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Reducing the number of periodic points in the smooth homotopy class of a self-map of a simply-connected manifold with periodic sequence of Lefschetz numbers

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Abstract. Let f be a smooth self-map of an m-dimensional ($m \geq 4$) closed connected and simply-connected manifold such that the sequence $\{L(f^n)\}_{n=1}^{\infty}$ of the Lefschetz numbers of its iterations is periodic. For a fixed natural r we wish to minimize, in the smooth homotopy class, the number of periodic points with periods less than or equal to r. The resulting number is given by a topological invariant $J[f]$ which is defined in combinatorial terms and is constant for all sufficiently large r. We compute $J[f]$ for self-maps of some manifolds with simple structure of homology groups.

1. Introduction. Let M be a compact m-dimensional manifold and f be a continuous self-map of M. The question of what is the minimal number of fixed points in the homotopy class of f is one of the most important problems in fixed point theory. The answer for manifolds of dimension $m \geq 3$ is given by the Nielsen number $N(f)$ [16]. The same answer remains true if we modify the question by taking smooth f and asking for the minimal number of fixed points in the smooth homotopy class of f [17].

However, there is a remarkable difference between the smooth and continuous cases when minimizing the number of periodic points instead of fixed points. In order to describe this difference let us fix a natural number r and consider

$$\min\{\#\mathrm{Fix}(g^r) : g \sim f\}. \tag{1.1}$$

In the continuous category the minimum in (1.1) is given by $NF_r(f)$, an invariant introduced by Jiang [16], while in the smooth category by $NJD_r[f]$, defined in [9]. It turns out that the invariants do not coincide: for smooth f, we have $NJD_r[f] \geq NF_r(f)$, and equality holds only in some exceptional situations [9], [14].

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The construction of $NJD_r[f]$ takes into account two kinds of obstacles in reducing the number of periodic points: the first one may be called "topological" and depends on the fundamental group of the manifold (more precisely, on so-called Reidemeister relations). The second one is related to smoothness and can be expressed in terms of local fixed point indices of iterations $\{\operatorname{ind}(f^n, x_0)\}_{n=1}^{\infty}$, where x_0 is a periodic point. Although it is difficult to compute $NJD_r[f]$ in general, it becomes an easier task for simply-connected manifolds, as then the first obstacle disappears. In such a case the invariant, denoted by $D_r[f]$, is defined only in terms of indices of iterations at periodic points [10].

The invariant $D_r[f]$ was computed in some particular cases, for example for each r and all self-maps of S^3 in [11] and under some specific assumptions on r and f for other simply-connected manifolds [7], [8], [12].

In this work we consider maps with periodic Lefschetz numbers, and examine the problem of determining

$$MF^{\mathrm{diff}}_{\leq r}(f) = \min\Big\{\#\bigcup_{k\leq r} \operatorname{Fix}(g^k) \colon g \overset{s}{\sim} f\Big\}, \tag{1.2}$$

where $\overset{s}{\sim}$ means that the maps g and f are C^1-homotopic.

The aim of this paper is to introduce a topological invariant $J[f]$ and prove that it is equal to $MF^{\mathrm{diff}}_{\leq r}(f)$ for all sufficiently large r (Theorem 2.7).

There are two reasons why we restrict our considerations to the class of maps for which $\{L(f^n)\}_{n=1}^{\infty}$ is periodic. Firstly, in that case $MF^{\mathrm{diff}}_{\leq r}(f)$ turns out to be independent of r for $r \geq R$, where R can be easily determined for a given f, so the invariant $J[f]$ does not depend on r. Secondly, the calculation of $J[f]$ is not very difficult (especially for manifolds with all homology groups low-dimensional, i.e. such that $\dim H_i(M;\mathbb{Q})$ is small), in contrast to other invariants mentioned above which usually need non-trivial combinatorial methods to be computed (see [8]).

The construction of $J[f]$ reduces to finding the minimal decomposition of $\{L(f^n)\}_{n=1}^{\infty}$ into sequences, each of which can be realized as fixed point indices (at a fixed point) of a smooth local map. In order to compute $J[f]$ one must know the representation of Lefschetz numbers in the form of so-called periodic expansion (Theorem 3.1) and the full list of periodic expansions of local indices of smooth maps in a given dimension (which has been recently provided in [13]; see Theorem 4.1). In the final part of the paper we find the invariant $J[f]$ for self-maps of some manifolds with low-dimensional homology, illustrating computational techniques which could also be applied for more complicated cases (Section 4).

The results obtained reveal an interesting relation between continuous and smooth categories for some of the classes of manifolds under consideration. Although usually the smooth and continuous cases are different, in some special situations they coincide. Namely, in a continuous homotopy class it is always possible to reduce the number of periodic points to no more than one fixed point. We prove that in the given setting the same is true in smooth homotopy classes for some manifolds having low-dimensional homology groups such as $S^m \times S^m$ with $m > 2$ (Corollary 4.8).

1.1. Periodic expansions of indices and Lefschetz numbers. Let M be an m-dimensional, closed, connected and simply-connected manifold. The sequence of indices of iterations plays a key role in reducing the number of periodic points in a smooth homotopy class. We will represent such sequences in the convenient form of so-called *periodic expansion*. In this section we also use periodic expansion to give a characterization of maps with $\{L(f^n)\}_{n=1}^{\infty}$ periodic.

Let U be an open subset of $\mathbb{R}^m$. For a map $f\colon U \to \mathbb{R}^m$ and its fixed point x_0, isolated for each iteration f^n, the sequence $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ of local fixed point indices is well-defined.

It turns out that there are always some congruences among the elements of this sequence, called the *Dold relations* [4],

$$\sum_{k|n} \mu(k)\,\mathrm{ind}(f^{(n/k)}, x_0) \equiv 0 \pmod{n}. \tag{1.3}$$

We denote by μ the Möbius function, i.e., $\mu\colon \mathbb{N} \to \mathbb{Z}$ is defined by the following three properties: $\mu(1) = 1$, $\mu(k) = (-1)^s$ if k is a product of s different primes, and $\mu(k) = 0$ otherwise.

For a given $k \in \mathbb{N}$ we define the *basic sequence*

$$\mathrm{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

Thus, reg_k is the periodic sequence

$$(0, \ldots, 0, k, 0, \ldots, 0, k, 0, \ldots),$$

where the non-zero entries appear for indices divisible by k.

Theorem 1.1 (cf. [18]). *The sequence* $\{\mathrm{ind}(f^n, x_0)\}_{n=1}^{\infty}$ (*and any other sequence of integers*) *can be uniquely represented in the form of a periodic expansion*

$$\mathrm{ind}(f^n, x_0) = \sum_{k=1}^{\infty} a_k \,\mathrm{reg}_k(n),$$

where

$$a_n = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right) \mathrm{ind}(f^k, x_0).$$

Remark 1.2. Notice that by the Dold relations (1.3) the coefficients a_n are always integers.

Remark 1.3. The Dold relations also hold for the sequence $\{L(f^n)\}_{n=1}^{\infty}$ of Lefschetz numbers, so the coefficients in the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$ are also integers [18].

The following remark will play an important role in defining the invariant $J[f]$ (see Section 2).

Remark 1.4. The periodicity of $\{L(f^n)\}_{n=1}^{\infty}$ is equivalent to $\{L(f^n)\}_{n=1}^{\infty}$ being a finite combination of basic sequences [18].

For a manifold M of dimension m we will consider $H_i(M;\mathbb{Q})$, where $i = 0,\dots,m$, the homology groups with coefficients in $\mathbb{Q}$, which are finite-dimensional linear spaces over $\mathbb{Q}$. For a self-map f of M we denote by f_{*i} the linear map induced by f on $H_i(M;\mathbb{Q})$ and by f_* the self-map $\bigoplus_{i=1}^m f_{*i}$ of $\bigoplus_{i=1}^m H_i(M;\mathbb{Q})$.

Definition 1.5 ([18]). Let M be an m-dimensional compact connected manifold. For integer $i \geq 0$ and $f\colon M \to M$, let $e_i(\lambda)$ be the algebraic multiplicity of λ as an eigenvalue of f_{*i}. Define

$$e(\lambda) := \sum_{i=0}^{m} (-1)^i e_i(\lambda).$$

We will call an eigenvalue $\lambda \neq 0$ *essential* provided $e(\lambda) \neq 0$.

Remark 1.6. Let us notice that only essential eigenvalues give a contribution to $\{L(f^n)\}_{n=1}^{\infty}$. Namely, let $\sigma(f)$ denote the spectrum of f_* and $\sigma_{\mathrm{es}}(f)$ be the set of essential eigenvalues. Then

$$L(f^m) = \sum_{\lambda\in\sigma(f)} e(\lambda)\lambda^m = \sum_{\lambda\in\sigma_{\mathrm{es}}(f)} e(\lambda)\lambda^m. \tag{1.4}$$

The following theorem gives a characterization of maps that have periodic Lefschetz numbers of iterations and shows that this is quite a large class of maps. Although the conclusion can be deduced from Theorem 2.2 in [1], it is hidden in some chains of implications, so for the convenience of the reader we give a straightforward proof below.

Theorem 1.7. *Let f be a self-map of a compact manifold M of dimension m. Then $\{L(f^n)\}_{n=1}^{\infty}$ is periodic if and only if all essential eigenvalues of the map f_* induced by f on homology are* (*primitive*) *roots of unity.*

Proof. "$\Rightarrow$" The Lefschetz number $L(f^n)$ is by definition

$$L(f^n) = \sum_{i=0}^{m} (-1)^i \operatorname{tr} f_{*i}^n.$$

Let $\rho = \max\{|\lambda| : e(\lambda)\neq 0,\ \lambda\in\sigma(f_*)\} = \max\{|\lambda| : \lambda \in \sigma_{\mathrm{es}}(f_*)\}$ (with the convention that the maximum over the empty set is zero).

We denote by $\lambda_1, \dots, \lambda_k$ all essential eigenvalues (not necessarily distinct) satisfying $|\lambda_i| < \rho$, and define $\tau = \max\{|\lambda_1|, \dots, |\lambda_k|\}$.

Suppose that $\rho > 1$. Then

$$|L(f^n)| \geq \rho^n - (|\lambda_1^n| + \dots + |\lambda_k^n|) \geq \rho^n - k\tau^n. \tag{1.5}$$

If $\tau \leq 1$, then the right-hand side of (1.5) tends to infinity as $n \to \infty$. If $\tau > 1$, then

$$\rho^n - k\tau^n = \tau^n[(\rho/\tau)^n - k]$$

also tends to infinity. On the other hand, by assumption the sequence $\{L(f^n)\}_{n=1}^{\infty}$ is periodic and thus bounded, so we get a contradiction.

As a result, we have $\rho \leq 1$. The classical theorem of Kronecker states that if all non-zero eigenvalues of an integral matrix have moduli less than or equal to one, then all the eigenvalues are (primitive) roots of unity (cf. [21]). Now, we consider a collection of algebraically conjugate eigenvalues in $\sigma(f)$. It is known that if one of the eigenvalues in the collection is essential then all other eigenvalues are also essential (cf. [18, Remark 3.1.54] for the details). As a result, we may associate with each collection of essential conjugate eigenvalues an integral matrix and apply the Kronecker theorem. This proves the first implication.

"$\Leftarrow$" If all essential eigenvalues of f_* are roots of unity of degrees $l_1, \dots, l_s$, then taking their least common multiple K we see that $L(f^n) = L(f^{n+K})$, thus $\{L(f^n)\}_{n=1}^{\infty}$ is periodic. ∎

The algebraic condition on the eigenvalues in Theorem 1.7 allows one to produce many examples of maps with periodic Lefschetz numbers of iterations. What is more, it turns out that any smooth map that can be deformed to a map with finitely many periodic points also satisfies this condition.

Theorem 1.8 ([3], [25]). *Let f be a C^1 self-map of a compact manifold M with a finite number of periodic points. Then $\{L(f^n)\}_{n=1}^{\infty}$ is periodic.*

Proof. Let us consider the periodic expansions of the sequences of the indices of iterations at $x \in \operatorname{Per}(f)$, where $\operatorname{Per}(f)$ denotes the set of periodic points of f:

$$\operatorname{ind}(f^n, x) = \sum_{k=1}^{\infty} a_k \operatorname{reg}_k(n). \tag{1.6}$$

Chow, Mallet-Paret and Yorke proved in [3] that if f is a C^1 map, then $\{\operatorname{ind}(f^n, x)\}_{n=1}^{\infty}$ is a finite combination of basic sequences:

$$\operatorname{ind}(f^n, x) = \sum_{k \in O(x)} a_k \operatorname{reg}_k(n), \tag{1.7}$$

where $O(x)$ is a finite set. By the Lefschetz–Hopf theorem for f^n we obtain

$$L(f^n) = \sum_{x \in \mathrm{Fix}(f^n)} \mathrm{ind}(f^n, x). \tag{1.8}$$

By assumption there are finitely many periodic points. As a consequence, by (1.8), $\{L(f^n)\}_{n=1}^{\infty}$ is a finite sum of sequences of the form (1.7), each of which is a finite combination of periodic sequences. ■

In particular, Morse–Smale diffeomorphisms constitute a well-known class of maps on compact manifolds with finitely many periodic points [22], and thus they are examples of maps with periodic sequence of Lefschetz numbers of iterations.

2. Definition of $J[f]$. In this section we will define, using the notion of so-called DD^m sequences, a new topological invariant $J[f]$ which is equal to the minimal number of periodic points with periods less than or equal to a given r in the smooth homotopy class of f and which does not depend on r for sufficiently large r.

In the rest of the paper we will consider smooth (i.e. C^1) self-maps of a manifold M with periodic sequence of Lefschetz numbers of iterations. Our assumptions on the manifold are the following: M is smooth, closed, connected and simply-connected, but in the formulation of the results we will not repeat the natural assumptions of smoothness and connectedness of M.

By Remarks 1.3 and 1.4 we can represent $\{L(f^n)\}_{n=1}^{\infty}$ in the form

$$L(f^n) = \sum_{k \in O} b_k \,\mathrm{reg}_k(n), \tag{2.1}$$

where $O = \{k : b_k \neq 0\}$ is finite and $b_k \in \mathbb{Z}$.

By a *p-orbit* we will understand an orbit consisting of points with minimal period equal to p.

Definition 2.1. A sequence $\{c_n\}_{n=1}^{\infty}$ of integers is called a $DD^m(p)$ *sequence* if there are: a C^1 map $\phi\colon U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^m$ is open, and P, an isolated p-orbit of ϕ, such that

$$c_n = \mathrm{ind}(\phi^n, P) \tag{2.2}$$

(notice that $c_n = 0$ if n is not a multiple of p).

Let r be a fixed natural number. The finite sequence $\{c_n\}_{n|r}$ will be called a $DD^m(p\,|\,r)$ *sequence* if (2.2) holds for $n\,|\,r$, and a $DD^m(p \le r)$ *sequence* if (2.2) holds for $n \le r$.

Definition 2.2. Let $\{L(f^n)\}_{n=1}^{\infty}$ be an infinite sequence of Lefschetz numbers. Suppose we can decompose $\{L(f^n)\}_{n=1}^{\infty}$ into the sum

$$L(f^n) = c_1(n) + \cdots + c_s(n), \tag{2.3}$$

where c_i is a $DD^m(l_i)$ sequence for $i = 1, \ldots, s$. Each such decomposition determines the number $l = l_1 + \cdots + l_s$. We define the number $J[f]$ as the smallest l which can be obtained in this way.

Analogously, we decompose $\{L(f^n)\}_{n\le r}$ and $\{L(f^n)\}_{n|r}$ into

$$L(f^n) = c_1(n) + \cdots + c_s(n), \tag{2.4}$$

where for each $i = 1, \ldots, s$, c_i is a $DD^m(l_i \le r)$ sequence or $DD^m(l_i \,|\, r)$ sequence, respectively. Again, each such decomposition determines the number $l = l_1 + \cdots + l_s$.

Then $D_r[f]$ is defined as the smallest l under decompositions into $DD^m(l_i \,|\, r)$ sequences; and $J_r[f]$ as the smallest l under decompositions into $DD^m(l_i \le r)$ sequences.

By convention, if the sequence of Lefschetz numbers consists only of zero elements, then it is a sum of 0 respective DD^m sequences.

The following theorem was the main result of [10].

Theorem 2.3. *Let f be a self-map of a closed simply-connected manifold of dimension $m \ge 3$ and $r \in \mathbb{N}$ a fixed number. Then*

$$D_r[f] = \min\{\#\mathrm{Fix}(g^r) \colon g \overset{s}{\sim} f\}, \tag{2.5}$$

where $\overset{s}{\sim}$ means that the maps g and f are C^1-homotopic.

Let us define

$$MF^{\mathrm{diff}}_{\le r}(f) = \min\Big\{\#\bigcup_{k\le r} \mathrm{Fix}(g^k) \colon g \overset{s}{\sim} f\Big\}. \tag{2.6}$$

Lemma 2.4. *Let f be a self-map of a closed simply-connected manifold of dimension $m \ge 3$ and $r \in \mathbb{N}$ a fixed number. Then*

$$J_r[f] \le MF^{\mathrm{diff}}_{\le r}(f) \le D_{r!}[f]. \tag{2.7}$$

Proof. By the Kupka–Smale theorem $f \colon M \to M$ is smoothly homotopic to a map g such that $\#\bigcup_{k\le r} \mathrm{Fix}(g^k)$ is finite [22]. For any such g,

$$\bigcup_{k\le r} \mathrm{Fix}(g^k) = a_1 \cup \cdots \cup a_s,$$

where a_i is an orbit of length l_i. By the Lefschetz–Hopf formula we get, for $n \le r$,

$$L(f^n) = L(g^n) = \sum_{i=1}^{s} \mathrm{ind}(g^n, a_i) = \sum_{i=1}^{s} c_i(n),$$

where $c_i(n) = \operatorname{ind}(g^n, a_i)$. Since g is smooth, each c_i is a $DD^m(l_i \le r)$ sequence. Furthermore,

$$J_r[f] \le l_1 + \cdots + l_s = \#a_1 + \cdots + \#a_s = \# \bigcup_{k \le r} \operatorname{Fix}(g^k).$$

As a consequence we get the first inequality in (2.7).

To prove the second, assume that g_1 achieves the minimum in (2.6) for r, and g_2 achieves the minimum in (2.5) for $r!$. Then

$$\begin{aligned} MF^{\mathrm{diff}}_{\le r}(f) &= \# \bigcup_{k \le r} \operatorname{Fix}(g_1^k) \le \# \bigcup_{k \le r} \operatorname{Fix}(g_2^k) \le \# \bigcup_{k \mid r!} \operatorname{Fix}(g_2^k) \\ &= \#\operatorname{Fix}(g_2^{r!}) = D_{r!}[f]. \ \blacksquare \end{aligned}$$

Remark 2.5. In the proof of Lemma 2.4 instead of $r!$ one can use the least common multiple of $\{1, \ldots, r\}$.

Lemma 2.6. *Let f be a self-map of a closed simply-connected manifold of dimension $m \ge 4$. Assume $R = \max\{k : k \in O\}$ in (2.1) and $r \ge R$.*

(1) *To calculate $J[f]$, $J_r[f]$ and $D_r[f]$ one can equivalently take the minimum in Definition 2.2 only over $DD^m(1)$ or $DD^m(1 \le r)$, $DD^m(1 \mid r)$ sequences, respectively.*
(2) *Furthermore, one can equivalently take the minimum in Definition 2.2 over sequences $c(n) = \sum_k a_k \operatorname{reg}_k(n)$ satisfying $a_k = 0$ for $k > R$.*

Proof. Part (1) was proved in [8] and is a consequence of the fact that every $DD^m(p)$ sequence with $p \ge 2$ is a sum of at most two $DD^m(1)$ sequences.

(2) Assume that a minimal decomposition (2.3) of $\{L(f^n)\}_{n=1}^{\infty}$ is given, where $c_i(n) = \sum_k a_k^i \operatorname{reg}_k(n)$, and assume that $a_k^i \ne 0$ for some $k > R$ and some $i = 1, \ldots, s$. We have $b_k = 0 = \sum_{i=1}^{s} a_k^i$ in (2.1). Thus, we can always replace $c_1, \ldots, c_s$ in the minimal decomposition by $c_1', \ldots, c_s'$ with the respective coefficients $a_k'^i = 0$. The same is obviously true for $\{L(f^n)\}_{n \le r}$ and $\{L(f^n)\}_{n \mid r}$. $\blacksquare$

Theorem 2.7. *Let f be a self-map of a closed simply-connected manifold M of dimension $m \ge 4$. Assume $\{L(f^n)\}_{n=1}^{\infty}$ is periodic and $R = \max\{k : k \in O\}$ in (2.1). Then for every $r \ge R$,*

$$J[f] = J_r[f] = MF^{\mathrm{diff}}_{\le r}(f).$$

Proof. We have

$$J[f] = J_r[f] \le MF^{\mathrm{diff}}_{\le r}(f) \le D_{r!}[f] = J[f]. \tag{2.8}$$

Here the equalities are straightforward consequences of Lemma 2.6(2), while the inequalities were proved in Lemma 2.4. $\blacksquare$

Remark 2.8. Notice that by Theorem 2.7 the value of $J_r[f]$ is constant for all $r \geq R$. As a consequence, $J_r[f] = J[f]$ is an invariant that does not depend on r (for r large enough), but only on the space M and f. In other words, $J[f]$ is equal to the minimal number of all periodic points with periods less than or equal to r in a smooth homotopy class of f for all sufficiently large r.

What is more, for a given manifold M we will determine the value of $R = \max\{k\colon k \in O\}$ and express it in terms of primitive roots of unity contained in the spectrum of the map f_* on homology (cf. Remark 3.4).

Remark 2.9. Assume that a decomposition of the Lefschetz numbers of the iterations of f into $DD^m(1\,|\,r)$ sequences is given. Then, by the construction described in [9], we can find, in the smooth homotopy class of f, a map g whose fixed points are in one-to-one correspondence to the $DD^m(1\,|\,r)$ sequences. Thus, by Lemma 2.6(1) and the equality $J[f] = D_{r!}[f]$ obtained in (2.8), the number $J[f]$ can be realized at fixed points in the smooth homotopy class of f.

3. Periodic expansion for maps with periodic Lefschetz numbers of iterations. In this section we will determine the periodic expansion of Lefschetz numbers for smooth maps having $\{L(f^n)\}_{n=1}^{\infty}$ periodic, which makes it possible to find the value of $J[f]$ effectively.

Let $\varepsilon_1, \dots, \varepsilon_{\varphi(d)}$ be all the dth primitive roots of unity, where φ denotes the Euler function, i.e. $\varphi(d)$ is the number of positive integers less than or equal to d that are coprime to d. For a given d we define $L_d(n) = \varepsilon_1^n + \cdots + \varepsilon_{\varphi(d)}^n$.

The cyclotomic polynomial $\prod_{i=1}^{\varphi(d)}(x - \varepsilon_i)$ has integer coefficients, thus $L_d(n)$ is equal to $\operatorname{tr} A^n$ for some integer matrix A, having the cyclotomic polynomial as the characteristic polynomial. On the other hand, the sequence $\operatorname{tr} A^n$ for an integer matrix A always satisfies the Dold relations (see Theorem 3.1.4 in [19]). As a consequence, by Theorem 1.2, $L_d(n)$ can be uniquely represented as an integral combination of basic sequences reg_k.

Let us consider an arbitrary map f with periodic Lefschetz numbers of iterations. Then by Theorem 1.7 all its essential eigenvalues are primitive roots of unity. Let P_d denote the set of all dth primitive roots of unity and let $\sigma_{\mathrm{es}}(f)$ be the set of essential eigenvalues of f. We define

$$e(d) = \sum_{\lambda \in P_d \cap \sigma_{\mathrm{es}}(f)} e(\lambda).$$

The essential dth primitive roots of unity appear in collections of $\varphi(d)$ elements, contributing $\frac{e(d)}{\varphi(d)} L_d(n)$ to $L(f^n)$, so we get

$$L(f^n) = \sum_d \frac{e(d)}{\varphi(d)} L_d(n).$$

As a consequence, to find the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$ it is enough to determine the expansions of each $\{L_d(n)\}_{n=1}^{\infty}$.

We represent $\{L_d(n)\}_{n=1}^{\infty}$, for d fixed, as an integral combination of basic sequences reg_k:

$$L_d(n) = \sum_{k=1}^{\infty} a_k^d \,\mathrm{reg}_k(n), \tag{3.1}$$

where a_k^d are integers.

The following theorem gives the value of a_k^d, and thus allows us to determine the periodic expansion of $\{L_d(n)\}_{n=1}^{\infty}$.

THEOREM 3.1. *The coefficient a_k^d of the periodic expansion of $\{L_d(n)\}_{n=1}^{\infty}$ is equal to*

$$a_k^d = \begin{cases} 0, & k \nmid d, \\ \mu(d/k), & k \mid d. \end{cases} \tag{3.2}$$

We precede the proof with two technical lemmas. Let (a, b) denote the greatest common divisor of a and b.

LEMMA 3.2. *Let $d \in \mathbb{N}$ and p be a prime number. Then:*

(1) *if $(d/p^\alpha, p) = 1$, then $\varphi(d)/\varphi(d/p^\alpha) = p^{\alpha-1}(p-1)$;*
(2) *if $(d/p^\alpha, p) = p$, then $\varphi(d)/\varphi(d/p^\alpha) = p^\alpha$.*

Proof. Let $d = p_1^{s_1} \cdot \ldots \cdot p_r^{s_r}$, where p_i for $i = 1, \ldots, r$ are different primes. Then $\varphi(d) = d\frac{(p_1-1)\cdot\ldots\cdot(p_r-1)}{p_1\cdot\ldots\cdot p_r}$ (cf. [2]).

We prove (2). If $(d/p^\alpha, p) = p$, then for some i, $p_i = p$ and $s_i \geq \alpha + 1$; assume for simplicity that $i = r$. Then

$$\frac{\varphi(d)}{\varphi(d/p^\alpha)} = \frac{d\frac{(p_1-1)\cdot\ldots\cdot(p_{r-1}-1)(p-1)}{p_1\cdot\ldots\cdot p_{r-1}p}}{\frac{d}{p^\alpha}\frac{(p_1-1)\cdot\ldots\cdot(p_{r-1}-1)(p-1)}{p_1\cdot\ldots\cdot p_{r-1}p}} = p^\alpha.$$

The proof of (1) is analogous. ∎

In the rest of this section we will repeatedly make use of the multiplicativity of the Möbius function: $\mu(pq) = \mu(p)\mu(q)$ for coprime p and q.

LEMMA 3.3. *Let $d, k \in \mathbb{N}$ and let p be a prime. If $kp \mid d$ then*

$$\sum_{l \mid k} \mu\left(\frac{d}{l}\right)\mu\left(\frac{kp}{l}\right)\frac{\varphi(d)}{\varphi(d/l)} = -\sum_{l \mid k} \mu\left(\frac{d}{l}\right)\mu\left(\frac{k}{l}\right)\frac{\varphi(d)}{\varphi(d/l)}.$$

Proof. We denote the left-hand side of the formula by A. Then

$$A = \sum_{l\mid k\wedge(k/l,p)=1} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}p\Big)\frac{\varphi(d)}{\varphi(d/l)} + \sum_{l\mid k\wedge(k/l,p)=p} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{lpsp}{l}\Big)\frac{\varphi(d)}{\varphi(d/l)},$$

where in the second term we substitute $k = lps$, $s \in \mathbb{N}$ since $(k/l, p) = p$. As $\mu(lpsp/l) = 0$, the second term vanishes. We get

$$(3.3)\qquad A = \mu(p)\Big(\sum_{l\mid k\wedge(k/l,p)=1} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}\Big)\frac{\varphi(d)}{\varphi(d/l)} + \sum_{l\mid k\wedge(k/l,p)=p} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}\Big)\frac{\varphi(d)}{\varphi(d/l)} - \sum_{l\mid k\wedge(k/l,p)=p} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}\Big)\frac{\varphi(d)}{\varphi(d/l)}\Big).$$

By assumption $d = kpt$, where $t \in \mathbb{N}$. Thus the third term in brackets in (3.3) is equal to zero because $\mu(d/l) = \mu(kpt/l) = \mu(lpspt/l) = 0$. Finally, we obtain

$$A = -\sum_{l\mid k} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}\Big)\frac{\varphi(d)}{\varphi(d/l)}. \ \blacksquare$$

Proof of Theorem 3.1. The following formula for a_k^d was proved in [6]:

$$(3.4)\qquad a_k^d = \begin{cases} 0, & k \nmid d, \\ \dfrac{1}{k}\displaystyle\sum_{l\mid k} \mu\Big(\frac{d}{l}\Big)\mu\Big(\frac{k}{l}\Big)\frac{\varphi(d)}{\varphi(d/l)}, & k \mid d. \end{cases}$$

We will simplify this formula so as to obtain (3.2). Clearly, we have to consider the case of $k \mid d$ only.

We will use induction on the number of primes in the decomposition of k. We denote by a_k^d the right-hand side of (3.4) and by A_k^d the right-hand side of (3.2).

First, for $k = 1$ and arbitrary d, we have

$$a_1^d = \mu(d) = A_1^d.$$

Next, let $k = p$ be a prime dividing d. Then

$$(3.5)\qquad a_p^d = \frac{1}{p}\Big(\mu(d)\mu(p) + \mu\Big(\frac{d}{p}\Big)\frac{\varphi(d)}{\varphi(d/p)}\Big).$$

Let us consider two cases. If $(d/p, p) = p$, then $p^2 \mid d$ so $\mu(d) = 0$ and by Lemma 3.2(2),

$$a_p^d = \frac{1}{p}\cdot\mu\Big(\frac{d}{p}\Big)p = A_p^d.$$

If $(d/p, p) = 1$, then by Lemma 3.2(1) formula (3.5) takes the form

$$a_p^d = \frac{1}{p}\Big(-\mu\Big(\frac{d}{p}\cdot p\Big) + \mu\Big(\frac{d}{p}\Big)(p-1)\Big) = \mu\Big(\frac{d}{p}\Big) = A_p^d.$$

Now we make our inductive assumption: for all natural k and d such that $k \mid d$ and k has r prime factors (i.e. $k = p_1 \cdot \ldots \cdot p_r$, where p_i are not necessarily different primes), we have

$$a_k^d = \mu\left(\frac{d}{k}\right) = A_k^d.$$

We will prove that the same is true for k having one factor more, i.e.

$$a_{kp}^d = A_{kp}^d. \tag{3.6}$$

We have

$$\begin{aligned} kp \cdot a_{kp}^d &= \sum_{l|kp} \mu\left(\frac{d}{l}\right)\mu\left(\frac{kp}{l}\right)\frac{\varphi(d)}{\varphi(d/l)} \\ &= \sum_{l|k} \mu\left(\frac{d}{l}\right)\mu\left(\frac{kp}{l}\right)\frac{\varphi(d)}{\varphi(d/l)} + \sum_{p|l|kp \wedge l\nmid k} \mu\left(\frac{d}{l}\right)\mu\left(\frac{kp}{l}\right)\frac{\varphi(d)}{\varphi(d/l)} \\ &=: X + Y. \end{aligned}$$

By Lemma 3.3 and the induction assumption we get

$$X = -\sum_{l|k} \mu\left(\frac{d}{l}\right)\mu\left(\frac{k}{l}\right)\frac{\varphi(d)}{\varphi(d/l)} = -\mu\left(\frac{d}{k}\right)k.$$

In order to compute Y we consider two cases: $(k,p) = 1$ to $(k,p) = p$.

(I) $(k,p) = 1$. Then we substitute $l = pj$, where $j \mid k$, and obtain

$$\begin{aligned} Y &= \sum_{j|k} \mu\left(\frac{d}{jp}\right)\mu\left(\frac{k}{j}\right)\frac{\varphi(d)}{\varphi(d/jp)} \\ &= \sum_{j|k} \mu\left(\frac{d}{jp}\right)\mu\left(\frac{k}{j}\right)\frac{\varphi(d/p)}{\varphi(d/jp)}\,\frac{\varphi(d)}{\varphi(d/p)}. \end{aligned} \tag{3.7}$$

By the induction assumption for k and d/p and by Lemma 3.2, we have

$$Y = \mu\left(\frac{d}{pk}\right)k\frac{\varphi(d)}{\varphi(d/p)} = \begin{cases} \mu\left(\dfrac{d}{kp}\right)kp - \mu\left(\dfrac{d}{kp}\right)k & \text{if } (d/p,p) = 1, \\ \mu\left(\dfrac{d}{kp}\right)kp & \text{if } (d/p,p) = p. \end{cases} \tag{3.8}$$

Thus a_k^d under the assumption that $(k,p) = 1$ is equal to

$$X + Y = \begin{cases} -\mu\left(\dfrac{d}{k}\right)k + \mu\left(\dfrac{d}{kp}\right)kp - \mu\left(\dfrac{d}{kp}\right)k & \text{if } (d/p,p) = 1, \\ -\mu\left(\dfrac{d}{k}\right)k + \mu\left(\dfrac{d}{kp}\right)kp & \text{if } (d/p,p) = p. \end{cases} \tag{3.9}$$

Notice that if $(d/p,p) = 1$, then $\mu(d/k) = -\mu(d/kp)\mu(p)$. As a consequence, $-\mu(d/k)k - \mu(d/kp)k = 0$ in the first case of (3.9). Furthermore, if

$(d/p, p) = p$, then $p^2 | \frac{d}{k}$, so $-\mu(d/k)k = 0$ in the second case of (3.9). Finally, always

$$X + Y = \mu\left(\frac{d}{kp}\right)kp, \tag{3.10}$$

which is (3.6).

(II) $(k, p) = p$. Then we substitute $l = p^2 j$, where $j \mid \frac{k}{p}$, to obtain,

$$\begin{aligned} Y &= \sum_{j|\frac{k}{p}} \mu\left(\frac{d}{jp^2}\right)\mu\left(\frac{k}{jp}\right)\frac{\varphi(d)}{\varphi(d/jp^2)} \\ &= \sum_{j|\frac{k}{p}} \mu\left(\frac{d}{jp^2}\right)\mu\left(\frac{k}{jp}\right)\frac{\varphi(d/p^2)}{\varphi(d/jp^2)}\,\frac{\varphi(d)}{\varphi(d/p^2)}. \end{aligned} \tag{3.11}$$

Then we repeat the same reasoning as in case (I). We consider two subcases: $(d/p^2, p) = 1$ and $(d/p^2, p) = p$, apply the induction assumption for k/p and d/p^2, and use Lemma 3.2 for $\alpha = 2$, obtaining the conclusion in the form (3.10). This ends the proof of Theorem 3.1. ■

Remark 3.4. Let $L(f^n) = \sum_{k\in O} b_k \operatorname{reg}_k(n)$, where $O = \{k : b_k \neq 0\}$. Then, by Theorem 2.7, $R = \max\{k \colon k \in O\}$, so R is the maximal index of non-vanishing basic sequences in the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$. Thus, by Theorem 3.1, R is equal to the maximal degree of all essential primitive roots of unity in the spectrum of the map f_*.

4. Applications: calculations of the invariant $J[f]$ for manifolds with low-dimensional homology groups

4.1. Indices of iterations of smooth maps. By definition the calculation of $J[f]$ reduces to finding the minimal decomposition of $\{L(f^n)\}_{n=1}^{\infty}$ into sequences of the form $\{\operatorname{ind}(g^n, x_0)\}_{n=1}^{\infty}$, where g is a smooth map and x_0 is a fixed point of g. Thus, to calculate $J[f]$ one must know all local indices of iterations of a smooth map at a fixed point. This information is provided in [13] and given below as Theorem 4.1.

Let us remark that finding the indices of iterations of a particular class of maps is a difficult task in general. Recently some important results in this direction were obtained for planar homeomorphisms [23], $\mathbb{R}^3$-homeomorphisms [19], [24] and holomorphic maps [26].

To describe the indices for smooth maps we will need some notation. For a set H of integers we denote by $\operatorname{LCM}(H)$ the least common multiple of all elements in H, with the convention that $\operatorname{LCM}(\emptyset) = 1$. We define $\overline{H} = \{\operatorname{LCM}(Q) : Q \subset H\}$.

Next, for natural s we denote by $L(s)$ every set of natural numbers of the form $\overline{L}$, where $\#L = s$ and $1, 2 \notin L$.

By $L_2(s)$ we denote every set of natural numbers of the form $\overline{L}$, where $\#L = s+1$ and $1 \notin L$, $2 \in L$.

THEOREM 4.1 ([13]). *Let g be a C^1 self-map of $\mathbb{R}^m$ $(m>1)$, having x_0 as an isolated fixed point for each iteration. Then the sequence $\{\mathrm{ind}(g^n, x_0)\}_{n=1}^{\infty}$ of local indices has one of the following forms.*

(I) *For m odd:*

(A^o) $\mathrm{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-3}{2})} a_k \,\mathrm{reg}_k(n)$.

$(B^o), (C^o), (D^o)$ $\mathrm{ind}(g^n, x_0) = \sum_{k \in L(\frac{m-1}{2})} a_k \,\mathrm{reg}_k(n)$,

$$\text{where } a_1 = \begin{cases} 1 & \text{in case } (B^o), \\ -1 & \text{in case } (C^o), \\ 0 & \text{in case } (D^o). \end{cases}$$

$(E^o), (F^o)$ $\mathrm{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-1}{2})} a_k \,\mathrm{reg}_k(n)$,

$$\text{where } a_1 = 1 \text{ and } a_2 = \begin{cases} 0 & \text{in case } (E^o), \\ -1 & \text{in the case } (F^o). \end{cases}$$

(II) *For m even:*

(A^e) $\mathrm{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-4}{2})} a_k \,\mathrm{reg}_k(n)$.

(B^e) $\mathrm{ind}(g^n, x_0) = \sum_{k \in L(\frac{m-2}{2})} a_k \,\mathrm{reg}_k(n)$.

$(C^e), (D^e), (E^e)$ $\mathrm{ind}(g^n, x_0) = \sum_{k \in L_2(\frac{m-2}{2})} a_k \,\mathrm{reg}_k(n)$,

$$\text{where } a_1 = \begin{cases} 1 & \text{in case } (C^e), \\ -1 & \text{in case } (D^e), \\ 0 & \text{in case } (E^e). \end{cases}$$

(F^e) $\mathrm{ind}(g^n, x_0) = \sum_{k \in L(\frac{m}{2})} a_k \,\mathrm{reg}_k(n)$, *where $a_1 = 1$.*

COROLLARY 4.2. *In particular, in dimension $m = 4$ we have six patterns of possible indices of iterations (cf. also Theorem 3.3 in [12]), which are listed in Table 1.*

By $[d, l]$ we denote the least common multiple of d and l.

Table 1. Sequences of local indices in dimension $m = 4$; in all formulas $d, l > 2$

Case	$m = 4$
(A)	$a_1 \,\mathrm{reg}_1(n) + a_2 \,\mathrm{reg}_2(n)$
(B)	$a_1 \,\mathrm{reg}_1(n) + a_d \,\mathrm{reg}_d(n)$
(C)	$\mathrm{reg}_1(n) + a_2 \,\mathrm{reg}_2(n) + a_d \,\mathrm{reg}_d(n) + a_{[d,2]} \,\mathrm{reg}_{[d,2]}(n)$
(D)	$-\mathrm{reg}_1(n) + a_2 \,\mathrm{reg}_2(n) + a_d \,\mathrm{reg}_d(n) + a_{[d,2]} \,\mathrm{reg}_{[d,2]}(n)$
(E)	$a_2 \,\mathrm{reg}_2(n) + a_d \,\mathrm{reg}_d(n) + a_{[d,2]} \,\mathrm{reg}_{[d,2]}(n)$
(F)	$\mathrm{reg}_1(n) + a_d \,\mathrm{reg}_d(n) + a_l \,\mathrm{reg}_l(n) + a_{[d,l]} \,\mathrm{reg}_{[d,l]}(n)$

4.2. Continuous versus smooth category. As mentioned in the introduction, there is a huge difference between the smooth and continuous categories in respect of minimization of the number of periodic points in a homotopy class. We will clarify this difference below.

Let us consider the numbers

$$MF_{\le r}(f) = \min\Big\{\# \bigcup_{k\le r} \mathrm{Fix}(g^k) \colon g \sim f\Big\}, \tag{4.1}$$

$$MF_r(f) = \min\{\#\mathrm{Fix}(g^r) \colon g \sim f\}, \tag{4.2}$$

where $\sim$ means that the maps g and f are (continuously) homotopic.

The following formula was proved in [15, Theorem 5.1] for any self-map of a simply-connected closed manifold of dimension $m \ge 3$:

$$MF_r(f) = \begin{cases} 0 & \text{if } L(f^k) = 0 \text{ for all } k \mid r, \\ 1 & \text{otherwise.} \end{cases}$$

Thus we get

$$MF_{\le r}(f) \le MF_{r!}(f) \le 1 \tag{4.3}$$

for any natural r. As a consequence, in dimension $m \ge 3$ one can always find in the homotopy class of f a map g with no more than one (fixed) point in the set $\bigcup_{k\le r} \mathrm{Fix}(g^k)$.

Applying Corollary 4.2 we will show that this is not true in the smooth category even for maps having periodic Lefschetz numbers of iterations. We exhibit a self-map f of a 4-dimensional manifold M with simple homology groups for which $J[f] = 2$. Our example also illustrates the method of calculating the invariant $J[f]$.

Example 4.3. Let us consider a 4-dimensional closed simply-connected manifold M with the following homology groups: $H_0(M;\mathbb{Q}) = \mathbb{Q}$, $H_1(M;\mathbb{Q}) = 0$, $H_2(M;\mathbb{Q}) = \mathbb{Q}^4$, $H_3(M;\mathbb{Q}) = 0$, $H_4(M;\mathbb{Q}) = \mathbb{Q}$. Assume that a smooth self-map f of M induces the identity on the 0th and 4th homology groups and that the eigenvalues of f_{*2} are primitive roots of unity of degree $d = 4$, each with multiplicity 2.

Now, we consider $L_4(n)$ (cf. (3.1)) generated by $\varphi(4) = 2$ primitive roots of unity. By Theorem 3.1, $L_4(n)$ has the following periodic expansion:

$$L_4(n) = \sum_{k\mid 4} \mu\left(\frac{4}{k}\right) \mathrm{reg}_k(n) = -\,\mathrm{reg}_2(n) + \mathrm{reg}_4(n).$$

Determining the periodic expansion of the Lefschetz numbers of the iterations of f we have to take into account the eigenvalues equal to 1 on

$H_1(M;\mathbb{Q})$ and $H_4(M;\mathbb{Q})$ and multiplicities on $H_2(M;\mathbb{Q})$. We obtain

$$L(f^n) = \mathrm{reg}_1(n) + 2L_4(n) + \mathrm{reg}_1(n) \\ = 2\,\mathrm{reg}_1(n) - 2\,\mathrm{reg}_2(n) + 2\,\mathrm{reg}_4(n). \tag{4.4}$$

As the dimension of the manifold is 4, we can use Table 1 to calculate $J[f]$. It is easy to observe that the sequence in (4.4) is none of the sequences in Table 1. Thus $J[f] > 1$. On the other hand, the sequence in (4.4) can be represented as a sum of two sequences listed in Table 1, of type (A) and (B) respectively:

$$c_{(\mathrm{A})} = 2\,\mathrm{reg}_1(n) - 2\,\mathrm{reg}_2(n), \qquad c_{(\mathrm{B})} = 2\,\mathrm{reg}_4(n).$$

Thus, finally, $J[f] = 2$.

REMARK 4.4. As an example of a map satisfying the assumptions of Example 4.3 one can take a self-map f of $M = S^2 \times S^2 \# S^2 \times S^2$, defined in the following way. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $g(x,y,z) = (-x,y,z)$. Assume that M is a result of attaching by the identity mapping two copies of $S^2 \times S^2$ along the boundary of a disk D such that if $(u,v) \in D$ then $(v,u) \in D$ and $(g(u),v) \in D$. Then also $(g(v),u) \in D$ for $(u,v) \in D$ and thus the map given by (4.5) below is well-defined.

Let us define the map f by

$$f((u,v)_1) = (g(v),u)_2, \qquad f((u,v)_2) = (g(v),u)_1, \tag{4.5}$$

where $(u,v)_i$, $i = 1,2$, denote the coordinates in the ith copy of $S^2 \times S^2$ in $S^2 \times S^2 \# S^2 \times S^2$. Then $f = \bar{g} \circ s \circ t$ is the composition of three self-maps $t, s, \bar{g}$ of $S^2 \times S^2 \# S^2 \times S^2$ defined as:

$$t((u,v)_i) = (v,u)_i, \tag{4.6}$$

$$s((u,v)_i) = (u,v)_{3-i}, \tag{4.7}$$

$$\bar{g}((u,v)_i) = (g(u),v)_i, \tag{4.8}$$

where $i = 1,2$. It is not difficult to verify that the induced map f_{*2} has eigenvalues $-i$ and i, each with multiplicity two, and that f is orientation preserving (because t is orientation preserving and $\bar{g}$ and s are orientation reversing).

REMARK 4.5. Notice that by Remark 3.4, we have $R = 4$ in Example 4.3. As a consequence, for each $r \geq 4$ there is a g in the smooth homotopy class of f such that g has two elements in the set $\bigcup_{k\leq r} \mathrm{Fix}(g^k)$ and by Remark 2.9 each of them is a fixed point. Furthermore, there is no h in the smooth homotopy class of f with less than two elements in $\bigcup_{k\leq r} \mathrm{Fix}(h^k)$.

4.3. Manifolds for which continuous and smooth categories coincide. In this subsection we identify a class of manifolds such that for any of their smooth self-maps f we have $J[f] = 1$, which means that one can always (as in the continuous case) reduce the number of periodic points of periods not exceeding a given r to just one fixed point.

Below we will call a map f *Lefschetz trivial* if $L(f^n) = 0$ for each n. Notice that for such a map $J[f] = 0$.

THEOREM 4.6. *Let M be a closed simply-connected manifold of dimension $m \geq 4$ such that $\dim H_i(M;\mathbb{Q}) \leq 1$ for each i and let f be a Lefschetz non-trivial C^1 self-map of M with periodic sequence of Lefschetz numbers of iterations. Then $J[f] = 1$.*

Proof. By Theorem 1.7, the Lefschetz numbers in this case are equal to $L(f^n) = \sum_i (-1)^i \lambda_i^n$, where each $\lambda_i \in \{-1, 1\}$ is an essential eigenvalue. On the other hand, we have

$$\lambda_i^n = \begin{cases} \operatorname{reg}_1(n) & \text{if } \lambda_i = 1, \\ -\operatorname{reg}_1(n) + \operatorname{reg}_2(n) & \text{if } \lambda_i = -1. \end{cases} \tag{4.9}$$

As a consequence, we may represent $L(f^n)$ in the form $L(f^n) = a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$ with some integers a_1 and a_2. By Table 1 we can realize each such sequence by one sequence of indices of iterations of type (A). Thus, for self-maps of a 4-dimensional manifold M we have $J[f] = 1$. On the other hand, every $DD^4(1)$ sequence is also a $DD^m(1)$ sequence for $m > 4$ (cf. [10]). Consequently, if $\dim M \geq 4$ then $J[f] = 1$. ∎

By Theorem 4.1 the larger the dimension m of the manifold M, the fewer sequences are required to obtain a given combination of basic sequences (Lefschetz numbers). As a consequence, we may generalize Theorem 4.6 to a larger class of manifolds, assuming the dimension of the manifold is higher. This technique is illustrated by the following

THEOREM 4.7. *Let M be a closed simply-connected manifold of dimension $m \geq 5$ such that $\dim H_q(M;\mathbb{Q}) = 2$ for some q and $\dim H_i(M;\mathbb{Q}) \leq 1$ for each $i \neq q$. Assume that f is a Lefschetz non-trivial C^1 self-map of M with periodic sequence of Lefschetz numbers of iterations. Then $J[f] = 1$.*

Proof. By an argument similar to that in the proof of Theorem 4.6 we may represent $L(f^n)$ in the form

$$L(f^n) = a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n) + L_d(n), \tag{4.10}$$

where $L_d(n) = \varepsilon_1^n + \cdots + \varepsilon_{\varphi(d)}^n$ is a sum of powers of all primitive roots of unity induced by f on $H_q(M;\mathbb{Q})$; $a_1, a_2 \in \mathbb{Z}$. As $H_q(M;\mathbb{Q})$ is two-dimensional, $\varphi(d) = 2$ and the only possible roots of unity are of degree $d = 1, 2, 3, 4, 6$. The roots of degree 1 and 2 give the contribution only to $a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n)$ so we must calculate the contribution of the roots of degrees $3, 4, 6$. Applying Theorem 3.1 we calculate the periodic expansion of $\{L(f^n)\}_{n=1}^{\infty}$ in each case; the results are given in Table 2.

Table 2. All possible sequences $\{L(f^n)\}_n$ given in (4.10)

Case	$L(f^n)$ for $d = 3, 4, 6$
$d = 3$	$a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n) + \operatorname{reg}_3(n)$
$d = 4$	$a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n) + \operatorname{reg}_4(n)$
$d = 6$	$a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n) - \operatorname{reg}_3(n) + \operatorname{reg}_6(n)$

The dimension of the manifold m is at least 5. We will show that each sequence in Table 2 is also a sequence of type (A^o) of Theorem 4.1 for $m = 5$, which proves that $J[f] = 1$ independently of f.

For $m = 5$ we have

$$L_2\left(\frac{m-3}{2}\right) = L_2(1) = \overline{\{2, d\}} = \{1, 2, d, [d, 2]\}.$$

Thus the sequence of type (A^o) is given by

$$a_1 \operatorname{reg}_1(n) + a_2 \operatorname{reg}_2(n) + a_d \operatorname{reg}_d(n) + a_{[d,2]} \operatorname{reg}_{[d,2]}(n),$$

and covers all forms of sequences listed in Table 2, which ends the proof. ∎

Corollary 4.8. *Let M be one of the following manifolds: S^m $(m \geq 4)$, $S^l \times S^k$ $(l \neq k;\ l, k \geq 2)$, $\mathbb{C}P^m$ $(m \geq 2)$, $\mathbb{H}P^m$ $(m \geq 1)$; $S^m \times S^m$ $(m > 2)$. Let f be a smooth self-map of M with periodic sequence of Lefschetz numbers of iterations. Then by Theorems* 4.6 *and* 4.7 *for each r there is a g in the smooth homotopy class of f such that either $\bigcup_{k \leq r} \operatorname{Fix}(g^k) = \{p\}$, where p is a fixed point of g, or $\bigcup_{k \leq r} \operatorname{Fix}(g^k) = \emptyset$.*

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