$

This proves the Hilbert relation (1.8).

(ii) At each $a\in S$ there exists an open neighborhood $U_0$ on which $g$ is a $q$-fibering. By Property 3.1, there exists a holomorphic map $\varphi: U_0\to\mathbb{C}^q$ such that $g$ is light along $\varphi$ at each $z\in S_{U_0}\cap\varphi^{-1}(0)$. A topological argument ([Tu$_1$, (1.1.5)]) shows that for a (possibly smaller) neighborhood $U\Subset U_0$ of $a$, $(\varphi,g,U)$ is a proper slicing of $\{(g_j)\}_{1\le j\le p}$ with

$$S\cap\varphi^{-1}(0)\cap\overline{U}=\{a\}.$$

The map $\varphi$ can be chosen with minimal covering index at a point $a$ (see (3.2) and (3.3)). If $a\in S\setminus(S_{\mathrm{sing}}\cap X_{\mathrm{sing}})$, then Properties 6.2 and 6.4 below imply that

$$\nu_{g,\varphi}(a;0,\varphi(a)) = \nu_g^0(a).$$

Finally, it follows from Property 6.1 and the expression (1.6) that the multiplicity $\nu_g^0(a)$ gives a Hilbert exponent for the ideal $\mathcal{I}(\mathfrak{S}_{U,Y})$. ∎

A reinterpretation of Tsikh's criterion for a minimal defining system ([Ts, pp. 119–120]) can be given as in the next corollary (where the proof offers an alternative verification of the second implication on p. 119 of [Ts]):

COROLLARY 3.1. *Let $X$ be a complex manifold of pure dimension $m>0$ and $S$ an analytic subset of pure codimension $q$ defined by a system $\{g_j\}_{1\le j\le q}\subset\mathcal{O}(X)$. Then the following conditions are equivalent*:

(i) *Every branch of $S$ contains a point $a\in S_{\mathrm{reg}}$ with $\nu_g^0(a)=1$ (where $g=(g_1,\dots,g_q)$).*
(ii) *The set $\mathcal{E}:=\{a\in S\mid \nu_g^0(a)>1\}$ is nowhere dense in $S$.*
(iii) *$\{g_j\}_{1\le j\le q}$ defines $S$ minimally.*

*Proof.* Assume first that the set $\mathcal{E}$ is nowhere dense in $S$. Then by Theorem 3.1, every point $a\in S$ admits a neighborhood $U$ such that the Hilbert relation (3.7) holds with the exponent $\mathfrak{h}=1$ for all $Q\in\mathcal{I}(\mathfrak{S}_{U,Y})$. In particular, if $a\in S$ and $f_a\in I_a(S_a)$, then $f_a\equiv 0$ $(\langle g_{1,a},\dots,g_{q,a}\rangle)$. Thus the set $\{g_j\}_{1\le j\le q}$ defines $S$ minimally.

Conversely, if the system $\{g_j\}_{1\le j\le q}$ defines $S$ minimally, then by Tsikh's criterion [Ts, p. 119], the set

$$\mathcal{T} := \{a \in S \mid (dg_1 \wedge \cdots \wedge dg_q)_a = 0\}$$

is nowhere dense in $S$. At every point $a \in S \setminus \mathcal{T}$ the map $g$ has Jacobian rank $q$, hence is locally (equivalent to) a projection at $a$ and $a \in S_{\mathrm{reg}}$. By Property 6.3, $\nu^0_g(a) = 1$ for all such $a$. Therefore $S \setminus \mathcal{T} = S \setminus \mathcal{E}$ and conditions (i) and (ii) hold. ∎

THEOREM 3.2. *If $\mathfrak{D}$ is a principal divisor in a normal space $X$ with defining equation $g(z) = 0$, then at each point $a \in S := \mathfrak{S}_{|\mathfrak{D}|}$, there is a neighborhood $U$ such that:*

(i) $\mathfrak{h}(g, a)$ *is given by formula* (1.9).
(ii) $\mathfrak{h}(g, a)$ *is equal to the smallest Hilbert exponent for the ideal* $\mathcal{I}(\mathfrak{S}_{U,Y})$.

*Proof.* Given a point $a \in S$, there is a neighborhood $U \subseteq X$ such that, for every branch $\mathfrak{B}_\kappa$ of $S_U$, $1 \le \kappa \le r$, there exists a holomorphic function $g_\kappa \in \mathcal{O}(U)$ with germ $(g_\kappa)_z$ generating the stalk of the ideal sheaf of $\mathfrak{B}_\kappa$ at every $z \in U$ ([GR$_2$, Theorem 5, p. 129]). By Corollary 3.1, one deduces that $\nu^0_{g_\kappa}(z) = 1$ for every $z \in (\mathfrak{B}_\kappa)_{\mathrm{reg}} \cap X_{\mathrm{reg}}$. The normality of $X$ implies that the same holds for all $z \in (\mathfrak{B}_\kappa)_{\mathrm{reg}}$. Hence by the divisibility Property 6.7, $g$ is divisible by $g_1^{\nu_1} \cdots g_r^{\nu_r}$ in $U$, where $\nu_\kappa$ is the multiplicity of $g$ at any point of $\mathfrak{B}_\kappa \cap (S_U)_{\mathrm{reg}}$. Hence one can write

$$g\rfloor U = u g_1^{\nu_1} \cdots g_r^{\nu_r}$$

for some $u \in \mathcal{O}^*(U)$. Similarly, if $G \in \mathcal{I}(\mathfrak{S}_{U,Y})$, then for each $y \in Y$,

$$G^{(y)}\rfloor U = G\rfloor\{y\} \times U = v^{(y)}\, g_1^{s_1} \cdots g_r^{s_r}$$

for some $v^{(y)} \in \mathcal{O}^*(U)$ and suitable integers $s_j > 0$. Thus, taking $\mathfrak{h} = \mathfrak{h}_S := \max\{\nu_\kappa \mid \kappa = 1, \dots, r\}$, one has

$$(G^{(y)})^{\mathfrak{h}}\rfloor U = (v^{(y)})^{\mathfrak{h}}\, g_1^{\mathfrak{h}\, s_1} \cdots g_r^{\mathfrak{h}\, s_r} = \tilde{v}^{(y)} g_1^{\nu_1} \cdots g_r^{\nu_r}$$

for some $\tilde{v}^{(y)} \in \mathcal{O}(U)$. Therefore $G^{(y)} \in \sqrt[\mathfrak{h}]{\langle g\rfloor U\rangle}$. The quotient $\lambda(y, z) := \tilde{v}^{(y)}(y, z)/u(z)$ is holomorphic in $Y \times U$ by the Hartog theorem ([GR$_1$, Satz 29]), and it satisfies the equation

$$G^{\mathfrak{h}}(y, z) = \lambda(y, z)\, g(z), \qquad (y, z) \in Y \times U. \tag{3.8}$$

Furthermore, the number $\mathfrak{h}$ is the smallest positive integer satisfying (3.8). Indeed, taking $l$ to be a positive integer less than $\mathfrak{h}_S$, the function $\psi := g_1 \cdots g_r$ vanishes on $S_U$ but the function $(\psi\rfloor U)^l$ is not divisible by $g\rfloor U$. Similarly the germ $(\psi^l)_a = (\psi_a)^l$ is not divisible by the germ $g_a$. Consequently, $\mathfrak{h}_S = \mathfrak{h}(g, a)$. ∎

If $S$ is a complete intersection of divisors in an affine algebraic variety, then the product subvariety $\mathfrak{S} = Y \times S$ admits an intrinsically determined

Hilbert exponent for $Y$-regular functions (cf. Płoski–Tworzewski [PT, Theorem 3.1]):

Theorem 3.3. *Let $V$ be an affine algebraic variety and $Y$ a complex space. Assume that $S$ is a complete intersection (of codimension $q$) of divisors $\mathfrak{D}_j$ in $V$ defined by regular functions $g_j$, $1 \leq j \leq q$. Then the integer*

$$\mathfrak{h}_S := \max\{\nu_g^0(c_\mu) \mid c_\mu \in S_{\mathrm{reg}} \cap \mathfrak{B}_\mu\}, \tag{3.9}$$

*where $g = (g_1, \ldots, g_q)$ and the maximum is taken over all branches $\mathfrak{B}_\mu$ of $S$, is a Hilbert exponent for the ideal $\mathcal{J}_Y(Y \times S)$.*

*Proof.* By Theorem 3.1, given $a_0 \in S$, there exists a neighborhood $U \subseteq V$ of $a_0$ such that, setting $\mathfrak{h}_{a_0} := \mathfrak{h}_{((\mathfrak{D}_j\rfloor U)_1 \cdots (\mathfrak{D}_p \rfloor U)_p), U}$, every element $f \in \mathcal{I}(\mathfrak{S}_{U,Y})$ satisfies a Hilbert relation

$$f^{\mathfrak{h}_{a_0}}(y,\zeta) = \alpha_1(y,\zeta)\, g_1(\zeta) + \cdots + \alpha_q(y,\zeta)\, g_q(\zeta), \qquad \forall (y,\zeta) \in Y \times U, \tag{3.10}$$

for suitable $\alpha_j \in \mathcal{O}(Y \times U)$, $1 \leq j \leq q$. This relation shows that

$$f^{\mathfrak{h}_{a_0}}{}_w \equiv 0 \ (\langle g_{1,w}, \ldots, g_{q,w} \rangle)$$

in $\mathcal{O}_{Y \times V, w}$ at every point $w = (y_0, a_0) \in Y \times S$. Since the function $\nu_g$ is locally constant on $S_{\mathrm{reg}}$ (Property 6.5 below), the integer $\mathfrak{h}_S$ given by (3.9) is well-defined and one has

$$\mathfrak{h}_{a_0} \leq \mathfrak{h}_S, \qquad \forall a_0 \in S_{\mathrm{reg}}.$$

This implies that

$$(f^{\mathfrak{h}_S})_w \equiv 0 \ (\langle g_{1,w}, \ldots, g_{q,w} \rangle)$$

in $\mathcal{O}_{Y \times V, w}$ at every point $w \in (Y \times S) \setminus \mathcal{A}$, where $\mathcal{A} := Y \times S_{\mathrm{sing}}$. Hence, if $f \in \mathcal{I}(Y \times S)$ is $Y$-regular, then it follows from the Noether stability of $\mathcal{O}_Y[V]$ that

$$f^{\mathfrak{h}_S} \equiv 0 \ (\langle g_1, \ldots, g_q \rangle_Y) \quad \text{in } \mathcal{O}_Y[V]. \ \blacksquare \tag{3.11}$$

Proposition 3.2. *Let $V$, $Y$ and $S$ be the same as in Theorem 3.3. If either $V$ is irreducible and $S$ is minimally defined by $\mathfrak{F} = \{g_j\}_{1 \leq j \leq q}$ or $Y$ is a Stein space and $\mathfrak{F}$ generates an ideal in $\mathcal{O}(Y \times V)$ equal to an intersection of prime ideals, then $\mathcal{J}_Y(Y \times S) = \langle g_1, \ldots, g_q \rangle_Y$.*

*Proof.* Assume first that $V$ is irreducible and $S$ is minimally defined by $\mathfrak{F}$. Then by Corollary 3.1, taking $a_0$ to be a point of $S_{\mathrm{reg}}$, we have (3.10) (with $\mathfrak{h}_{a_0} = \nu_g^0(a_0) = 1$), hence also (3.11) (with $\mathfrak{h}_S = 1$). This proves that $\mathcal{J}_Y(Y \times S) = \langle g_1, \ldots, g_q \rangle_Y$.

In the remaining case the desired conclusion follows from the same argument as that for the corresponding assertion in Theorem 2.3. ∎

If $Y_0 \subseteq Y$ and $\mathfrak{S} \subset Y \times \mathbb{C}^N$ (respectively, $\mathfrak{S} \subset Y \times \mathbb{P}^N(\mathbb{C})$), set $\mathfrak{S}_{|Y_0} := \mathfrak{S} \cap (Y_0 \times \mathbb{C}^N)$ (respectively, $\mathfrak{S}_{|Y_0} := \mathfrak{S} \cap (Y_0 \times \mathbb{P}^N(\mathbb{C}))$).

PROPOSITION 3.3. *Let $\mathfrak{D}$ be a principal divisor in $Y \times \mathbb{C}^N$ defined by a primitive element $g \in \mathcal{O}_Y[X_1, \ldots, X_N]$ over a normal, irreducible space $Y$. Assume that at every point of $Y$ off an almost thin subset $\mathcal{T}$ of codimension* 2 ([AS, p. 14]), *there is a Stein neighborhood $U$ such that $g$ has simple, irreducible factors in $\mathcal{O}_U[X_1, \ldots, X_N]$. Then $\mathcal{J}_Y(\mathfrak{S}_{|\mathfrak{D}|}) = \langle g \rangle_Y$.*

*Proof.* Let $\mathfrak{S} := \mathfrak{S}_{|\mathfrak{D}|}$. If $\mathfrak{S} = \emptyset$, then we have $1 = u\, g$, where $u = (1/g) \in \mathcal{O}(Y \times \mathbb{C}^N)$. The subalgebra $\mathcal{O}_Y[X_1, \ldots, X_N]$ being strictly stable in $\mathcal{O}(Y \times \mathbb{C}^N)$, the quotient function $u$ actually belongs to $\mathcal{O}_Y[X_1, \ldots, X_N]$. Hence one has $\langle g \rangle_Y = \mathcal{O}_Y[X_1, \ldots, X_N]$. If $\mathfrak{Z} \neq \emptyset$, then $\deg(g) > 0$. Choose a Stein neighborhood $U \subseteq Y \setminus \mathcal{T}$ such that $g$ has simple, irreducible factors in $\mathcal{O}_U[X_1, \ldots, X_N]$. By the principal ideal theorem for relative hypersurfaces ([Tu$_2$, Theorem 4.2(2)]), the subvariety $\mathfrak{S}_{|U}$ is minimally defined by an equation $g_1 \cdots g_r = 0$ where each $g_j \in \mathcal{O}_U[X_1, \ldots, X_N]$ has positive degree and the set $\{\mathcal{V}(g_j)\}_{1 \leq j \leq r}$ gives all branches of $\mathfrak{S}_{|U}$. Moreover, by Properties 6.5 and 6.7,

$$g = u(\pi^* \phi)\, g_1^{\lambda_1} \cdots g_r^{\lambda_r}$$

for some $u \in \mathcal{O}^*(U \times \mathbb{C}^N)$, $\phi \in \mathcal{O}(U)$, and suitable positive integers $\lambda_j$. Since $g$ is primitive over $U$ and has simple irreducible factors in $\mathcal{O}_U[X_1, \ldots, X_N]$, the function $\phi$ is nonvanishing in $U$ and each $\lambda_j$ equals 1. As above, the strict stability of the subalgebra $\mathcal{O}_U[X_1, \ldots, X_N]$ implies that $g = v\, g_1 \cdots g_r$ for some $v \in \mathcal{O}_U[X_1, \ldots, X_N]$. Let $Q \in \mathcal{J}_Y(\mathfrak{S}_{|\mathfrak{D}|})$. Then by Corollary 3.1 and Theorem 3.2, the quotient $h := Q/g$ is holomorphic in $U \times \mathbb{C}^N$. It follows that $h$ is holomorphic in $(Y \setminus \mathcal{T}) \times \mathbb{C}^N$, hence (by normality of $Y$) also in $Y \times \mathbb{C}^N$. Therefore (as in the preceding) $h \in \mathcal{O}_Y[X_1, \ldots, X_N]$, thereby proving the desired conclusion. ∎

**4. Nullstellensatz for relative projective varieties.** An element $g$ in $\mathcal{O}_Y[X_1, \ldots, X_N]$ (or $\mathcal{P}ol_{Y,N}$) of (generic) positive degree is said to be *irreducible at* $y_0 \in Y$ if there exists a Stein neighborhood $U$ such that $g$ is irreducible in $\mathcal{O}_U[X_1, \ldots, X_N]$. To each element $g \in \mathcal{P}ol_{Y,N+1}$ is associated a subset $\mathcal{V}(g)$ in $Y \times \mathbb{P}^N(\mathbb{C})$:

$$\mathcal{V}(g) := \{(y, [z_0, \ldots, z_N]) \in Y \times \mathbb{P}^N(\mathbb{C}) \mid g^{(y)}(z_0, \ldots, z_N) = 0\}.$$

A subset $\mathfrak{S} \subseteq Y \times \mathbb{P}^N(\mathbb{C})$ is called a *relative algebraic set* (over $Y$) if at each point of $Y$ there exist an open neighborhood $Y_0$ and (finitely many) elements $g_j \in \mathcal{P}ol_{Y_0,N+1}$, $1 \leq j \leq p$, such that the restriction $\mathfrak{S}_{|Y_0}$ is given by the common zero set $\mathcal{V}(g_1, \ldots, g_p) := \mathcal{V}(g_1) \cap \cdots \cap \mathcal{V}(g_p)$. The relative Chow Theorem asserts that *a subset $\mathfrak{S} \subseteq Y \times \mathbb{P}^N(\mathbb{C})$ is relative algebraic* *over $Y$ if and only if it is analytic in $Y \times \mathbb{P}^N(\mathbb{C})$* (for a proof see Fischer [Fi, 4.3] or [Tu$_2$, Theorem 3.1]).

LEMMA 4.1. *Let $Y$ be irreducible and $G \in \mathcal{O}_Y[X_1, \ldots, X_N]$ (respectively, $G \in \mathcal{P}ol_{Y,N+1}$) be primitive over $Y$. If $G$ is irreducible at some point of $Y_{\mathrm{reg}}$, then $\mathcal{V}(G)$ is an irreducible hypersurface in $Y \times \mathbb{C}^N$ (respectively, $Y \times \mathbb{P}^N(\mathbb{C})$).*

*Proof.* Observe that if $g \in \mathcal{O}_Y[X_1, \ldots, X_N]$, the irreducibility of the affine variety $\mathcal{V}(g)$ is equivalent to that of its completion $\mathcal{V}({}^h g)$ in $\mathbb{P}^N(\mathbb{C})$. Also, an element $G \in \mathcal{P}ol_{Y,N+1}$ is reducible in $\mathcal{O}_U[X_0, \ldots, X_N]$ for some open subset $U \subseteq Y_{\mathrm{reg}}$ if and only if so is $g := {}^a G$ in $\mathcal{O}_U[X_1, \ldots, X_N]$. Therefore to prove the lemma it suffices to consider the case of a primitive element $g \in \mathcal{O}_Y[X_1, \ldots, X_N]$. Let $\mathfrak{S}_j$, $1 \le j \le r$, be the irreducible components of $\mathfrak{S} := \mathcal{V}(g)$. Assume that there exists a Stein neighborhood $U \subseteq Y_{\mathrm{reg}}$ over which $g$ is irreducible. Then (as in the proof of Proposition 3.3) the restriction $(\mathfrak{S}_j)_{|U}$ is minimally defined by an equation $g_j = 0$ of positive degree, for every $j$. Hence it follows from Corollary 3.1 and Theorem 3.2 that the quotient $u = g/g_1 \cdots g_r$ is holomorphic in $U \times \mathbb{C}^N$. Therefore, by Theorem 2.1, if $r > 1$, then $g$ is reducible in $\mathcal{O}_U[X_1, \ldots, X_N]$, hence a contradiction. This proves that $\mathfrak{S} = \mathfrak{S}_1$, so it is an irreducible variety. ∎

Let $U^{\{k\}} := \{a = [a_0, \ldots, a_N] \mid a_k \neq 0\}$ be the $k$th canonical chart and $\alpha^{[k]}$ the associated coordinate map on $U^{\{k\}}$ given by $\alpha^{[k]} : a \mapsto a^{[k]} := \left(\frac{a_0}{a_k}, \ldots, \widehat{\frac{a_k}{a_k}}, \ldots, \frac{a_N}{a_k}\right)$ (here "$\widehat{\ }$" denotes omission). The *$k$th dehomogenization* (for $1 \le k \le N$) of an element $g \in \mathcal{P}ol_{Y,N+1}$ is the holomorphic function $g^{[k]} : Y \times \mathbb{C}^N \to \mathbb{C}$ defined by $(y, \zeta) \mapsto g^{[k],\{y\}}(\zeta)$, where

$$g^{[k],\{y\}}(\zeta_0, \ldots, \zeta_{k-1}, \zeta_{k+1}, \ldots, \zeta_N) := g^{\{y\}}(\zeta_0, \ldots, \zeta_{k-1}, 1, \zeta_{k+1}, \ldots, \zeta_N);$$

the dehomogenization ${}^a g = g^{[0]}$ is similarly defined. Setting $\tilde{g}^{[k]} := g^{[k]} \circ (\mathrm{id}_Y, \alpha^{[k]})$, the system $\{(Y \times U^{\{k\}}, \tilde{g}^{[k]})\}_{0 \le k \le N}$ defines a divisor in $Y \times \mathbb{P}^N(\mathbb{C})$, called the *principal divisor associated to $g$*.

If $g_j \in \mathcal{P}ol_{Y_0,N+1}$, $1 \le j \le p$, and $\mathfrak{S} := \mathcal{V}(g_1, \ldots, g_p)$, then the set $\mathfrak{I}_Y(\mathfrak{S})$ of all elements $f \in \mathcal{P}ol_{Y,N+1}$ vanishing on $\mathfrak{S}$ is a homogeneous ideal in $\mathcal{O}_Y[X_0, \ldots, X_N]$, as can be seen as follows: if $f \in \mathfrak{I}_Y(\mathfrak{S})$, and $w = (y, a) \in \mathfrak{S}$, then $f^{(y)}$ vanishes at all homogeneous coordinates of $a$. It follows that all homogeneous components $f_j^{(y)}$ of $f^{(y)}$ vanish at $a$. Thus every such component belongs to $\mathfrak{I}_Y(\mathfrak{S})$, proving that $\mathfrak{I}_Y(\mathfrak{S})$ is homogeneous. A standard argument shows that a nonvoid relative algebraic set $\mathfrak{S}$ is irreducible in $Y \times \mathbb{P}^N(\mathbb{C})$ if and only if $\mathfrak{I}_Y(\mathfrak{S})$ is a prime ideal in $\mathcal{O}_Y[X_0, \ldots, X_N]$. It is easy to see that if $\mathfrak{I}_Y(\mathfrak{S}) = \langle P_1, \ldots, P_r \rangle_Y$ for some $P_j \in \mathcal{P}ol_{Y,N+1}$, $1 \le j \le r$, then $\mathfrak{S} = \mathcal{V}(P_1, \ldots, P_r)$. The precise determination of the ideal $\mathfrak{I}_Y(\mathfrak{S})$ for a given relative projective variety $\mathfrak{S}$ will be considered in the rest of this paper in several cases.

Proposition 4.1. *Let $\mathfrak{D}$ be a principal divisor in $Y \times \mathbb{P}^N(\mathbb{C})$ associated to a primitive element $G \in \mathcal{P}ol_{Y,N+1}$ over a normal, irreducible space $Y$, and $\mathfrak{S} = \mathfrak{S}_{|\mathfrak{D}|}$. If at every point of $Y$ off an almost thin subset $\mathcal{T}$ of codimension 2, there is a neighborhood $U$ such that $G$ has simple, irreducible factors in $\mathcal{O}_U[X_0, \ldots, X_N]$, then $\mathfrak{I}_Y(\mathfrak{S}) = \langle G \rangle_Y$.*

*Proof.* The pseudopolynomial $f = {}^aG$ is primitive over $Y$ with $\mathcal{V}(f) \neq \emptyset$. Let $U \subseteq Y \setminus \mathcal{T}$ be a Stein neighborhood over which $G$ is irreducible. Then $f$ has simple, irreducible factors in $\mathcal{O}_U[X_1, \ldots, X_N]$ (see Lemma 6.1 below). Hence by Lemma 4.1, ${}^a\mathfrak{S} = \mathcal{V}(f)$ is a union of irreducible branches of codimension 1 in $Y \times \mathbb{C}^N$. Consequently, the same is true for $\mathfrak{S} = \mathcal{V}(G)$ in $Y \times \mathbb{P}^N(\mathbb{C})$. Observe that by Proposition 3.3, $\mathcal{J}_Y({}^a\mathfrak{S}) = \langle f \rangle$. If $Q \in \mathfrak{I}_Y(\mathfrak{S})$, then ${}^aQ \in \mathcal{J}_Y({}^a\mathfrak{S})$, hence ${}^aQ = uf$ for some $u \in \mathcal{O}_Y[X_1, \ldots, X_N]$. Therefore $Q = ({}^hu)\, G \in \langle G \rangle_Y$. ∎

According to Gunning [Gu, Theorem 2, p. 42], if in a product space $W \times \mathbb{C}$, where $W$ is an open subset of $\mathbb{C}^N$, a subvariety $\mathcal{S}$ is realizable as a finite branched analytic covering of $W$ under the natural projection $W \times \mathbb{C} \to W$, then a set of global generators can be constructed for the ideal $\mathcal{I}(\mathcal{S})$. A generalization of this assertion to the case of a divisor in a vector bundle on a Stein manifold is given in [Tu$_2$, 4.2(2)]. By virtue of the latter and Lemma 6.1, conditions under which the associated ideal of a divisor in $Y \times \mathbb{P}^N(\mathbb{C})$ is principal can be ascertained:

Theorem 4.1. *Assume that $Y$ is a connected Stein manifold (of dimension $n \geq 0$) with $H^2(Y, \mathbb{Z}) = 0$, and $\mathfrak{D}$ an (arbitrary effective) divisor in $Y \times \mathbb{P}^N(\mathbb{C})$ with $\mathfrak{S}_{|\mathfrak{D}|} \cap (Y \times \mathbb{C}^N) \neq \emptyset$. Then:*

- (i) *$\mathfrak{D}$ is a principal divisor associated to an element $f \in \mathcal{P}ol_{Y,N+1}$.*
- (ii) *$\mathfrak{S}_{|\mathfrak{D}|}$ consists of finitely many branches $\mathfrak{S}_j$, $1 \leq j \leq l$, each being defined minimally by an equation $g_j = 0$ for some irreducible $g_j \in \mathcal{P}ol_{Y,N+1}$.*
- (iii) *$\mathfrak{I}_Y(\mathfrak{S}_{|\mathfrak{D}|}) = \langle g_1 \ldots g_l \rangle_Y$, where, if $\deg(g_j) = 0$, then $\mathfrak{S}_j = \phi_j^{-1}(0) \times \mathbb{P}^N(\mathbb{C})$ for some irreducible $\phi_j \in \mathcal{O}(Y)$.*

Theorem 4.2 (Relative projective Nullstellensatz; cf. [ZS, Theorem 15, pp. 171–172], [CLO, Theorem 9, p. 384]). *Let $Y$ be a connected complex space and $\mathfrak{D}_j$ a principal divisor in $Y \times \mathbb{P}^N(\mathbb{C})$ associated to $g_j \in \mathcal{P}ol_{Y,N+1}(l_j)$, for $j = 1, \ldots, p$, and $\mathfrak{S} := \bigcap_{j=1}^p \mathfrak{S}_{|\mathfrak{D}_j|}$.*

(1) *If $p \leq N$, then the restriction $\mathfrak{S}_{|\{y\}}$ is not empty for all $y \in Y$ and there is a positive integer $\mathfrak{h}$ such that*

$$\mathfrak{I}_Y(\mathfrak{S}) = \sqrt[\mathfrak{h}]{\langle g_1, \ldots, g_p \rangle_Y}. \tag{4.1}$$

(2) *If $p > N$ and if $\mathfrak{S}$ is not empty, then the relation* (4.1) *remains valid for $\mathfrak{S}$.*

*Proof.* Let $\mathcal{C}(\mathfrak{S}) \subseteq Y \times \mathbb{C}^{N+1}$ be the affine cone of $\mathfrak{S}$. If $p \leq N$, the restriction $\mathfrak{S}_{|\{y_0\}}$ is nonempty for all $y_0 \in Y$, by a result of Lang [La, p. 43, Corollary]. Thus, according to [CLO, p. 384, (2)], the ideal $\mathfrak{I}_Y(\mathfrak{S})$ coincides with that (in $\mathcal{O}_Y[X_0, \ldots, X_N]$) associated to the analytic cone $\mathcal{C}(\mathfrak{S})$. Therefore Theorem 2.3 implies (by resorting to homogeneous expansions) that there exists a positive integer $\mathfrak{h}$ with the following property: for each $y_0 \in Y$ there is a neighborhood $Y_0$ such that every element $G \in \mathfrak{I}_Y(\mathfrak{S}) \cap \mathcal{P}ol_{Y,N+1}(d)$ satisfies the equation

$$G^{\mathfrak{h}} = \lambda_1 g_1 + \cdots + \lambda_q g_q \quad \text{in } Y_0 \times \mathbb{C}^{N+1}$$

for some $\lambda_j \in \mathcal{P}ol_{Y_0,N+1}(\mathfrak{h}d - l_j)$, $1 \leq j \leq p$. This proves (4.1). ∎

In the following, let $\mathfrak{D}_j$ be a divisor in $\mathbb{P}^N(\mathbb{C})$ with support $\mathfrak{F}_j$, for $1 \leq j \leq q \leq N$. By Theorem 4.1, each $\mathfrak{D}_j$ admits a global defining equation $g_j(z_0, \ldots, z_N) = 0$ for some $g_j \in \mathcal{P}ol_{N+1}$. Also, by Theorem 4.2, the projective variety $\mathfrak{F} := \bigcap_{j=1}^q \mathfrak{F}_j$ is nonempty. If $\mathfrak{F}$ has codimension $q$ at $a$, define the *intersection number* of the divisors $\mathfrak{D}_j$ at $a \in \mathfrak{F} \cap U^{\{k\}}$ by

$$(4.2) \qquad (\mathfrak{D}_1 \cdots \mathfrak{D}_q)_a := \nu^0((\tilde{g}_1^{[k]}, \ldots, \tilde{g}_q^{[k]}), a^{[k]})$$

(cf. [AY, p. 180]). Observe that, by the invariance of multiplicity under an invertible holomorphic matrix transformation ([$\mathrm{St}_2$, Theorem 6.1]), the intersection number (4.2) is a positive integer intrinsically determined by the divisors $\mathfrak{D}_j$ (independent of their local representations).

A projective variety $\mathfrak{F}$ of pure codimension $q > 0$ in $\mathbb{P}^N(\mathbb{C})$ is called a *complete intersection* if there exist (effective) divisors $\mathfrak{D}_1, \ldots, \mathfrak{D}_q$ such that $\mathfrak{F} = \bigcap_{j=1}^q \mathfrak{S}_{|\mathfrak{D}_j|}$. Thus, for such a complete intersection one may choose for each $\mathfrak{D}_j$ a defining homogeneous polynomial $f_j$. The integer

$$(4.3) \qquad \mathfrak{h}_{\{\mathfrak{D}_1,\ldots,\mathfrak{D}_q\}} := \max\{(\mathfrak{D}_1 \cdots \mathfrak{D}_q)_{c_\mu} \mid c_\mu \in \mathfrak{B}_\mu \cap \mathfrak{F}_{\mathrm{reg}}\},$$

where the maximum is taken over all branches $\mathfrak{B}_\mu$ of $\mathfrak{F}$, is intrinsically determined. For a system $\mathcal{F} = \{f_j\}_{1 \leq j \leq q}$ (as above) defining a complete intersection, also set

$$(4.4) \qquad \mathfrak{h}_{\{\mathcal{F}\}} := \max\Big(\mathfrak{h}_{\{\mathfrak{D}_1,\ldots,\mathfrak{D}_q\}}, \max_{1 \leq j,\, k \leq q}\{m_k(f_j)\}\Big),$$

where $m_k(P)$ denotes the highest power such that $z_k^{m_k(P)}$ divides $P$. The system $\mathcal{F}$ is said to be *minimal* for $\mathfrak{F}$ if $\mathfrak{h}_{\{\mathcal{F}\}} = 1$.

THEOREM 4.3 (Hilbert exponent for a projective complete intersection). *Assume that:*

(i) $\mathfrak{F}$ *is a complete intersection of divisors in* $\mathbb{P}^N(\mathbb{C})$ *defined by* $\mathcal{F} = \{f_j\}_{1\le j\le q}$, *and*

(ii) $\mathfrak{F} \cap \{z = [z_0, \dots, z_N] \mid z_k = 0\}$ *is thin in* $\mathfrak{F}$ *for each* $k \in \mathbb{Z}[0, N]$.

*Then for any complex space* $Y$*:*

(a) *Every nonzero element* $G \in \mathcal{P}ol_{Y,N+1}(d) \cap \mathfrak{I}_Y(Y \times \mathfrak{F})$ *satisfies the equation*

$$G^{\mathfrak{h}_{\{\mathcal{F}\}}} = \lambda_1 f_1 + \cdots + \lambda_q f_q \quad \text{in } Y_0 \times \mathbb{C}^{N+1} \tag{4.5}$$

*for some* $\lambda_j \in \mathcal{P}ol_{Y_0,N+1}(\mathfrak{h}_{\{\mathcal{F}\}} d - l_j)$, $1 \le j \le q$.

(b) *If either* $\langle f_1, \dots, f_q\rangle_Y$ *is a prime ideal or the system* $\mathcal{F}$ *defines* $\mathfrak{F}$ *minimally, then* $\mathfrak{I}_Y(Y \times \mathfrak{F}) = \langle f_1, \dots, f_q\rangle_Y$.

*Proof.* Let $G \in \mathcal{P}ol_{Y,N+1}(d) \cap \mathfrak{I}_Y(Y \times \mathfrak{F})$ be given. Choose $k \in \mathbb{Z}[0, N]$ such that $m_k(G) > 0$ and $\mathfrak{F}$ contains a point $a \in U^{\{k\}}$. By Theorem 3.1(ii), given $y_0 \in Y$, there exists a product neighborhood $Y_0 \times \varDelta \subseteq Y \times \mathbb{C}^N$ of $(y_0, a^{[k]})$ such that, for an exponent $\mathfrak{h} \ge \mathfrak{h}_{\{\mathfrak{D}_1,\dots,\mathfrak{D}_q\}}$, the equation

$$(G^{[k]}(y, \zeta))^{\mathfrak{h}} = \sum_{j=1}^{q} \alpha_j^{(y)}(\zeta) f_j^{[k]}(\zeta), \quad \forall (y, \zeta) \in Y_0 \times \varDelta, \tag{4.6}$$

holds for some $\alpha_j \in \mathcal{O}(Y_0 \times \varDelta)$, $1 \le j \le q$. The subalgebra $\mathcal{O}_Y[X_1, \dots, X_N]$ of pseudopolynomials being Noether-stable in $\mathcal{O}(Y \times \mathbb{C}^N)$, the coefficient functions $\alpha_j$ in (4.6) may be chosen to be elements of $\mathcal{P}ol_{Y_0,N}$. Hence by the identity theorem for holomorphic functions, the relation (4.6) remains valid for all $(y, \zeta') \in Y \times \mathbb{C}^N$. If we denote by $d(P)$ the degree, respectively, "${}^hP_j$" the $k$th homogenization, of a nonzero polynomial $P_j \in \mathcal{O}[X_1, \dots, X_N]$, then the identity

$$X_k^{d(P_1)+d(P_2)}\, {}^h(P_1 + P_2) = X_k^{d(P_1+P_2)}[X_k^{d(P_2)}\, {}^hP_1 + X_k^{d(P_1)}\, {}^hP_2]$$

holds for all such $P_j$, $j = 1, 2$ ([ZS, (3), p. 179]). By use of this formula it can be shown that the relation (4.6) implies that

$$[({}^hG^{[k]})^{(y)}(z_0, \dots, z_N)]^{\mathfrak{h}} = \sum_{j=1}^{q} \hat{\lambda}_j^{(y)}(z_0, \dots, z_N)\, {}^h(f_j^{[k]})(z_0, \dots, z_N)$$

for suitable pseudopolynomials $\hat{\lambda}_j \in \mathcal{P}ol_{Y_0,N+1}$, $1 \le j \le q$, for all $(y, z) \in Y_0 \times \{z \in \mathbb{C}^{N+1} \mid z_k \ne 0\}$. Thus for such $(y, z)$, the identity

$$X_k{}^{-m_k(P)}\, P = {}^h(P^{(k)})$$

([ZS, (5′), p. 180]) implies that

$$z_k^{-m_k(G)\mathfrak{h}}(G^{(y)}(z_0, \dots, z_N))^{\mathfrak{h}} = \sum_{j=1}^{q} \hat{\lambda}_j^{(y)}(z_0, \dots, z_N)\, z_k^{-m_k(f_j)} f_j(z_0, \dots, z_N).$$

Therefore, choosing $\mathfrak{h} = \mathfrak{h}_{\{\mathcal{F}\}}$ as in (4.4), one has

$$(G^{(y)}(z_0,\dots,z_N))^{\mathfrak{h}_{\{\mathcal{F}\}}} = \sum_{j=1}^{q} \hat{\lambda}_j^{(y)}(z_0,\dots,z_N)\, z_k^{\mu_{k,j}} f_j(z_0,\dots,z_N),$$

where the integer $\mu_{k,j} := m_k(G)\mathfrak{h}_{\{\mathcal{F}\}} - m_k(f_j)$ is nonnegative. Hence (by continuity) the global relation (4.5) follows.

If $\langle f_1,\dots,f_q\rangle_Y$ is a prime ideal, then the above relation implies that $G \in \langle f_1,\dots,f_q\rangle_Y$. Clearly, if $\mathfrak{h}_{\{\mathcal{F}\}} = 1$, then the same conclusion holds for $\mathfrak{F}$. Thus assertion (b) is proved. ∎

A consequence of the above theorem and the Bézout property of projective hypersurfaces ([Tu$_3$, Corollary 4.1]) is the following:

Corollary 4.1. *Let $\mathfrak{F}_j(c)$ be a hypersurface in $\mathbb{P}^N(\mathbb{C})$ defined by a homogeneous equation $f_j(z;c) = 0$, where $c$ varies in a locally connected Hausdorff space $W$. Assume that for some $c = c^* \in W$, the set $\mathfrak{F}(c) := \bigcap_{j=1}^N \mathfrak{F}_j(c)$ does not meet any of the hyperplanes $\{z = [z_0,\dots,z_N] \mid z_k = 0\}$ for $0 \le k \le N$. Then for all $c \in W$ sufficiently close to $c^*$, the system $\mathcal{F}(c) = \{f_j(z;c)\}_{1\le j\le N}$ defines a finite intersection $\mathfrak{F}(c)$ with*

$$\mathfrak{h}_{\{\mathcal{F}(c)\}} \le \deg(f_1)\cdots\deg(f_N).$$

**5. Gauss decomposition of pseudopolynomials.** In this section, let $(X,p)$ be a semi-Riemann domain of dimension $m > 0$ (see [Tu$_4$, §2 and §3] for notation), $a$ an arbitrary point of $X$, and $Y$ an irreducible complex space. A (holomorphic) *strictly $a$-homogeneous pseudopolynomial on $X$ over $Y$* (of degree $d$) is a holomorphic function $\psi : Y \times X \to \mathbb{C}$ such that

$$\psi^{(y)} = G^{(y)} \circ p^{[a]}$$

for some $G \in \mathcal{P}ol_{Y,N}(d)$, where $N = 2m$, and $p^{[a]} : X \to \mathbb{C}^m \equiv \mathbb{R}^{2m}$ has components $\zeta_{2j-1}^{[a]} := \tilde{x}_j - \tilde{x}_j(a)$, $\zeta_{2j}^{[a]} := \tilde{y}_j - \tilde{y}_j(a)$, $1 \le j \le m$; call such $G$ a *strict push-forward* of $\psi$. Denote by $\mathcal{P}ol_{Y,N}(a,d)$ the set of all such elements $\psi$ of degree $d$. In particular, an element $\psi_a^{(k,\xi)} \in \mathcal{P}ol_{Y,N}(a,k)$, where $k \ge 1$, with strict push-forward

$$F^{(k,\xi)}(X_1,\dots,X_N) := (\xi_1 X_1 + \dots + \xi_N X_N)^k, \qquad \xi = (\xi_1,\dots,\xi_N) \in \mathbb{C}^N,$$

is called an *elementary $a$-pseudospherical harmonics* (in view of Proposition 3.1 and Example 3.2 of [Tu$_4$, §3]). To characterize the submodule over $\mathcal{O}(Y)$ generated by such elements, it is helpful to introduce (as follows) a differential operator associated to each element $\psi \in \mathcal{P}ol_{Y,N}(a,d)$.

For any $G \in \mathcal{P}ol_{Y,N}(d)$ of positive degree, substituting $\bar{a}_\mu$ for each $a_\mu$ in the expression (2.1) defines an element $\hat{G} \in \mathcal{P}ol_{Y,N}(d)$. Given $\psi \in \mathcal{P}ol_{Y,N}(a,d)$ with strict push-forward $G$, by substituting further (in (2.1)) the symbolic operator $\partial/\partial X_j$ for each variable $X_j$ (holding $y$ fixed in $Y \setminus \Delta(\psi)$), a linear operator $\hat{G}[D] : \mathcal{P}ol_Y(k) \to \mathcal{P}ol_Y(k-d)$ is defined, hence also a linear operator $\hat{\psi}^{[a]}[D] : \mathcal{P}ol_{Y,N}(a,k) \to \mathcal{P}ol_{Y,N}(a,k-d)$, for integers $k$, $d$ with $0 < d \leq k$, with the property that

$$\hat{\psi}^{[a]}[D]\eta = (\hat{G}[D]H) \circ p^{[a]},$$

where $H$ is a strict push-forward of $\eta$. A bilinear mapping

$$\mathcal{P}ol_{Y,N}(a,d) \times \mathcal{P}ol_{Y,N}(a,k) \to \mathcal{P}ol_{Y,N}(a,k-d)$$

(between abelian groups) is given by the rule

$$\langle \psi, \eta \rangle := \hat{\psi}^{[a]}[D]\eta. \tag{5.1}$$

Note that when applied to the space $\mathcal{P}ol_N(k) \times \mathcal{P}ol_N(k)$, this bilinear map defines a Hermitian symmetric scalar product. Also, for every element $Q \in \mathcal{P}ol_N(d)$, $1 \leq d \leq k$, the definition (5.1) implies that

(5.2)
$$\hat{Q}[D]F^{(k,\xi)}(z_1,\ldots,z_N) = k(k-1)(k-d+1)\,\hat{Q}(\xi)F^{(k-d,\xi)}(z_1,\ldots,z_N).$$

If $\psi \in \mathcal{P}ol_{Y,N}(a,d)$, $\eta \in \mathcal{P}ol_{Y,N}(a,k)$ and $\xi \in \mathcal{P}ol_{Y,N}(a,k-d)$, then the following adjoint formula holds:

$$\langle \hat{\psi}^{[a]}[D]\eta, \xi \rangle = \langle \eta, \psi\xi \rangle \tag{5.3}$$

(cf. [H, p. 30]). This formula and Proposition 4.1 give rise to a generalization of the *Gauss decomposition* rule:

Proposition 5.1.

(1) *If* $\mathfrak{M} \subseteq \mathcal{P}ol_{Y,N}(a,k)$ *is a submodule over* $\mathcal{O}(Y)$ *and* $g \in \mathcal{P}ol_{Y,N}(a,d)$ *with* $0 < d < k$, *then*

$$\mathfrak{M} = \ker(\hat{g}^{[a]}[D]\lfloor\mathfrak{M}) \oplus g \cdot \Im(\hat{g}^{[a]}[D]\lfloor\mathfrak{M}); \tag{5.4}$$

*in particular,*

$$\mathcal{P}ol_{Y,N}(a,k) = \ker(\hat{g}^{[a]}[D]) \oplus g \cdot \mathcal{P}ol_{Y,N}(a,k-d). \tag{5.5}$$

(2) *For each* $\psi \in \mathcal{P}ol_{Y,N}(a,k)$,

$$\psi = \sum_{j=0}^{[k/d]} g^j\, \mathcal{Z}_{k-jd} \tag{5.6}$$

*where* $\mathcal{Z}_j \in \ker(\hat{g}^{[a]}[D]) \cap \mathcal{P}ol_{Y,N}(a,j)$ *and* $\mathcal{Z}_j$ *is not divisible by* $g$.

*Proof.* Observe that if $\psi = g\eta$, where $\eta \in \mathcal{P}ol_{Y,N}(a,k-d)$, then

$$\langle \psi,\, \psi \rangle = \hat{\psi}[D]\,\psi = \ (\hat{\eta}[D]\hat{g}^{[a]}[D])\,\psi = \langle \eta,\, \hat{g}^{[a]}[D]\,\psi \rangle.$$

Suppose that $\psi \in \ker(\hat{g}^{[a]}[D]\lfloor\mathfrak{M}) \cap g \cdot \Im(\hat{g}^{[a]}[D]\rfloor\mathfrak{M})$. Then $\psi = g\eta$ for some $\eta \in \Im(\hat{g}^{[a]}[D]\lfloor\mathfrak{M})$, and consequently $\langle \psi,\, \psi \rangle = 0$, whence $\psi = 0$. This proves that the kernel space of $\hat{g}^{[a]}[D]\lfloor\mathfrak{M}$ and the space $g\cdot\Im(\hat{g}^{[a]}[D]\lfloor\mathfrak{M})$ have trivial intersection, hence are of complementary dimensions in $\mathfrak{M}$. From this the relation (5.4) follows.

By virtue of the adjoint formula (5.3), the mapping $\hat{g}^{[a]}[D] : \mathcal{P}ol_{Y,N}(a,k) \to \mathcal{P}ol_{Y,N}(a,k-d)$ is surjective. Hence the decomposition (5.4) implies that the representation (5.5) holds. By iteration of this formula it is easy to show that every element $\psi \in \mathcal{P}ol_{Y,N}(a,k)$ admits (for each $d \in \mathbb{Z}(0,k)$) the Gauss decomposition (5.6). ■

Given $c = (c^1, \ldots, c^q)$ with $c^j = (c^j_1, \ldots, c^j_N) \in \mathbb{C}^N$, $1 \le j \le q$, define the *Fermat variety* $\mathfrak{F}^{\{d,c\}} := \mathcal{V}(F^{\{d,c^1\}}, \ldots, F^{\{d,c^q\}})$, where

$$F^{\{d,c^j\}} := c^j_1 X^d_1 + \cdots + c^j_N X^d_N.$$

The *solid a-pseudospherical harmonics over* $Y$ (of type $\{d,c\}$ and degree $k \ge 1$) are the members of the submodule (over $\mathcal{O}(Y)$)

$$\mathfrak{H}^{\{d,c\}}_Y(a,k) := \{\mathcal{Z} \in \mathcal{P}ol_{Y,N}(a,k) \mid \langle \psi, \mathcal{Z}\rangle = 0,\ \forall \psi \in \langle \Psi_{a,\{1\}}, \ldots, \Psi_{a,\{q\}}\rangle_Y\},$$

where $\Psi_{a,\{j\}} = \Psi^{\{d,c^j\}}_a := F^{\{d,c^j\}} \circ p^{[a]}$.

PROPOSITION 5.2. *Assume that $\mathfrak{F}$ is a complete intersection of Fermat divisors in $\mathbb{P}^{N-1}(\mathbb{C})$ of codimension $q$ defined by a system $\mathcal{F} = \{F^{\{d,c^j\}}\}_{1\le j\le q}$, and one of the following conditions holds:*

(i) *$\mathcal{F}$ defines $\mathfrak{F}$ minimally,*
(ii) *$\langle F_{\{1\}}, \ldots, F_{\{q\}}\rangle_Y$, where $F_{\{j\}} = F^{\{d,c^j\}}$, is a prime ideal,*
(iii) *$N > 2,\ q = 1,\ c^1 \ne 0 \in \mathbb{C}^N$, and $Y$ is a normal space.*

*Then*

$$\mathcal{P}ol_{Y,N}(a,k) = \mathfrak{H}^{\{d,c\}}_Y(a,k) \bigoplus_{1\le j\le q} \Psi^{\{d,c^j\}}_a\, \mathcal{P}ol_{Y,N}(a,k-d); \tag{5.7}$$

*furthermore, $\mathfrak{H}^{\{d,c\}}_Y(a,k)$ is generated over $\mathcal{O}(Y)$ by the set*

$$H_{a,k} = H^{\{d,c\}}_{a,k} := \{\psi^{(k,\xi)}_a \mid [\bar{\xi}] = [\bar{\xi}_1, \ldots, \bar{\xi}_N] \in \mathfrak{F}^{\{d,c\}}\}$$

*of elementary pseudospherical harmonics of degree $k$ parametrized by $\mathfrak{F}^{\{d,c\}}$.*

*Proof.* Repeated application of formula (5.4) yields the decomposition formula (5.7). By (5.2), an elementary pseudospherical harmonics $\psi^{(k,\xi)}_a$ belongs to $\mathfrak{H}^{\{d,c\}}_Y(a,k)$ whenever $\xi \in \mathfrak{F}^{\{d,c\}}$. Owing to the decomposition (5.7), it suffices to prove that the submodule over $\mathcal{O}(Y)$ generated by the set $H_{a,k}$ in $\mathcal{P}ol_{Y,N}(a,k)$ has an "orthogonal complement" given by the direct sum

$$\bigoplus_{1\le j\le q} \Psi^{\{d,c^j\}}_a\, \mathcal{P}ol_{Y,N}(a,k-d).$$

By the identity (5.2), each element $\psi$ of this direct sum satisfies the equation $\langle\psi,\, \psi_a^{(k,\xi)}\rangle = 0$, provided $[\bar\xi] \in \mathfrak{F}^{\{d,c\}}$. Conversely, if $\mathcal{Z} \in \mathcal{P}ol_{Y,N}(a,k)$ and $\mathcal{Z}$ is "orthogonal" to $H_{a,k}$, then, for any strict push-forward $Z$ of $\mathcal{Z}$, $\hat Z^{(y)}(\xi) = 0$, hence also $Z^{(y)}(\bar\xi) = 0$ for each $y \in Y$. By Theorem 4.3 there exist $\lambda_j \in \mathcal{P}ol_{Y,N}(s-d)$, $1 \le j < q$, such that

$$Z^{\mathfrak{h}_{\{\mathcal{F}\}}} = \lambda_1 F^{\{d,c^1\}} + \cdots + \lambda_q F^{\{d,c^q\}} \qquad \text{on } Y \times \mathbb{C}^N.$$

Hence, if either $\langle F_{\{1\}}, \ldots, F_{\{q\}}\rangle_Y$ is a prime ideal or $\mathfrak{h}_{\{\mathcal{F}\}} = 1$, then

$$Z \in \bigoplus_{1\le j\le q} F^{\{d,c^j\}}\, \mathcal{P}ol_{Y,N}(k-d). \tag{5.8}$$

If $N > 2$, $q = 1$ and $c^1 \neq 0$, then $F^{\{d,c^1\}}$ is irreducible in $\mathcal{O}_Y[X_1, \ldots, X_N]$ ([P, Theorem 1]), hence, if $Y$ is a normal space, Proposition 4.1 asserts that the relation (5.8) remains valid (with $q = 1$). Thus in either case the second assertion follows. ∎

Let $D \subset X$ be an open set and $a \in D$. The (induced) Laplace operator $\Delta_p$ in $D^*$ can be expressed in the form

$$\Delta_p = r_a^{-2}\Delta_{\mathrm{sph}} + r_a^{1-N}\frac{\partial}{\partial r_a}\left(r_a^{N-1}\frac{\partial}{\partial r_a}\right),$$

where $\Delta_{\mathrm{sph}}$ is, by definition, the "pseudospherical Laplacian". Let $S_a(\rho_0) := \partial D_{[a]}(\rho_0)$ for sufficiently small $\rho_0 > 0$.

PROPOSITION 5.3.

(1) *For each fixed $(a,\xi) \in X \times \mathbb{C}^N$, the (surface pseudospherical harmonics) $\mathfrak{Y}_a^{(k,\xi)} := r_a^{-k}\, \psi_a^{(k,\xi)}$ is an eigenvector of $\Delta_{\mathrm{sph}}$ belonging to the eigenvalue $-\,k\,(k+N-2)$.*

(2) *If $N > 2$, each eigenspace of the pseudospherical Laplacian on $S_a(\rho_0)$ is spanned by the functions $\mathfrak{Y}_a^{(k,\xi)}$ with $\xi_1^2 + \cdots + \xi_N^2 = 0$, for some $k \ge 0$.*

(3) *$\mathcal{L}^2(S_a(\rho_0)) = \bigoplus_{k\ge 0} \hat E_{a,k}(\rho_0)$, where $\hat E_{a,k}(\rho_0)$ denotes the span of the set $E_{a,k}(\rho_0) := {\rho_0}^{-k}\, H_{a,k}^{\{2,c^{[1]}\}}\rfloor S_a(\rho_0)$, with $c^{[1]} = (1, \ldots, 1)$.*

*Proof.* Observe that the function $\mathfrak{Y}_a^{(k,\xi)}$ is $a$-radially symmetric ([Tu$_4$, §3, Remark 2]). Since every $a$-radially symmetric function $\mathfrak{Y}$ ([Tu$_4$, §3]) satisfies the equation

$$\Delta_{\mathrm{sph}}(\mathfrak{Y}) = r_a^{2-k}\, \Delta_p(r_a^k\, \mathfrak{Y}) - k(k+N-2)\, \mathfrak{Y}$$

for any integer $k > 0$, it follows that the surface pseudospherical harmonic $\mathfrak{Y}_a^{(k,\xi)}$ is an eigenfunction of $\Delta_{\mathrm{sph}}$ with eigenvalue $-k(k+N-2)$. The remaining assertion follows from the standard argument (see [H, p. 32]) by considering a "real" decomposition formula (5.6) with $g := r_a^2$. ∎

**6. Appendix: Multiplicity and relative cancellation rules.** Some basic properties of the multiplicity of a holomorphic map are summarized below. For complete proofs of Properties 3.1 and 6.1–6.5, see [Tu$_1$].

PROPERTY 6.1 (Cf. [Tu$_1$, (1.2.17)]). *If $(\varphi, g, D)$ is a proper slicing of divisors $\{\mathfrak{D}_j\}_{1\le j\le p}$ in $M$ (in the sense of Definition 1.1) then the sum $\sum\{\nu_{g,\varphi}(z; w, t) \mid z \in D\}$ is a positive integer independent of $(w, t) \in N \times Q$.*

PROPERTY 6.2 ([Tu$_1$, (2.2.1)–(2)]). *If $a \in (F_a)_{\mathrm{reg}}$, then $\nu_f(a) = \nu_{f,\varphi}(a)$ for all $\varphi \in \Phi_a^2(f)$.*

PROPERTY 6.3 ([Tu$_1$, (2.2.2)]). *If $N$ is normal at $f(a)$, then $\nu_f(a) = 1$ if and only if $f$ is (equivalent to) a (local) projection at $a$ and $a \in (F_a)_{\mathrm{reg}}$.*

PROPERTY 6.4 ([Tu$_1$, (2.2.5)]). *If $a \in M_{\mathrm{reg}}$, then $\nu_f(a) = \tilde{\nu}_f(a) = \nu_{f,\varphi}(a)$ for all $\varphi \in \Phi_a^2(f)$.*

PROPERTY 6.5 ([Tu$_1$, (2.2.6)]). $\nu_f(z) = \mathrm{const}$ *for all $z \in (F_a)_{\mathrm{reg}}$.*

PROPERTY 6.6 ([AS, pp. 266–267]). *If $a \in M_{\mathrm{reg}}$ and $f$, $g \in \mathcal{O}_{M,a} \setminus \{0\}$, then $\nu_{fg}^0(a) = \nu_f^0(a) + \nu_g^0(a)$, where $\nu_f^0(a) := \nu_f(a)$ if $f(a) = 0$, and $\nu_f^0(a) := 0$ otherwise.*

PROPERTY 6.7 ([Tu$_2$, Lemma, p. 132]). *Let $M$ be a normal complex space and $f$, $g \in \mathcal{O}(M)$ with $\mathcal{S} := \mathcal{V}(f)$ a thin subset of $M$. Then $g \in \langle f\rangle$ whenever $\nu_g^0(w) \ge \nu_f^0(w)$ for every $w \in Y_{\mathrm{reg}} \cap \mathcal{S}_{\mathrm{reg}}$.*

In the following let $Y$ denote an irreducible complex space, and $f, g, h, P \in \mathcal{O}_Y[X_1, \dots, X_N]$ be of positive degree. Some relative factoring, cancellation (and therewith divisibility) rules (of recurring use) are gathered below.

LEMMA 6.1. *If $f$ is primitive over $Y$, then $f$ has simple, irreducible factors in $\mathcal{O}_Y[X_1, \dots, X_N]$ if and only if so does the homogenization $F = {}^h f$ in $\mathcal{O}_Y[X_0, \dots, X_N]$.*

*Proof.* Since $\mathcal{S} = \mathcal{V}(f)$ is rational over $Y$, the set $\overline{\mathcal{S}} \cap (Y \times \mathbb{P}^{N-1}(\mathbb{C}))$ is thin on (the projective closure) $\overline{\mathcal{S}}$, by [Tu$_2$, Theorem 3.2]. Suppose that $F = G_1 G_2$, $G_j \in \mathcal{O}_Y[X_0, \dots, X_N]$, with $\deg(G_j) > 0$, $j = 1, 2$. Then each $G_j$ is homogeneous and ${}^a F = ({}^a G_1)\, ({}^a G_2)$, where each ${}^a G_j$ is of positive degree. Thus $f$ is reducible over $Y$. The converse assertion is proved similarly. ∎

In the remainder of this section assume that $Y$ is a normal complex space. The next assertion is an easy consequence of the unique factorization property of pseudopolynomials [Tu$_2$, Theorem 4.2(2)]:

Proposition 6.1. *Assume that $g$ and $P$ are relatively prime over some pseudoball $U_0$ ([Tu$_4$, §2]) at every point of $Y$ off an almost thin subset $\mathcal{T}$ of codimension 2 (namely, admitting no common factor in $\mathcal{O}_{U_0}[X_1, \ldots, X_N]$ with nonvoid zero set). Then $fP \equiv hP$ $(\langle g \rangle)$ (in the ring $\mathcal{O}(Y \times \mathbb{C}^N)$) if and only if $f \equiv h$ $(\langle g \rangle_Y)$.*

Corollary 6.1. *Let $F, G, H, Q \in \mathcal{P}ol_{Y,N+1}$ be of positive degree. Assume that $Q$ and $G$ are relatively prime over some pseudoball $U_0$ at every point of $Y$ off an almost thin subset of codimension $\geq 2$. Then $FQ \equiv HQ$ $(\langle G \rangle)$ (in $\mathcal{O}(Y \times \mathbb{C}^{N+1})$) if and only if $F \equiv H$ $(\langle G \rangle_Y)$.*

Proposition 6.2. *Assume that $g$ is primitive over $Y$ and irreducible at every point of $Y$ off an almost thin subset $\mathcal{T}$ of codimension $\geq 2$. If $P$ is not divisible by $g$ over some pseudoball at each point of $Y \setminus \mathcal{T}$, then $fP \equiv hP$ $(\langle g \rangle)$ (in $\mathcal{O}(Y \times \mathbb{C}^N)$) if and only if $f \equiv h$ $(\langle g \rangle_Y)$.*

*Proof.* Without loss of generality assume that $h \equiv 0$. Suppose that $fP \equiv 0$ $(\langle g \rangle)$. Let $U_0 \subseteq Y \setminus \mathcal{T}$ be a pseudoball such that $P \notin \langle g \rangle_{U_0}$. Suppose that $P = u\hat{P}$ and $g = u\hat{g}$ for some $u, \hat{P}, \hat{g} \in \mathcal{O}_{U_0}[X_1, \ldots, X_N]$. If $\deg \hat{g} = 0$, then the primitivity of $g$ implies that $\hat{g}$ is nonvanishing, thus contradicting the fact that $P$ is not divisible by $g$. It then follows from the local irreducibility of $g$ in $Y \setminus \mathcal{T}$ (and the primitivity of $g$) that the function $u$ is nonvanishing. Thus $g$ and $P$ are relatively prime over $U_0$. Hence by Proposition 6.1, $f \equiv 0$ $(\langle g \rangle_Y)$. ∎

Of possible use in the theory of algebraic functions is the following cancelation property:

Proposition 6.3. *If $P$ is irreducible in $\mathcal{O}_Y[X_1, \ldots, X_N]$ with $\deg(P) > \deg(g)$, then $fP \equiv hP$ $(\langle g \rangle)$ (in $\mathcal{O}(Y \times \mathbb{C}^N)$) if and only if $f \equiv h$ $(\langle g \rangle_Y)$.*

**Acknowledgements.** The author is indebted to the referee for comments and suggestions which led to improvements of this paper.

Support by the Minnesota State University "Faculty Reassigned Time For Research" Project is gratefully acknowledged.

## References

[AY] L. A. Aĭzenberg and A. P. Yuzhakov, *Integral Representations and Residues in Multidimensional Complex Analysis*, Transl. Math. Monogr. 58, Amer. Math. Soc., Providence, RI, 1983.

[AS] A. Andreotti and W. Stoll, *Analytic and Algebraic Dependence of Meromorphic Functions*, Lecture Notes in Math. 234, Springer, Berlin, 1971.

[BD] C. A. Berenstein and M. A. Dostal, *The Ritt theorem in several variables*, Ark. Mat. 12 (1974), 267–280.

[BH] A. Borel et A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France 89 (1961), 461–513.

[CP] P. Cassou-Noguès and A. Płoski, *Un théorème des zéros effectif*, Bull. Polish Acad. Sci. Math. 44 (1996), 61–70.

[CLO] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, 3rd ed., Springer, New York, 2007.

[D] R. N. Draper, *Intersection theory in analytic geometry*, Math. Ann. 180 (1969), 175–204.

[Fi] G. Fischer, *Complex Analytic Geometry*, Springer, Berlin, 1976.

[Fo] O. Forster, *Primärzerlegung in Steinschen Algebren*, Math. Ann. 154 (1964), 307–329.

[$GR_1$] H. Grauert und R. Remmert, *Komplexe Räume*, Math. Ann. 136 (1958), 245–318.

[$GR_2$] H. Grauert and R. Remmert, *Theory of Stein Spaces*, Grundlehren Math. Wiss. 236, Springer, Berlin, 1979.

[$GR_3$] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Grundlehren Math. Wiss. 265, Springer, Berlin, 1984.

[GH] Ph. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Gu] R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables. Vol. II: Local Theory*, Wadsworth & Brooks/Cole, Belmont, CA, 1990.

[GuRo] R. C. Gunning and H Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[H] S. Helgason, *Topics in Harmonic Analysis on Homogeneous Spaces*, Birkhäuser, Boston, 1981.

[La] S. Lang, *Introduction to Algebraic Geometry*, Interscience, New York, 1964.

[Łoj] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, Baseln, 1991.

[Lor] F. Lorenz, *Algebra. Vol. I: Fields and Galois Theory*, Springer, New York, 2006.

[P] A. Płoski, *On the irreducibility of polynomials in several complex variables*, Bull. Polish Acad. Sci. Math. 39 (1991), 241–247.

[PT] A. Płoski and P. Tworzewski, *Effective Nullstellensatz on analytic and algebraic varieties*, Bull. Polish Acad. Sci. Math. 46 (1998), 31–38.

[S] Y. T. Siu, *Hilbert Nullstellensatz in global complex-analytic case*, Proc. Amer. Math. Soc. 19 (1968), 296–298.

[$St_1$] W. Stoll, *The multiplicity of aholomorphic map*, Invent. Math. 2 (1966), 15–58.

[$St_2$] W. Stoll, *Value Distribution on Parabolic Spaces*, Lecture Notes in Math. 600, Springer, Berlin, 1977.

[Ts] A. K. Tsikh, *Multidimensional Residues and Their Applications*, Transl. Math. Monogr. 103, Amer. Math. Soc., Providence, RI, 1992.

[$Tu_1$] C. Tung, *The first main theorem of value distribution on complex spaces*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. $I^a$ (8) 15 (1979), 91–263.

[$Tu_2$] C. Tung, *Minimal equations, global reducibility of holomorphic functions, and relative rationality of analytic sets*, Rend. Sem. Mat. Univ. Padova 63 (1980), 127–143.

[$Tu_3$] C. Tung, *On the degree theory of holomorphic mappings*, in: Geometrical and Algebraical Aspects in Several Complex Variables (Cetraro, 1989), Sem. Conf. 8, EditEl, Rende, 1991, 341–360.

[$Tu_4$] C. Tung, *On generalized integral means and Euler type vector fields*, Complex Anal. Oper. Theory 5 (2011), 701–730.

[Tw] P. Tworzewski, *A note on the Max Noether theorem*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. 39 (1989), no. 61, 3 pp.

[W] B. L. van der Waerden, *Modern Algebra*, Vol. II, Ungar, New York, 1950.
[ZS] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, NJ, 1960.

Chia-chi Tung
Department of Mathematics and Statistics
Minnesota State University, Mankato
Mankato, MN 56001, U.S.A.
E-mail: chia.tung@mnsu.edu

*Received 14.8.2011*
*and in final form 14.5.2012* (2518)