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Curvature properties of a semi-symmetric metric connection on S-manifolds

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Abstract. In this study, S-manifolds endowed with a semi-symmetric metric connection naturally related with the S-structure are considered and some curvature properties of such a connection are given. In particular, the conditions of semi-symmetry, Ricci semi-symmetry and Ricci-projective semi-symmetry of this semi-symmetric metric connection are investigated.

1. Introduction. In 1963, Yano [28] introduced the notion of f-structure on an m-dimensional C^∞ manifold M, as a non-vanishing tensor field φ of type $(1,1)$ on M which satisfies $\varphi^3+\varphi=0$ and has constant rank r. It is known that r is even, say $r=2n$. Moreover, TM splits into two complementary subbundles $\operatorname{Im}\varphi$ and $\ker\varphi$ and the restriction of φ to $\operatorname{Im}\varphi$ determines a complex structure on this subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to $U(n)\times O(s)$ (see [3]), where $s=m-2n$. Almost complex $(s=0)$ and almost contact $(s=1)$ are well-known examples of f-structures. The case $s=2$ appeared in the study of hypersurfaces in almost contact manifolds [5, 12], which motivated Goldberg and Yano [13] to define globally framed f-manifolds (also called metric f-manifolds or f.pk-manifolds).

A wide class of globally framed f-manifolds was introduced by Blair in [3] according to the following definition: a metric f-structure is said to be a *K-structure* if the fundamental 2-form Φ given by $\Phi(X,Y)=g(X,\varphi Y)$ for any vector fields X and Y on M is closed and the normality condition holds, that is, $[\varphi,\varphi]+2\sum_{i=1}^{s} d\eta^i\otimes\xi_i=0$, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, ξ_i are the structure vector fields and η^i their dual 1-forms, $i=1,\dots,s$ (see Section 2 for further details). A K-manifold is called an *S-manifold*

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if $d\eta^k = \Phi$ for all $k = 1, \ldots, s$. S-manifolds have been studied by several authors (see, for example, [4, 6, 14, 17]).

Further, in 1924 Friedmann and Schouten [11] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Later, Hayden [15] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, Yano [29] made a systematic study of semi-symmetric metric connections on a Riemannian manifold. More precisely, if ∇ is a linear connection in a differentiable manifold M, then the *torsion tensor* T of ∇ is given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ for any vector fields X and Y on M. The connection ∇ is said to be *symmetric* if the torsion tensor T vanishes, otherwise it is said to be *non-symmetric*. The connection ∇ is said to be *semi-symmetric* if T is of the form $T(X, Y) = \eta(Y)X - \eta(X)Y$ for any X, Y, where η is a 1-form on M. Moreover, if g is a (pseudo)-Riemannian metric on M, then ∇ is called a *metric connection* if $\nabla g = 0$, otherwise it is called *non-metric*. It is well known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold.

It is worth pointing out here that (pseudo)-Riemannian manifolds endowed with a semi-symmetric metric connection are a particular case of the so-called Riemann–Cartan spaces (see, for instance, [23]), which have many physical applications. Thus, in the framework of general relativity theory, space-time is supplied with torsion in addition to curvature due to a known relationship between the torsion of an asymmetric metric connection and the spin tensor of matter. More physical applications of the notion of torsion were also discovered by Penrose [19]. There are various physical problems involving specifically semi-symmetric metric connections; for instance, the displacement on the earth surface following a fixed point is metric and semi-symmetric [22]. In this context, the interesting report of Suhendro [24] can be consulted. On the other hand, several authors have studied semi-symmetric metric connections on different types of Riemannian and semi-Riemannian manifolds (see, among many others, [2, 7, 8, 10, 18, 20, 25]).

The purpose of this paper is to link the two notions commented above by investigating the curvature properties of a certain semi-symmetric metric connection defined on S-manifolds and naturally related to the S-structure. To this end, in Section 2 we give a brief introduction to S-manifolds and in Section 3 we define a semi-symmetric metric connection on an S-manifold, obtaining some general results. In Section 4, we investigate the curvature and the Ricci tensor fields of such a connection. In particular, we prove that an S-manifold has constant f-sectional curvature with respect to this semi-symmetric metric connection if and only if it also has constant f-sectional curvature with respect to the Riemannian connection, giving the relationship between both constants. Consequently, the curvature of this semi-symmetric metric connection is completely determined by its f-sectional curvature.

Finally, in the last section, we present some results concerning the semi-symmetry, Ricci semi-symmetry and Ricci-projective semi-symmetry properties of a semi-symmetric metric connection. In particular, we prove that if an S-manifold is semi-symmetric with respect to such a connection, then it is of constant f-sectional curvature zero. We point out that the results obtained in the final section establish a clear difference between the cases $s \leq 2$ and $s > 2$.

2. Preliminaries on S-manifolds. A $(2n+s)$-dimensional differentiable manifold M is called a *metric f-manifold* if there exist a $(1,1)$ type tensor field φ, vector fields $\xi_1, \dots, \xi_s$, 1-forms $\eta^1, \dots, \eta^s$ and a Riemannian metric g on M such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta_{ij}, \quad \varphi\xi_i = 0, \quad \eta^i \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \tag{2.2}$$

for any $X, Y \in \mathcal{X}(M)$, $i, j \in \{1, \dots, s\}$, and moreover

$$\eta^i(X) = g(X, \xi_i), \qquad g(X, \varphi Y) = -g(\varphi X, Y). \tag{2.3}$$

Then, a 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$ for any $X, Y \in \mathcal{X}(M)$, called the *fundamental 2-form*. In what follows, we denote by $\mathcal{M}$ the distribution spanned by the structure vector fields $\xi_1, \dots, \xi_s$, and by $\mathcal{L}$ its orthogonal complementary distribution. Thus, $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$. If $X \in \mathcal{M}$, then $\varphi X = 0$, and if $X \in \mathcal{L}$, then $\eta^i(X) = 0$ for any $i \in \{1, \dots, s\}$, that is, $\varphi^2 X = -X$.

In a metric f-manifold, special local orthonormal bases of vector fields can be considered. Let U be a coordinate neighborhood and E_1 a unit vector field on U orthogonal to the structure vector fields. Then, from (2.1)–(2.3), φE_1 is also a unit vector field on U orthogonal to E_1 and the structure vector fields. Next, if possible, let E_2 be a unit vector field on U orthogonal to E_1, φE_1 and the structure vector fields and so on. The local orthonormal basis

$$\{E_1, \dots, E_n, \varphi E_1, \dots, \varphi E_n, \xi_1, \dots, \xi_s\}$$

so obtained is called an *f-basis*. Moreover, a metric f-manifold is *normal* if

$$[\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0,$$

where $[\varphi, \varphi]$ denotes the Nijenhuis tensor field associated to φ. A metric f-manifold is said to be an *S-manifold* if it is normal and

$$\eta^1 \wedge \cdots \wedge \eta^s \wedge (d\eta^i)^n \neq 0 \quad \text{and} \quad \Phi = d\eta^i, \ 1 \leq i \leq s.$$

Observe that, if $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds can be found in [3, 4, 14].

The following results are known for the Riemannian connection of an S-manifold:

Theorem 2.1 ([3]). *An S-manifold $(M, \varphi, \xi_i, \eta^i, g)$ satisfies the condition*

$$(\nabla^*_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, \varphi Y)\xi_i + \eta^i(Y)\varphi^2 X\} \tag{2.4}$$

for all $X, Y \in \mathcal{X}(M)$, where ∇^* denotes the Riemannian connection with respect to g.

Thus, from (2.4) we deduce that

$$\nabla^*_X \xi_i = -\varphi X \tag{2.5}$$

for any $X \in \mathcal{X}(M)$, $i \in \{1, \dots, s\}$.

Finally, for the curvature tensor field of the Riemannian connection of an S-manifold, we recall:

Theorem 2.2 ([6]). *Let $(M, \varphi, \xi_i, \eta^i, g)$ be an S-manifold of dimension $2n + s$. Then,*

$$R^*(X, Y)\xi_i = \sum_{j=1}^{s} \{\eta^j(X)\varphi^2 Y - \eta^j(Y)\varphi^2 X\}, \tag{2.6}$$

$$R^*(X, \xi_i)Y = -\sum_{j=1}^{s} \{g(\varphi X, \varphi Y)\xi_j + \eta^j(Y)\varphi^2 X\}, \tag{2.7}$$

for all $X, Y \in \mathcal{X}(M)$, $i, j \in \{1, \dots, s\}$, where R^* denotes the curvature tensor field of the Riemannian connection.

Corollary 2.3 ([6]). *Let $(M, \varphi, \xi_i, \eta^i, g)$ be an S-manifold of dimension $2n + s$. Then*

$$R^*(\xi_i, X, \xi_j, Y) = -g(\varphi X, \varphi Y), \tag{2.8}$$

$$K^*(\xi_i, X) = g(\varphi X, \varphi X), \tag{2.9}$$

$$S^*(X, \xi_i) = 2n \sum_{i=1}^{s} \eta^i(X), \tag{2.10}$$

for all $X, Y \in \mathcal{X}(M)$, $i, j \in \{1, \dots, s\}$, where K^* and S^* denote respectively the sectional curvature and the Ricci tensor field of the Riemannian connection.

Consequently, from (2.9), if $s \geq 2$, an S-manifold cannot have constant sectional curvature. For this reason, it is necessary to introduce a more restrictive curvature. In general, a plane section π on a metric f-manifold

$(M, \varphi, \xi_i, \eta^i, g)$ is said to be an *f-section* if it is determined by a unit vector X, normal to the structure vector fields and φX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has constant f-sectional curvature c; it is then denoted by $M(c)$. The curvature tensor field R^* of $M(c)$ satisfies (see [17])

$$R^*(X,Y,Z,W) = \sum_{i,j=1}^{s} \{g(\varphi X, \varphi W)\eta^i(Y)\eta^j(Z) - g(\varphi X, \varphi Z)\eta^i(Y)\eta^j(W) + g(\varphi Y, \varphi Z)\eta^i(X)\eta^j(W) - g(\varphi Y, \varphi W)\eta^i(X)\eta^j(Z)\} + \frac{c+3s}{4}\{g(\varphi X, \varphi W)g(\varphi Y, \varphi Z) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)\} + \frac{c-s}{4}\{\Phi(X,W)\Phi(Y,Z) - \Phi(X,Z)\Phi(Y,W) - 2\Phi(X,Y)\Phi(Z,W)\} \tag{2.11}$$

for any $X, Y, Z, W \in \mathcal{X}(M)$.

3. A semi-symmetric metric connection on S-manifolds. From now on, let M denote an S-manifold $(M, \varphi, \xi_i, \eta^i, g)$ of dimension $2n + s$. We define a new connection on M by

$$\nabla_X Y = \nabla^*_X Y + \sum_{j=1}^{s} \eta^j(Y)X - \sum_{j=1}^{s} g(X,Y)\xi_j \tag{3.1}$$

for any $X, Y \in \mathcal{X}(M)$. It is easy to show that ∇ is a linear connection on M. Moreover, we can prove:

THEOREM 3.1. *Let M be an S-manifold. The linear connection ∇ defined in* (3.1) *is a semi-symmetric metric connection on M.*

Proof. By (3.1) and the fact that the Riemannian connection is torsion-free, the torsion tensor T of the connection ∇ is given by

$$T(X,Y) = \sum_{j=1}^{s} \{\eta^j(Y)X - \eta^j(X)Y\} \tag{3.2}$$

for any $X, Y \in \mathcal{X}(M)$. Moreover, by using (3.1) again, for all $X, Y, Z \in \mathcal{X}(M)$ and since ∇^* is a metric connection, we have

$$(\nabla_X g)(Y,Z) = 0. \tag{3.3}$$

From (3.2) and (3.3) we conclude that the linear connection ∇ is a semi-symmetric metric connection on M. ■

For example, let us consider $\mathbb{R}^{2n+s}$ with its standard S-structure given in [14]:

$$\eta^a = \frac{1}{2}\Big(dz^a - \sum_{i=1}^{n} y^i dx^i\Big), \qquad \xi_a = 2\frac{\partial}{\partial z^a},$$

$$g = \sum_{\alpha=1}^{s} \eta^a \otimes \eta^a + \frac{1}{4}\Big(\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)\Big),$$

$$\varphi\Big(\sum_{i=1}^{n}\Big(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\Big) + \sum_{a} Z_a\frac{\partial}{\partial z^\alpha}\Big)$$
$$= \sum_{i=1}^{n}\Big(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\Big) + \sum_{\alpha=1}^{s}\sum_{i=1}^{n} Y_i y^i \frac{\partial}{\partial z^\alpha},$$

where (x^i, y^i, z^a), $i = 1, \dots, n$ and $\alpha = 1, \dots, s$, are the cartesian coordinates. It is known that, with this structure, $\mathbb{R}^{2n+s}$ is an S-space-form of constant f-sectional curvature $c = -3s$. If, following [14], we denote

$$(x^1, \dots, x^n, y^1, \dots, y^n, z^1, \dots, z^s) = (x^1, \dots, x^{2n+s}),$$

the Christoffel symbols of the semi-symmetric metric connection defined in (3.1) are given by

$$\Gamma^b_{ai} = \Gamma^{*b}_{ai} - \frac{1}{2}sy_i\delta_{ab} - 2\sum_{\alpha=2n+1}^{2n+s} g_{ai}\delta_{\alpha b}, \qquad \Gamma^b_{a\lambda} = \Gamma^{*b}_{a\lambda} - 2\sum_{\alpha=2n+1}^{2n+s} g_{a\lambda}\delta_{\alpha b},$$

$$\Gamma^b_{a\beta} = \Gamma^{*b}_{a\beta} + \frac{1}{2}\delta_{ab} - 2\sum_{\alpha=2n+1}^{2n+s} g_{a\beta}\delta_{\alpha b},$$

for any $a, b \in \{1, \dots, 2n+s\}$, $i \in \{1, \dots, n\}$, $\lambda \in \{n+1, \dots, 2n\}$ and $\beta \in \{2n+1, \dots, 2n+s\}$, where Γ^{*b}_{ai}, $\Gamma^{*b}_{a\lambda}$ and $\Gamma^{*b}_{a\alpha}$ denote the Christoffel symbols of the Riemannian connection of $\mathbb{R}^{2n+s}$ (see [14] for the details).

Throughout this paper, we always use the letter ∇ to denote the semi-symmetric metric connection defined in (3.1). Observe that, following the notation of [2, 29], in this case the 1-form π and the vector field P which define the connection ∇ are

$$\pi = \sum_{i=1}^{s} \eta^i \quad \text{and} \quad P = \sum_{i=1}^{s} \xi_i.$$

Proposition 3.2. *Let M be an S-manifold. Then*

$$\nabla_X \xi_i = -\varphi X + X - \sum_{j=1}^{s} \eta^i(X)\xi_j, \tag{3.4}$$

$$(\nabla_X \eta_i)Y = g(X, \varphi Y) + g(X, Y) - \sum_{j=1}^{s} \eta^i(X)\eta^j(Y), \tag{3.5}$$

for any $X, Y \in \mathcal{X}(M)$ and $i \in \{1, \dots, s\}$.

Proof. First, (3.4) is a direct consequence of (3.1), taking into account (2.5). Now, by using (3.3) and (3.4), since

$$(\nabla_X \eta^i)(Y) = X\eta^i(Y) - \eta^i(\nabla_X Y) = g(Y, \nabla_X \xi_i),$$

we deduce (3.5). ∎

THEOREM 3.3. *Let M be an S-manifold. Then*

$$(\nabla_X \varphi)Y = \sum_{i=1}^{s} \{(g(\varphi X, \varphi Y) - g(X, \varphi Y))\xi_i + \eta^i(Y)(\varphi^2 X - \varphi X)\} \tag{3.6}$$

for all $X, Y \in \mathcal{X}(M)$.

Proof. From (3.1), we get

$$(\nabla_X \varphi)Y = (\nabla^*_X \varphi)Y - \sum_{i=1}^{s} \eta^i(Y)\varphi X - \sum_{i=1}^{s} g(X, \varphi Y)\xi_i.$$

Therefore, we obtain the result from (2.4). ∎

By using (2.1) and (3.6), we easily prove:

COROLLARY 3.4. *Let M be an S-manifold. Then*

$$(\nabla_X \varphi)\xi_i = -\varphi \nabla_X \xi_i = \varphi^2 X - \varphi X, \tag{3.7}$$

$$\nabla_{\xi_i} \varphi X = \varphi \nabla_{\xi_i} X, \tag{3.8}$$

for all $X \in \mathcal{X}(M)$, $i \in \{1, \dots, s\}$.

4. The curvature of ∇. Let M be an S-manifold endowed with the semi-symmetric metric connection ∇ defined in (3.1). From formula (2.3) in [2], if R and R^* denote the curvature tensor fields of ∇ and ∇^*, respectively, then

$$\begin{aligned} R(X,Y)Z = R^*(X,Y)Z &+ s\{g(X, \varphi Z)Y \\ &- g(Y, \varphi Z)X + g(Y,Z)\varphi X) \\ &- g(X,Z)\varphi Y + g(X,Z)Y - g(Y,Z)X\} \\ &+ \sum_{i,j=1}^{s} \{\eta^i(Y)\eta^j(Z)X - \eta^i(X)\eta^j(Z)Y \\ &+ g(Y,Z)\eta^i(X)\xi_j - g(X,Z)\eta^i(Y)\xi_j\} \end{aligned} \tag{4.1}$$

for all $X, Y, Z \in \mathcal{X}(M)$.

First, we want to investigate the sectional curvature associated with ∇. To this end, we need to establish the following symmetry for R which can be deduced from (4.1):

Proposition 4.1. *Let M be an S-manifold. Then*

$$R(X,Y,Z,W)-R(Z,W,X,Y)=2s\{g(X,\varphi Z)g(Y,W) - g(Y,\varphi Z)g(X,W)-g(X,\varphi W)g(Y,Z)+g(Y,\varphi W)g(X,Z)\} \tag{4.2}$$

for any $X,Y,Z,W\in\mathcal{X}(M)$.

Moreover, from (2.6), (2.7) and (4.1), we get some formulas involving the structure vector fields:

Proposition 4.2. *Let M be an S-manifold. Then*

$$R(X,Y)\xi_i=\sum_{j=1}^{s}\{\eta^i(X)\nabla_Y\xi_j-\eta^i(Y)\nabla_X\xi_j + \eta^j(X)(\varphi^2Y-Y)-\eta^j(Y)(\varphi^2X-X)\}, \tag{4.3}$$

$$\begin{aligned} R(X,\xi_i)Y&=2\sum_{j=1}^{s}\{\eta^j(Y)X-g(X,Y)\xi_j\}\\ &+s\{(g(X,\varphi Y)+g(X,Y))\xi_i+\eta^i(Y)(\varphi X-X)\}\\ &+\sum_{j,k=1}^{s}\{\eta^j(X)(\eta^j(Y)+\eta^i(Y))\xi_k\\ &-\eta^j(X)\eta^k(Y)(\xi_j+\xi_i)\}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} R(X,\xi_i)\xi_j&=2X-\sum_{k=1}^{s}\{\eta^k(X)(\xi_k+\xi_i)+\eta^j(X)\xi_k\}\\ &+s\{\eta^j(X)\xi_i+\delta_{ij}(\varphi X-X)\}+\delta_{ij}\sum_{k,l=1}^{s}\eta^k(X)\xi_l, \end{aligned} \tag{4.5}$$

$$\begin{aligned} R(\xi_i,\xi_j)X&=\sum_{k=1}^{s}\{\eta^k(X)(\xi_i-\xi_j)+(\eta^j(X)-\eta^i(X))\xi_k\}\\ &+s(\eta^i(X)\xi_j-\eta^j(X)\xi_i), \end{aligned} \tag{4.6}$$

$$R(\xi_i,\xi_j)\xi_k=\xi_i-\xi_j-(\delta_{ik}-\delta_{jk})\sum_{l=1}^{s}\xi_l+s(\delta_{ik}\xi_j-\delta_{jk}\xi_i), \tag{4.7}$$

for all $X,Y\in\mathcal{X}(M)$ and $i,j,k\in\{1,\dots,s\}$.

Now, by using the above propositions, we can prove the following theorem for the sectional curvature K of ∇.

Theorem 4.3. *Let M be an S-manifold. Then the sectional curvature of ∇ satisfies*

(i) $K(X,Y)=K^*(X,Y)-s$,
(ii) $K(X,\xi_i)=K(\xi_i,X)=2-s$,

(iii) $K(\xi_i,\xi_j)=K(\xi_j,\xi_i)=2-s$,

for any $X,Y\in\mathcal{L}$ *and* $i,j\in\{1,\ldots,s\}$, $i\neq j$.

Proof. First, from (4.1), if $X,Y\in\mathcal{L}$, then

$$R(X,Y,Y,X)=R^*(X,Y,Y,X)+s(g(X,Y)^2-g(X,X)g(Y,Y)),$$

and we deduce (i). Now, from (4.4), if $X\in\mathcal{L}$,

$$R(\xi_i,X)X=g(X,X)\Big\{2\sum_{j=1}^{s}\xi_j-s\xi_i\Big\}$$

for any $i\in\{1,\ldots,s\}$. Then, taking into account (4.2), we obtain (ii). Finally, (iii) is a direct consequence of (4.7). ■

Therefore, if $s\neq 2$, an S-manifold cannot be of constant sectional curvature with respect to the semi-symmetric metric connection defined in (4.1). But, what about the f-sectional curvature? First, we have:

PROPOSITION 4.4. *Let* M *be an* S*-manifold. Then*

$$R(\varphi X,\varphi Y,\varphi Z,\varphi W)=R(X,Y,Z,W) \tag{4.8}$$

for any $X,Y,Z,W\in\mathcal{L}$.

Proof. This is a direct computation from (4.1) taking into account that (see [3])

$$R^*(\varphi X,\varphi Y,\varphi Z,\varphi W)=R^*(X,Y,Z,W)$$

for any $X,Y,Z,W\in\mathcal{L}$. ■

Consequently, the f-sectional curvature of ∇ is well defined, since, by using (4.8), we find that, for any unit vector field $X\in\mathcal{L}$,

$$R(X,\varphi X,\varphi X,X)=R^*(X,\varphi X,\varphi X,X)-s. \tag{4.9}$$

Then, taking into account (2.11), from (4.1) and (4.9) we can deduce the following theorem:

THEOREM 4.5. *Let* M *be an* S*-manifold. Then the* f*-sectional curvature associated with the semi-symmetric metric connection* ∇ *is constant if and only if the* f*-sectional curvature associated with the Riemannian connection is constant. In this case, if* c *denotes the constant* f*-sectional curvature of the Riemannian connection, then* $c-s$ *is the constant* f*-sectional curvature of* ∇*. Moreover, the curvature tensor field of* ∇ *is completely determined by* c *and it is given by*

$$
\begin{aligned}
R(X,Y,Z,W) = & \sum_{i,j=1}^{s}\{2g(X,W)\eta^i(Y)\eta^j(Z) - 2g(Y,W)\eta^i(X)\eta^j(Z) \\
& + 2g(Y,Z)\eta^i(X)\eta^j(W) - 2g(X,Z)\eta^i(Y)\eta^j(W)\} \\
& + \sum_{i,j,k=1}^{s}\{\eta^i(X)\eta^k(Y)\eta^j(Z)\eta^k(W) \\
& - \eta^k(X)\eta^i(Y)\eta^j(Z)\eta^k(W)\} \\
& + \eta^k(X)\eta^i(Y)\eta^k(Z)\eta^j(W) - \eta^i(X)\eta^k(Y)\eta^j(W)\eta^k(Z)\} \\
& + \frac{c+3s}{4}\{g(\varphi X,\varphi W)g(\varphi Y,\varphi Z) - g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\} \\
& + \frac{c-s}{4}\{\Phi(X,W)\Phi(Y,Z) \\
& - \Phi(X,Z)\Phi(Y,W) - 2\Phi(X,Y)\Phi(Z,W)\} \\
& + s\{g(\varphi Z,X)g(Y,W) - g(X,W)g(\varphi Z,Y) \\
& + g(Y,Z)g(\varphi X,W) + g(X,Z)g(Y,W) \\
& - g(Y,Z)g(X,W) - g(X,Z)g(\varphi Y,W)\}
\end{aligned}
$$

for any $X,Y,Z,W \in \mathcal{X}(M)$.

For the Ricci tensor field S of the connection ∇, from formula (2.6) in [2] we deduce that

$$
\begin{aligned}
\text{(4.10)}\qquad S(X,Y) = & S^*(X,Y) \\
& + (2n+s-2)\Big\{\sum_{i,j=1}^{s}\eta^i(X)\eta^j(Y) - sg(X,\varphi Y) - sg(X,Y)\Big\}
\end{aligned}
$$

for any $X,Y \in \mathcal{X}(M)$, where S^* denotes the Ricci tensor field of the Riemannian connection and, as before, $\dim(M) = 2n+s$. Since S^* is a symmetric tensor field, we deduce that

$$
S(X,Y) - S(Y,X) = -2(2n+s-2)g(X,\varphi Y) \tag{4.11}
$$

for any $X,Y \in \mathcal{X}(M)$. Therefore, S is not a symmetric tensor field. Moreover, by using (2.10) we obtain

PROPOSITION 4.6. *Let M be an S-manifold. Then*

$$
S(X,\xi_i) = S(\xi_i,X) = (4n+s-2)\sum_{j=1}^{s}\eta^j(X) - s(2n+s-2)\eta^i(X) \tag{4.12}
$$

for any $X \in \mathcal{X}(M)$ and $i \in \{1,\dots,s\}$.

COROLLARY 4.7. *Let M be an S-manifold. Then*

$$
S(\xi_j,\xi_i) = (4n+s-2) - s(2n+s-2)\delta_{ij} \tag{4.13}
$$

for any $i,j = \{1,\dots,s\}$.

Now, we can prove:

PROPOSITION 4.8. *Let M be an S-manifold. Then*

$$S(\varphi X, \varphi Y) = S(X, Y) \tag{4.14}$$

for any $X, Y \in \mathcal{L}$.

Proof. This is a direct consequence of (4.10) taking into account that

$$S^*(\varphi X, \varphi Y) = S^*(X, Y)$$

(see Proposition 3.7 in [6]). ∎

COROLLARY 4.9. *Let M be an S-manifold. Then*

$$S(X, Y) = S(\varphi X, \varphi Y) + \sum_{i,j=1}^{s} \eta^i(X)\eta^j(Y)S(\xi_i, \xi_j) \tag{4.15}$$

for all $X, Y \in \mathcal{X}(M)$.

Proof. We can put

$$X = X_0 + \sum_{i=1}^{s} \eta^i(X)\xi_i \quad \text{and} \quad Y = Y_0 + \sum_{j=1}^{s} \eta^j(Y)\xi_j,$$

where $X_0, Y_0 \in \mathcal{L}$. Then, since from (2.3) and (4.12), $S(X_0, \xi_j) = S(\xi_i, Y_0) = 0$, we obtain

$$S(X, Y) = S(X_0, Y_0) + \sum_{i,j=1}^{s} \eta^i(X)\eta^j(Y)S(\xi_i, \xi_j). \tag{4.16}$$

Now, by (2.1) and (4.14), $S(X_0, Y_0) = S(\varphi X_0, \varphi Y_0) = S(\varphi X, \varphi Y)$ and the proof is complete. ∎

5. Semi-symmetry properties of an S-manifold with respect to ∇. Let us recall that, given a Riemannian manifold (M, g) of dimension $n \geq 3$ endowed with a linear connection ∇ whose curvature tensor field is denoted by R, for any $(0, k)$-tensor field T on M, $k \geq 1$, the $(0, k+2)$-tensor field $R.T$ is defined by

$$(R.T)(X_1, \ldots, X_k, X, Y) = -\sum_{i=1}^{k} T(X_1, \ldots, X_{i-1}, R(X, Y)X_i, X_{i+1}, \ldots, X_k) \tag{5.1}$$

for any $X, Y, X_1, \ldots, X_k \in \mathcal{X}(M)$. In this context, M is said to be *semi-symmetric with respect to* ∇ if $R.R = 0$, and *Ricci semi-symmetric* if $R.S = 0$, where S denotes the Ricci tensor field of ∇. For the Riemannian connection it is known that semi-symmetry implies Ricci semi-symmetry (for more details, [9, 26] and references therein can be consulted; specifically, for the contact geometry case we recommend the papers [16, 21, 27]).

In this context, for the semi-symmetric metric connection defined in (3.1) on an S-manifold M we can prove:

THEOREM 5.1. *Let M be a $(2n+s)$-dimensional S-manifold ($n \geq 1$) which is a semi-symmetric manifold with respect to the semi-symmetric metric connection ∇. Then M has constant f-sectional curvature $c = 0$ with respect to ∇.*

Proof. If $R.R = 0$, then from (5.1) we deduce that

$$\begin{aligned} R(R(X,\xi_i)X,\varphi X,\varphi X,\xi_j) + R(X,R(X,\xi_i)\varphi X,\varphi X,\xi_j) \\ + R(X,\varphi X,R(X,\xi_i)\varphi X,\xi_j) + R(X,\varphi X,\varphi X,R(X,\xi_i)\xi_j) = 0 \end{aligned} \tag{5.2}$$

for any unit vector field $X \in \mathcal{L}$ and any $i,j = 1,\ldots,s$. By using (4.4) and (4.5), a direct expansion of (5.2) gives $(2-s\delta_{ij})R(X,\varphi X,\varphi X,X) = 0$, which completes the proof. ∎

Therefore, from Theorem 4.5 we deduce:

COROLLARY 5.2. *A semi-symmetric $(2n+s)$-dimensional ($n \geq 1$) S-manifold with respect to the semi-symmetric metric connection ∇ is an S-space-form of constant f-sectional curvature equal to s.*

We point out that it is known that if an S-manifold is semi-symmetric with respect to the Riemannian connection ∇^*, then it is also an S-space-form of constant f-sectional curvature equal to s ([1]).

Moreover, we have:

THEOREM 5.3. *Let M be a $(2n+s)$-dimensional S-manifold with $n \geq 1$ and $s \geq 3$. Then M cannot be a Ricci semi-symmetric manifold with respect to the semi-symmetric metric connection ∇.*

Proof. Suppose that $R.S = 0$. Then, from (5.1),

$$S(R(X,\xi_i)\xi_j,\varphi X) + S(\xi_j,R(X,\xi_i)\varphi X) = 0 \tag{5.3}$$

for any unit vector field $X \in \mathcal{L}$ and any $i,j \in \{1,\ldots,s\}$, $i \neq j$. Now, by using (4.5),

$$S(R(X,\xi_i)\xi_j,\varphi X) = 2S(X,\varphi X). \tag{5.4}$$

Next, from (4.4) and (4.13),

$$S(\xi_j,R(X,\xi_i)\varphi X) = -s(4n+s-2). \tag{5.5}$$

Consequently, if we insert (5.4) and (5.5) into (5.3), we get

$$2S(X,\varphi X) = (4n+s-2)s. \tag{5.6}$$

But, since from (2.1) and (4.14), $S(\varphi X, X) = -S(X, \varphi X)$, using (5.6), we deduce $S(X, \varphi X) - S(\varphi X, X) = (4n + s - 2)s$. But, from (4.11) we obtain that $S(X, \varphi X) - S(\varphi X, X) = 2(2n + s - 2)s$, which is a contradiction. ∎

Concerning the case $s = 2$, we can prove the following theorem.

THEOREM 5.4. *Let M be a $(2n + 2)$-dimensional S-manifold, $n \geq 1$. If M is a Ricci semi-symmetric manifold with respect to the semi-symmetric metric connection ∇, then the Ricci tensor field of ∇ satisfies*

$$S(X, Y) = -4ng(X, \varphi Y) + \sum_{i,j=1}^{2} \eta^i(X)\eta^j(Y)S(\xi_i, \xi_j)$$

for any $X, Y \in \mathcal{X}(M)$.

Proof. By (2.1) and (4.16), it is sufficient to prove that $S(X, Y) = -4ng(X, \varphi Y)$ for any $X, Y \in \mathcal{L}$.

So let $X, Y \in \mathcal{L}$. Then, since $R.S = 0$, from (5.1) we obtain

$$S(R(X, \xi_1)\xi_2, Y) + S(\xi_2, R(X, \xi_1)Y) = 0. \tag{5.7}$$

But, by using (4.5) we get $S(R(X, \xi_1)\xi_2, Y) = 2S(X, Y)$, and by using (4.4) and (4.13) we obtain $S(\xi_2, R(X, \xi_1)Y) = 8ng(X, \varphi Y)$. Consequently, (5.7) yields the assertion. ∎

For Sasakian manifolds (case $s = 1$), we can prove:

THEOREM 5.5. *Let M be a $(2n + 1)$-Sasakian manifold, $n \geq 1$. If M is a Ricci semi-symmetric manifold with respect to the semi-symmetric metric connection ∇, then the Ricci tensor field of ∇ satisfies*

$$S(X, Y) - S(X, \varphi Y) = 2n\{g(X, Y) - g(X, \varphi Y)\}$$

for any $X, Y \in \mathcal{L}$.

Proof. Since $R.S = 0$, the definition (5.1) gives

$$S(R(X, \xi)\xi, Y) + S(\xi, R(X, \xi)Y) = 0. \tag{5.8}$$

But, from (4.5) and (4.14) we get $S(R(X, \xi)\xi, Y) = S(X, Y) - S(X, \varphi Y)$, and from (4.4) and (4.13), $S(\xi, R(X, \xi)Y) = 2n\{g(X, \varphi Y) - g(X, Y)\}$. Now, (5.8) gives the conclusion. ∎

Finally, we consider the Weyl projective curvature tensor field of ∇ given by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{\dim(M) - 1}\{S(Y, Z)X - S(X, Z)Y\} \tag{5.9}$$

for any $X, Y, Z, \in \mathcal{X}(M)$. Then the S-manifold M is said to be *Ricci-projectively semi-symmetric* with respect to the semi-symmetric metric connection ∇ if $P.S = 0$, where, taking into account (5.9),

$$(P.S)(X,Y,Z,W) = -S(P(X,Y)Z,W) - S(Z,P(X,Y)W) \tag{5.10}$$
$$= (R.S)(X,Y,Z,W)$$
$$+\frac{1}{2n+s-1}\{S(X,W)(S(Y,Z)-S(Z,Y)) + S(Y,W)(S(Z,X)-S(X,Z))\}$$

for any $X,Y,Z,W \in \mathcal{X}(M)$. We can prove the following theorem.

THEOREM 5.6. *Let M be a $(2n+s)$-dimensional S-manifold, $n \geq 1$. Then:*

(i) *If $s \geq 3$, M cannot be Ricci-projectively semi-symmetric with respect to ∇.*
(ii) *If $s = 2$ and M is Ricci-projectively semi-symmetric with respect to ∇, then the Ricci tensor field of ∇ satisfies*

$$S(X,Y) = -4ng(X,\varphi Y) + \sum_{i,j=1}^{2} \eta^i(X)\eta^j(Y)S(\xi_i,\xi_j)$$

for any $X,Y \in \mathcal{X}(M)$.
(iii) *If M is a Ricci-projectively semi-symmetric Sasakian manifold (that is, if $s = 1$) with respect to ∇, then the Ricci tensor field of ∇ satisfies*

$$S(X,Y) - S(X,\varphi Y) = 2n\{g(X,Y) - g(X,\varphi Y)\}$$

for any $X,Y \in \mathcal{L}$.

Proof. By using (2.1), (4.11) and (5.10), we get

$$(P.S)(X,\xi_i,\xi_j,Y) = (R.S)(X,\xi_i,\xi_j,Y)$$

for any $X,Y \in \mathcal{L}$ and $i,j \in \{1,\ldots,s\}$. Consequently, we complete the proof by using the same line of reasoning as in Theorems 5.3–5.5. ■

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