ANNALES
POLONICI MATHEMATICI
107.1 (2013)

Non-degenerate quadric surfaces of Weingarten type

by Dae Won Yoon (Jinju), Yilmaz Tunçer (Usak) and
Murat Kemal Karacan (Usak)

Abstract. We study quadric surfaces in Euclidean 3-space with non-degenerate second fundamental form, and classify them in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature and the second mean curvature.

1. Introduction. Let M be a surface in Euclidean 3-space $\mathbb{E}^3$. If M has non-degenerate second fundamental form II, we can regard this form as a new Riemannian (or pseudo-Riemannian) metric on M. In this case, we can define the Gaussian curvature and mean curvature of (M, II), denoted by K_{II} and H_{II} respectively..

For $X, Y \in \{K, H, K_{II}, H_{II}\}$, $X \neq Y$, if M satisfies the Jacobi equation

$$\Phi(X,Y) = \det\begin{pmatrix} X_u & X_v \\ Y_u & Y_v \end{pmatrix} = 0$$

or a linear equation $\alpha X + \beta Y = \gamma$, then it said to be an (X, Y)-*Weingarten surface* or an (X, Y)-*linear Weingarten surface*, respectively, where $X_u = \partial X/\partial u$, $X_v = \partial X/\partial v$ and $\alpha, \beta, \gamma \in \mathbb{R}$.

The inner geometry of the second fundamental form has been a popular research topic for a long time. W. Kühnel [11] and G. Stamou [13] investigated ruled (X, Y)-Weingarten surfaces in Euclidean 3-space $\mathbb{E}^3$. C. Baikoussis and Th. Koufogiorgos [1] studied helicoidal (H, K_{II})-Weingarten surfaces. M. I. Munteanu and A. I. Nistor [12] and D. W. Yoon [17] classified the polynomial translation (X, Y)-Weingarten surfaces in Euclidean 3-space, and F. Dillen and W. Kühnel [4] and F. Dillen and W. Sodsiri [5, 6] gave a classification of ruled (X, Y)-Weingarten surfaces in Minkowski 3-space $\mathbb{E}^3_1$, where $X, Y \in \{K, H, K_{II}\}$. D. Koutroufiotis [10] investigated closed ovaloid (X, Y)-linear Weingarten surfaces in $\mathbb{E}^3$. D. W. Yoon [16] and D. E. Blair

2010 *Mathematics Subject Classification*: Primary 53A05; Secondary 53B25.
Key words and phrases: Gaussian curvature, mean curvature, second Gaussian curvature, second mean curvature, quadric surface, Weingarten surface.

DOI: 10.4064/ap107-1-4

and Th. Koufogiorgos [2] classified ruled (X, Y)-linear Weingarten surfaces in $\mathbb{E}^3$. Recently, M. H. Kim and D. W. Yoon [8] studied (K, H)-Weingarten quadric surfaces in Euclidean 3-space.

An interesting geometric question is:

Classify all surfaces in Euclidean 3*-space and a Minkowski* 3*-space satisfying the condition*

$$\alpha X + \beta Y = \gamma,$$

where $X, Y \in \{K, H, K_{II}, H_{II}\}$, $X \neq Y$ *and* $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.

In this paper, we contribute to the solution of the above question, by studying it for quadric surfaces in Euclidean 3-space $\mathbb{E}^3$. We prove the following theorem:

THEOREM 1.1. *Let* α *and* β *be non-zero constants. Let* M *be a quadric surface with non-degenerate second fundamental form in Euclidean* 3*-space satisfying*

$$\alpha X + \beta Y = 0,$$

where $X \in \{K, H\}$, $Y \in \{H, K_{II}, H_{II}\}$. *Then* M *is an open part of an ordinary sphere or a hyperbolic paraboloid.*

2. Preliminaries. We describe a surface M in Euclidean 3-space $\mathbb{E}^3$ by

$$\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)).$$

Let $\mathbf{n}$ be the standard unit normal vector field on M defined by $\mathbf{n} = \mathbf{x}_u \times \mathbf{x}_v / \|\mathbf{x}_u \times \mathbf{x}_v\|$, where $\mathbf{x}_u = \partial \mathbf{x}(u, v)/\partial u$. Then the first fundamental form I and the second fundamental form II of M are defined by

$$I = E du^2 + 2F du dv + G dv^2, \qquad II = e du^2 + 2f du dv + g dv^2,$$

where

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle,$$
$$e = \langle \mathbf{x}_{uu}, \mathbf{n} \rangle, \quad f = \langle \mathbf{x}_{uv}, \mathbf{n} \rangle, \quad g = \langle \mathbf{x}_{vv}, \mathbf{n} \rangle.$$

The Gaussian curvature K and the mean curvature H are given by

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}.$$

By Brioschi's formula in Euclidean 3-space $\mathbb{E}^3$ (cf. [14]) we are able to define K_{II} of M by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g, respectively ([1], [2], [12], [17] etc.). Then

$$K_{II} = \frac{1}{(eg-f^2)^2} \times \left\{ \begin{vmatrix} -\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_u & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_u \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_u & f & g \end{vmatrix} \right\}.$$

It is said to be the *second Gaussian curvature* of M.

Next, we explain the second mean curvature H_{II} of M in $\mathbb{E}^3$. Let ∇ and $\widehat{\nabla}$ be the Levi-Civita connections of the metric tensors $I = g_{ij}dx^i dx^j$ and $II = L_{ij}dx^i dx^j$, respectively, and let Γ_{ij}^k and $\widehat{\Gamma}_{ij}^k$ be the Christoffel symbols of ∇ and $\widehat{\nabla}$, respectively. The difference tensor T is defined by

$$\mathrm{T}_{ij}^k = \widehat{\Gamma}_{ij}^k - \Gamma_{ij}^k \quad \text{for all } i,j,k \in \{1,2\}.$$

It is known that

$$\Gamma_{ik}^k = (\ln\sqrt{|\det I|})_{|i}, \quad \widehat{\Gamma}_{ik}^k = (\ln\sqrt{|\det II|})_{|i}, \quad \mathrm{T}_{ik}^k = (\ln\sqrt{|K|})_{|i},$$

where $\Phi_{|i}$ denotes the partial derivative $\partial\Phi/\partial u^i$.

Let D be a bounded connected open set whose closure $\overline{D}$ is contained in $U \subset \mathbb{R}^2$, let $\gamma : \overline{D} \to \mathbb{R}$ be a C^2-function such that $\gamma \equiv \partial\gamma/\partial s \equiv \partial\gamma/\partial t \equiv 0$ on the boundary of D and let $\widetilde{M} := \mathbf{x}(\overline{D})$ be a portion of M determined by $\mathbf{x}|_{\overline{D}} : \overline{D} \to \mathbb{E}^3$. Let $a \in \mathbb{R}^+$. The normal variation $\varphi : \overline{D} \times (-a,a) \to \mathbb{E}^3$ of $\widetilde{M}$ determined by γ is given by

$$\varphi(s,t,v) = \mathbf{x}(s,t) + v\gamma(s,t)\mathbf{n}(s,t)$$

for all $(s,t) \in \overline{D}$ and all $v \in (-a,a)$. For all $v \in (-a,a)$, define $\mathbf{x}^v : \overline{D} \to \mathbb{E}^3$ by

$$\mathbf{x}^v(s,t) = \varphi(s,t,v) \quad \text{for all } (s,t) \in \overline{D}.$$

If a is small enough, we can assume that $M^v := \mathbf{x}^v(\overline{D})$ is a portion of non-developable surface determined by $\mathbf{x}^v$ with II-area A_{II}^v is defined by

$$A_{II}^v = \iint\limits_{\overline{D}} \sqrt{|\det II^v|}\, ds\, dt,$$

where $II^v = L_{ij}^v dx^i dx^j$ is the second fundamental form of M^v. By a straightforward computation, we get

$$\left.\frac{\partial}{\partial v}\right|_{v=0} L_{ij}^v = \gamma K g_{ij} + \nabla_i \gamma_{|j} - 2\gamma H L_{ij},$$

which implies

$$\left.\frac{\partial}{\partial v}\right|_{v=0} \sqrt{|\det II^v|} = \left(\frac{1}{2}L^{ij}\nabla_i\gamma_{|j} - \gamma H\right)\sqrt{|\det II|}.$$

The first variation of A^v_{II} is

$$\frac{\partial}{\partial v}\bigg|_{v=0} A^v_{II} = \iint_{\overline{D}} \left(\frac{1}{2} L^{ij}\nabla_i \gamma_{|j} - \gamma H\right) dA_{II}.$$

On the other hand,

$$\iint_{\overline{D}} L^{ij}\nabla_i\gamma_{|j}\, dA_{II} = -\iint_{\overline{D}} \gamma \widehat{\nabla}_k (L^{ij}\mathrm{T}^k_{ij})\, dA_{II}.$$

Thus, we get,

$$\frac{\partial}{\partial v}\bigg|_{v=0} A^v_{II} = -\iint_{\overline{D}} \gamma\left(H + \frac{1}{2}L^{ij}\widehat{\nabla}_k \mathrm{T}^k_{ij}\right) dA_{II}.$$

We define

$$H_{II} := H + \frac{1}{2}L^{ij}\widehat{\nabla}_k \mathrm{T}^k_{ij}.$$

Furthermore,

$$L^{ij}\widehat{\nabla}_k \mathrm{T}^k_{ij} = L^{ij}\widehat{\nabla}_j \mathrm{T}^k_{ik} = L^{ij}\widehat{\nabla}_j (\ln\sqrt{|K|})_{|i} = \widehat{\Delta}(\ln\sqrt{|K|}),$$

where $\widehat{\Delta}$ is the Laplacian with respect to II. It follows that ([7])

$$\begin{aligned} H_{II} &= H + \frac{1}{2}\widehat{\Delta}(\ln\sqrt{|K|}) \\ &= H - \frac{1}{2\sqrt{|\det(L_{ij})|}} \sum_{i,j}^{2} \frac{\partial}{\partial x^i}\left(\sqrt{|\det(L_{ij})|}L^{ij}\frac{\partial}{\partial x^j}(\ln\sqrt{|K|})\right), \end{aligned}$$

where $\{x_i\}$ is a rectangular coordinate system in $\mathbb{E}^3$. The quantity H_{II} is called the *second mean curvature* of M.

Now, we define a quadric surface in $\mathbb{E}^3$. A subset M of Euclidean 3-space $\mathbb{E}^3$ is called a *quadric surface* if it is the set of points (x_1, x_2, x_3) satisfying the following equation of second degree:

$$\sum_{i=1}^{3} a_{ij}x_i x_j + \sum_{i=1}^{3} b_i x_i + c = 0,$$

where a_{ij}, b_i, c are all real numbers. Suppose that M is not a plane. Then $A = (a_{ij})$ is not a zero matrix and we may assume without loss of generality that it is symmetric. Possibly after applying a coordinate transformation in $\mathbb{E}^3$, M is either a ruled surface, or one of the following two kinds ([3]):

(2.1) $$x_3^2 - ax_1^2 - bx_2^2 = c, \qquad abc \neq 0,$$

or

(2.2) $$x_3 = \frac{a}{2}x_1^2 + \frac{b}{2}x_2^2, \qquad a > 0, b > 0.$$

If a surface satisfies (2.1), it is said to be a *quadric surface of the first kind*, and a surface satisfying (2.2) is called a *quadric surface of the second kind.*

3. Linear Weingarten quadric surfaces of the first kind. In this section, we investigate quadric surfaces of the first kind satisfying

$$\alpha X + \beta Y = 0,$$

where $X \in \{K, H\}, Y \in \{H, K_{II}, H_{II}\}$.

Let M_1 be a quadric surface of the first kind in $\mathbb{E}^3$ corresponding to $x_3 > 0$. Then M_1 can be parametrized by

$$\mathbf{x}(u,v) = (u, v, (c + au^2 + bv^2)^{1/2}).$$

Denote the function $c + au^2 + bv^2$ by ω. Then, using the natural frame $\{\mathbf{x}_u, \mathbf{x}_v\}$ of M_1 defined by $\mathbf{x}_u = (1, 0, au/\sqrt{\omega})$ and $\mathbf{x}_v = (0, 1, bv/\sqrt{\omega})$, the components E, F and G of the first fundamental form I of the surface are

$$E = 1 + a^2u^2/\omega, \quad F = abuv/\omega, \quad G = 1 + b^2v^2/\omega.$$

Moreover, the unit normal vector $\mathbf{n}$ of the surface M_1 is given by

$$\mathbf{n} = (-au/\sqrt{q}, -bv/\sqrt{q}, \sqrt{\omega}/\sqrt{q}),$$

where $q = a(a+1)u^2 + b(b+1)v^2 + c$. From this, the components e, f and g of the second fundamental form II are

$$e = q^{-1/2}\omega^{-1}A_0, \quad f = q^{-1/2}\omega^{-1}B_0, \quad g = q^{-1/2}\omega^{-1}C_0,$$

where $A_0 = a(bv^2 + c), B_0 = -abuv$ and $C_0 = b(au^2 + c)$.

Hence, the Gaussian curvature K and the mean curvature H are

$$K = \frac{1}{q^2}abc, \tag{3.1}$$

$$H = \frac{1}{2q^{3/2}}H_1, \tag{3.2}$$

where $H_1 = (a + b)c + (ab + a^2b)u^2 + (ab + ab^2)v^2$.

To find the second Gaussian curvature, we must compute the derivatives of the functions e, f and g with respect to u and v.

$$\begin{aligned} e_u &= q^{-3/2}\omega^{-2}A_1, \quad & e_v &= q^{-3/2}\omega^{-2}A_2, \quad & e_{vv} &= q^{-5/2}\omega^{-3}A_3, \\ f_u &= q^{-3/2}\omega^{-2}B_1, \quad & f_v &= q^{-3/2}\omega^{-2}B_2, \quad & f_{uv} &= q^{-5/2}\omega^{-3}B_3, \\ g_u &= q^{-3/2}\omega^{-2}C_1, \quad & g_v &= q^{-3/2}\omega^{-2}C_2, \quad & g_{uu} &= q^{-5/2}\omega^{-3}C_3, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
A_1 &= (abv^2 + ac)(-a(a+1)u\omega - 2auq),\\
A_2 &= 2abvq\omega + (abv^2 + ac)(-b(b+1)v\omega - 2bvq),\\
A_3 &= (-3b(b+1)v\omega - 4bvq)(2abv\omega q + (abv^2 + ac)(-b(b+1)v\omega - 2bvq))\\
&\quad + \omega q(2ab\omega q + ab^2(b+1)v^2\omega - 6ab^3(b+1)v^4 - 2ab^2v^2q\\
&\quad - ab(b+1)c\omega - 6ab^2(b+1)cv^2 - 2abcq),\\
B_1 &= -abv\omega q + a^2b(a+1)u^2v\omega + 2a^2bu^2vq,\\
B_2 &= -abu\omega q + ab^2(b+1)uv^2\omega + 2ab^2uv^2q,\\
B_3 &= (-3a(a+1)u\omega - 4auq)(-abu\omega q + ab^2(b+1)uv^2\omega + 2ab^2uv^2q)\\
&\quad + \omega q(-ab\omega q - 2a^2(a+1)bu^2\omega - 2a^2bu^2q + ab^2(b+1)v^2\omega\\
&\quad + 2a^2b^2(b+1)u^2v^2 + 2ab^2qv^2 + 4a^2(a+1)b^2u^2v^2),\\
C_1 &= 2abu\omega q + (abu^2 + bc)(-a(a+1)u\omega - 2auq),\\
C_2 &= (abu^2 + bc)(-b(b+1)v\omega - 2bvq),\\
C_3 &= (-3a(a+1)u\omega - 4auq)(2abu\omega q + (abu^2 + bc)(-a(a+1)u\omega - 2auq)\\
&\quad + \omega q(2ab\omega q + a^2(a+1)bu^2\omega - 6a^3(a+1)bu^4 - 2a^2bu^2q\\
&\quad - a(a+1)bc\omega - 6a^2(a+1)bcu^2 - 2abcq).
\end{aligned}$$

Thus, the second Gaussian curvature K_{II} of M_1 with the help of (3.3) turns out to be

$$K_{II} = \frac{1}{a^2b^2c^2q^{3/2}\omega^3}K_2, \tag{3.4}$$

where

$$\begin{aligned}
K_2 &= abc\omega(-\tfrac{1}{2}A_3 + B_3 - \tfrac{1}{2}C_3) + \tfrac{1}{4}A_1B_0C_2\\
&\quad + (B_1 - \tfrac{1}{2}A_2)(B_0B_2 - \tfrac{1}{2}B_0C_1 - \tfrac{1}{2}A_0C_2) - \tfrac{1}{2}A_1C_0(B_2 - \tfrac{1}{2}C_1)\\
&\quad - \tfrac{1}{2}A_2B_0C_1 + \tfrac{1}{4}A_0C_1^2 + \tfrac{1}{4}A_2^2C_0.
\end{aligned}$$

By straightforward computation, the second mean curvature H_{II} of M_1 is

$$H_{II} = \frac{1}{2q^{3/2}}H_1 + \frac{1}{cq^{3/2}}H_2, \tag{3.5}$$

where

$$\begin{aligned}
H_2 = {}&\\
&(a^4 + 2a^3 + a^2)u^4 + (b^4 + 2b^3 + b^2)v^4 + (2a^2b^2 + 2a^2b + 2ab + 2ab^2)u^2v^2\\
&+ (3ac + 2a^2c - a^3c + a^2bc + abc)u^2 + (2b^2c + ab^2c + 3bc + abc - b^3c)v^2\\
&+ ac^2 + bc^2 + 2c^2.
\end{aligned}$$

First, we investigate (K, H)-linear Weingarten quadric surfaces of the first kind in Euclidean 3-space.

Suppose that a quadric surface M_1 in $\mathbb{E}^3$ satisfies the linear equation

$$\alpha K + \beta H = 0. \tag{3.6}$$

By (3.1) and (3.2), equation (3.6) becomes

$$4\alpha^2 a^2 b^2 c^2 - \beta^2 q H_1^2 = 0. \tag{3.7}$$

The direct computation of the left hand side of (3.7) gives a polynomial in u and v with constant coefficients by adjusting the power of the functions q and H_1. The coefficients of u^6 and v^6 in (3.7) give, respectively,

$$\beta^2 a^3 b^2 (a+1)^3 = 0, \qquad \beta^2 a^2 b^3 (b+1)^3 = 0.$$

Thus, $a = -1$, $b = -1$ and $\alpha^2 = c\beta^2$. Therefore, M_1 is a sphere.

Secondly, we study a quadric surface M_1 in $\mathbb{E}^3$ satisfying the linear equation

$$\alpha K + \beta K_{II} = 0. \tag{3.8}$$

By (3.1) and (3.4), equation (3.8) becomes

$$\beta^2 q K_2^2 - \alpha^2 a^6 b^6 c^6 \omega^6 = 0. \tag{3.9}$$

By inserting the functions q, ω and K_2, equation (3.9) becomes polynomial in u and v with constant coefficients. From the coefficients of u^{22} and v^{22}, we have, respectively,

$$\tfrac{1}{4}\beta^2 a^{15} b^4 c^2 (a+1)^5 = 0, \qquad \tfrac{1}{4}\beta^2 a^4 b^{15} c^2 (b+1)^5 = 0,$$

so $a = -1$ and $b = -1$. In this case, from the coefficient of u^{12} in (3.9) we have $\alpha^2 = c\beta^2$, which implies equation (3.9) holds identically. Thus, M_1 is a sphere.

Thirdly, suppose that a quadric surface M_1 in $\mathbb{E}^3$ satisfies

$$\alpha H + \beta K_{II} = 0. \tag{3.10}$$

Then, by (3.2) and (3.4), equation (3.10) becomes

$$(\alpha a^2 b^2 c^2 H_1 \omega^3 + 2\beta K_2)^2 q^5 - 4\alpha^2 a^4 b^4 c^2 H_2^2 \omega^6 = 0. \tag{3.11}$$

The coefficients of u^{30} and v^{30} in (3.11) give, respectively,

$$\beta^2 a^{19} b^4 c^2 (a+1)^9 = 0, \qquad \beta^2 a^4 b^{19} c^2 (b+1)^9 = 0.$$

Thus, $a = -1$, $b = -1$ because $abc \neq 0$ and $\beta \neq 0$. In this case, the coefficient of u^{12} in (3.11) is given by $4c^{11}(\alpha + \beta)^2$. Since $c \neq 0$, $\alpha = -\beta$. Then from the conditions of a, b, α and β, equation (3.11) clearly holds.

Fourthly, we consider a quadric surface M_1 in $\mathbb{E}^3$ satisfying

$$\alpha K + \beta H_{II} = 0. \tag{3.12}$$

By using (3.1) and (3.5), equation (3.12) can be written as

$$\beta^2 c^2 H_1^2 q^5 - (2\beta H_2 + 2\alpha abc^2 q^2)^2 = 0, \tag{3.13}$$

and the coefficients of u^{14} and v^{14} in (3.13) give, respectively,

$$\beta^2 a^7 b^2 c^2 (a+1)^7 = 0, \quad \beta^2 a^2 b^7 c^2 (b+1)^7 = 0.$$

Thus, clearly, $a = -1$, $b = -1$. In this case, the surface M_1 is a sphere. On the other hand, from the values of a and b, equation (3.13) becomes

$$-4c^8(\alpha^2 - c\beta^2) = 0.$$

From this, $\alpha^2 = c\beta^2$, thus equation (3.13) clearly holds.

Fifthly, we consider a quadric surface M_1 in $\mathbb{E}^3$ satisfying

$$\alpha H + \beta H_{II} = 0. \tag{3.14}$$

By using (3.2) and (3.5), equation (3.14) can be written as

$$4\beta^2 H_2^2 - c^2(\alpha+\beta)^2 H_1^2 q^5 = 0, \tag{3.15}$$

and the coefficients of u^{14} and v^{14} in (3.15) give, respectively

$$-a^7 b^2 c^2 (\alpha+\beta)^2 (a+1)^7 = 0, \quad -a^2 b^7 c^2 (\alpha+\beta)^2 (b+1)^7 = 0,$$

which imply $a = b = -1$ or $\alpha = -\beta$. If $a = b = -1$, then the coefficient of the constant term in (3.15) is $-4c^9(\alpha+\beta)^2$. From this, we get $\alpha = -\beta$, in which case equation (3.15) clearly holds. So, M_1 is a sphere.

Consequently, we have the following theorem.

Theorem 3.1. *Let α and β be non-zero constants. If M_1 is a quadric surface of the first kind with non-degenerate second fundamental form in Euclidean 3-space satisfying the equation*

$$\alpha X + \beta Y = 0,$$

where $X \in \{K, H\}$, $Y \in \{H, K_{II}, H_{II}\}$, then M_1 is an open part of an ordinary sphere.

Remark. The unit sphere with radius 1 satisfies $K = -H = -K_{II} = -H_{II} = 1$.

4. Linear Weingarten quadric surfaces of the second kind. In this section, we study quadric surfaces of the second kind satisfying

$$\alpha X + \beta Y = 0,$$

where $X \in \{K, H\}, Y \in \{H, K_{II}, H_{II}\}$.

Let $\mathbf{x} : U \to \mathbb{E}^3$ be a quadric surface of the second kind in $\mathbb{E}^3$. Then

$$\mathbf{x}(u,v) = \left(u, v, \frac{a}{2}u^2 + \frac{b}{2}v^2\right).$$

From this, the components E, F and G of the first fundamental form are

$$E = 1 + a^2u^2, \qquad F = abuv, \qquad G = 1 + b^2v^2.$$

We define a smooth function q as follows:

$$q = \|\mathbf{x}_u \times \mathbf{x}_v\|^2 = 1 + a^2u^2 + b^2v^2,$$

so the unit normal vector field $\mathbf{n}$ of so M_2 is

$$\mathbf{n} = \frac{1}{\sqrt{q}}(-au, -bv, 1). \tag{4.1}$$

The components of the second fundamental form on M_2 are

$$e = a/\sqrt{q}, \qquad f = 0, \qquad g = b/\sqrt{q}.$$

On the other hand, the Gaussian curvature K and the mean curvature H are

$$K = \frac{ab}{q^2}, \qquad H = \frac{1}{2q^{3/2}}H_1, \tag{4.2}$$

where $H_1 = a^2bu^2 + ab^2v^2 + a + b$. By definitions, the second Gaussian curvature K_{II} and the second mean curvature H_{II} are

$$K_{II} = \frac{1}{2q^{3/2}}K_2, \qquad H_{II} = \frac{1}{q^{3/2}}\left(\frac{1}{2}H_1 - H_2\right), \tag{4.3}$$

where $K_2 = (a^2b - a^3)u^2 + (ab^2 - b^3)v^2 + a + b$ and $H_2 = (a^3 - a^2b)u^2 + (b^3 - ab^2)v^2 - a - b$.

Firstly, we suppose that M_2 satisfies the equation $\alpha K + \beta H = 0$. Then from (4.2) we have

$$4\alpha^2a^2b^2 - \beta^2qH_1^2 = 0.$$

Since the above equation depends on the variables u and v, all the coefficients of the powers of u and v must vanish. For the leading coefficients of u^6 and v^6, we have $-\beta^2a^6b^2 = 0$ and $-\beta^2a^2b^6 = 0$ respectively, which imply $a = 0$ or $b = 0$. This is a contradiction. Therefore, there is no (K, H)-linear Weingarten quadric surface.

Secondly, we study quadric surfaces M_2 in $\mathbb{E}^3$ satisfying $\alpha K + \beta K_{II} = 0$. By (4.2) and (4.3), we obtain

$$\beta^2qK_2^2 - 4\alpha^2a^2b^2 = 0. \tag{4.4}$$

The coefficient of u^6 in (4.4) is $\beta^2a^6(a - b)^2$, which implies $a = b$. In this case, equation (4.4) becomes

$$4\beta^2b^4u^2 + 4\beta^2b^4v^2 + 4\beta^2b^2 - 4\alpha^2b^4 = 0.$$

Therefore, $\alpha b = 0$ and $\beta b = 0$, a contradiction. Thus, there is no (K, K_{II})-linear Weingarten quadric surface.

Thirdly, we suppose that a quadric surface M_2 in $\mathbb{E}^3$ satisfies $\alpha H + \beta K_{II} = 0$. Then, by (4.2) and (4.3), we get

$$(\alpha a^2 b - \beta a^3 + \beta a^2 b)u^2 + (\alpha ab^2 + \beta ab^2 - \beta b^3)v^2 + \alpha a + \alpha b + \beta a + \beta b = 0,$$

which easily implies $a = -b$ and $\alpha = -2\beta$. Thus, the implicit equation of M_2 is given by $z = \frac{a}{2}x^2 - \frac{a}{2}y^2$, that is, a hyperbolic paraboloid.

Fourthly, we consider a quadric surface M_2 in $\mathbb{E}^3$ satisfying $\alpha K + \beta H_{II} = 0$. By using (4.2) and (4.3), we obtain

$$4\alpha^2 a^2 b^2 q - \beta^2 (qH_1 - 2H_2)^2 = 0, \tag{4.5}$$

and the coefficient of u^8 in (4.5) gives $-\beta^2 a^8 b^2 = 0$. In this case, we have $\beta ab = 0$, which is a contradiction. Therefore, there is no (K, H_{II})-linear Weingarten quadric surface.

Fifthly, we consider a quadric surface M_2 in $\mathbb{E}^3$ satisfying $\alpha H + \beta H_{II} = 0$. By using (4.2) and (4.3), we obtain

$$(\alpha + \beta)qH_1 - 2\beta H_2 = 0. \tag{4.6}$$

From the coefficient of u^4 in (4.6), we have $a^4 b(\alpha + \beta) = 0$, which implies $\alpha = -\beta$. In this case, equation (4.6) becomes

$$(-2\beta a^3 + 2\beta a^2 b)u^2 + (-2\beta b^3 + 2\beta ab^2)v^2 + 2\beta a + 2\beta b = 0,$$

which implies $a = b = 0$, a contradiction.

Consequently, we have the following theorems.

Theorem 4.1. *Let α and β be non-zero constants. If M_2 is a quadric surface of the second kind with non-degenerate second fundamental form in Euclidean 3-space satisfying $\alpha H + \beta K_{II} = 0$, then M_2 is an open part of a hyperbolic paraboloid. Furthermore, the hyperbolic paraboloid satisfies $K_{II} = 2H$.*

Theorem 4.2. *Let α and β be non-zero constants. There is no quadric surface of the second kind with non-degenerate second fundamental form in Euclidean 3-space satisfying $\alpha K + \beta H = 0$, $\alpha K + \beta K_{II} = 0$, $\alpha K + \beta H_{II} = 0$ or $\alpha H + \beta H_{II} = 0$.*

Combining Theorems 3.1, 4.1, 4.2 and the result of [5], we obtain the following

Theorem 4.3 (Characterization). *Let α and β be non-zero constants. Let M be a quadric surface with non-degenerate second fundamental form in Euclidean 3-space satisfying*

$$\alpha X + \beta Y = 0,$$

where $X \in \{K, H\}$, $Y \in \{H, K_{II}, H_{II}\}$. Then M is an open part of an ordinary sphere or a hyperbolic paraboloid.

Acknowledgements. The authors are grateful to the referee for his/her useful comments which improved the first version of the paper.

References

[1] C. Baikoussis and Th. Koufogiorgos, *On the inner curvature of the second fundamental form of helicoidal surfaces*, Arch. Math. (Basel) 68 (1997), 169–176.
[2] D. E. Blair and Th. Koufogiorgos, *Ruled surfaces with vanishing second Gaussian curvature*, Monatsh. Math. 113 (1992), 177–181.
[3] B.-Y. Chen and F. Dillen, *Quadrics of finite type*, J. Geom. 38 (1990), 16–22.
[4] F. Dillen, F. and W. Kühnel, *Ruled Weingarten surfaces in Minkowski 3-space*, Manuscripta Math. 98 (1999), 307–320.
[5] F. Dillen and W. Sodsiri, *Ruled surfaces of Weingarten type in Minkowski 3-space*, J. Geom. 83 (2005), 10–21.
[6] F. Dillen and W. Sodsiri, *Ruled surfaces of Weingarten type in Minkowski 3-space II*, J. Geom. 84 (2005), 37–44.
[7] E. Glässner, *Über die Minimalflächen der zweiten Fundamentalform*, Monatsh. Math. 78 (1974), 193–214.
[8] M. H. Kim and D. W. Yoon, *Weingarten quadric surfaces in a Euclidean 3-space*, Turkish J. Math. 35 (2011), 479–485.
[9] Th. Koufogiorgos and T. Hasanis, *A characteristic property of the sphere*, Proc. Amer. Math. Soc. 67 (1977), 303–305.
[10] D. Koutroufiotis, *Two characteristic properties of the sphere*, Proc. Amer. Math. Soc. 44 (1974), 176–178.
[11] W. Kühnel, *Ruled W-surfaces*, Arch. Math. (Basel) 62 (1994), 475–480.
[12] M. I. Munteanu and A. I. Nistor, *Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space*, in Differential Geometry (Santiago de Compostella, 2008), World Sci. 2009), 316–320.
[13] G. Stamou, *Regelflächen vom Weingarten-Typ*, Colloq. Math. 79 (1999), 77–84.
[14] D. J. Struik, *Differential Geometry*, Addison-Wesley, Reading, MA, 1961.
[15] S. Verpoort, *The geometry of the second fundamental form: Curvature properties and variational aspects*, PhD Thesis, Katholieke Univ. Leuven, 2008.
[16] D. W. Yoon, *Some properties of the helicoid as ruled surfaces*, JP J. Geom. Topology 2 (2002), 141–147.
[17] D. W. Yoon, *Polynomial translation surfaces of Weingarten type in Euclidean 3-space*, Centr. Eur. J. Math. 8 (2010), 430–436.

Dae Won Yoon
Department of Mathematics Education
and RINS
Gyeongsang National University
Jinju 660-701, South Korea
E-mail: dwyoon@gnu.ac.kr

Yılmaz Tunçer, Murat Kemal Karacan
Department of Mathematics
Faculty of Sciences and Arts
Usak University
1 Eylul Campus
64200, Usak, Turkey
E-mail: yilmaz.tuncer@usak.edu.tr
murat.karacan@usak.edu.tr

Received 29.9.2011
and in final form 10.1.2012 (2561)