ANNALES
POLONICI MATHEMATICI
100.2 (2011)

Existence and uniqueness of periodic solutions for odd-order ordinary differential equations

by Yongxiang Li and He Yang (Lanzhou)

Abstract. The paper deals with the existence and uniqueness of 2π-periodic solutions for the odd-order ordinary differential equation

$$u^{(2n+1)} = f(t, u, u', \dots, u^{(2n)}),$$

where $f : \mathbb{R} \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is continuous and 2π-periodic with respect to t. Some new conditions on the nonlinearity $f(t, x_0, x_1, \dots, x_{2n})$ to guarantee the existence and uniqueness are presented. These conditions extend and improve the ones presented by Cong [Appl. Math. Lett. 17 (2004), 727–732].

1. Introduction and main results. In this paper, we extend the existence and uniqueness results of [4] for periodic solutions of the odd-order ordinary differential equation

$$u^{(2n+1)}(t) = f(t, u(t), u'(t), \dots, u^{(2n)}(t)), \tag{1.1}$$

where $n \geq 1$ is an integer, and $f : \mathbb{R} \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is continuous and 2π-periodic with respect to t.

The existence and uniqueness of periodic solutions is an important topic in the qualitative theory of ordinary differential equations. For first and second order differential equations, the problem has been widely and deeply investigated. In recent years, there has been increasing interest in the case of higher order equations (see [1–5, 7–10] for more details). However, odd-order equations are studied relatively rarely. In [5], Cong, Huang and Shi considered the special odd-order differential equation

$$u^{(2n+1)}(t) + \sum_{i=0}^{n-1} c_i u^{(2i+1)}(t) = g(t, u(t)), \tag{1.2}$$

2010 *Mathematics Subject Classification*: Primary 34C25; Secondary 34B15.
Key words and phrases: odd-order differential equation, periodic solution, existence and uniqueness, equivalent norm, fixed-point theorem.

DOI: 10.4064/ap100-2-1

Under the condition that

$$m \le |g_x(t,x)| \le M, \qquad \forall (t,x) \in \mathbb{R}\times\mathbb{R}$$

(where m and M are positive constants) and other assumptions they obtained the existence and uniqueness of periodic solutions for (1.2). In [4], Cong extended this work to the more general odd-order differential equation (1.1).

We recall some notations of [4]. Let $a_i, i = 0,1,\dots,2n$, be positive constants. Let D_i be the subset of $\mathbb{R}^{i+1}$ defined by

$$D_i = \{(x_0,x_1,\dots,x_i) \in \mathbb{R}^{i+1} : |x_j| < a_j,\ j = 0,1,\dots,i\}$$

for $i = 0,1,\dots,2n$. Define

$$f_{i+1}(t,x_{i+1},\dots,x_{2n}) = \sup_{(x_0,\dots,x_i)\in D_i} |f(t,x_0,x_1,\dots,x_{2n})|$$

for $i = 0,1,\dots,2n-1$. In [4], under the following assumptions:

(H1) there exist positive constants $b_0, c_i, i = 0,1,\dots,2n$, such that for $|x_i| \ge a_i$ and $(x_{i+1},\dots,x_{2n}) \in \mathbb{R}^{2n-i}$,

$$|f_i/x_i| \le c_i, \qquad i = 1,\dots,2n, \tag{1.3}$$

and for $|x_0| \ge a_0$ and $(x_1,\dots,x_{2n}) \in \mathbb{R}^{2n}$,

$$b_0 \le f/x_0 \le c_0 \quad \text{ or } -c_0 \le f/x_0 \le -b_0; \tag{1.4}$$

(H2) the constants b_0 and c_i satisfy the inequality

$$\left(1 + \frac{c_0}{b_0}\right)\sum_{i=1}^{2n} 2^i c_i < 1, \tag{1.5}$$

Cong obtained the following result on existence of periodic solutions:

Theorem A. *If conditions* (H1) *and* (H2) *above are satisfied, then equation* (1.1) *has a* 2π*-periodic solution.*

Furthermore, replacing (H1) by the Lipschitz-type condition

(H3) there exist positive constants $c_0, c_1,\dots,c_{2n}$ such that

$$|f(t,x_0,x_1,\dots,x_{2n}) - f(t,y_0,y_1,\dots,y_{2n})| \le \sum_{i=0}^{2n} c_i|x_i - y_i|$$

for any $(t,x_0,x_1,\dots,x_{2n}), (t,y_0,y_1,\dots,y_{2n}) \in \mathbb{R}\times\mathbb{R}^{2n+1}$, and there exists a positive constant $b_0 < c_0$ such that $b_0 \le |f_{x_0}| \le c_0$ on $\mathbb{R}\times\mathbb{R}^{2n+1}$,

Cong obtained the following existence theorem:

Theorem B. *Assume that* f *satisfies condition* (H3) *and the constants* b_0 *and* c_i *satisfy condition* (H2). *Then equation* (1.1) *has a* 2π*-periodic solution.*

The purpose of this paper is to improve Theorems A and B above. We shall substantially weaken conditions (H1) and (H2) in the two theorems and show that the periodic solution in Theorem B is unique. We shall replace (H1) by the following growth condition:

(F1) There exist constants $\alpha \neq 0$, $\beta \in (0, |\alpha|)$ and $c_1, \ldots, c_{2n}$, M such that

$$|f(t, x_0, x_1, \ldots, x_{2n}) - \alpha x_0| \leq \beta |x_0| + \sum_{i=1}^{2n} c_i |x_i| + M \tag{1.6}$$

for any $(t, x_0, x_1, \ldots, x_{2n}) \in \mathbb{R} \times \mathbb{R}^{2n+1}$.

Condition (F1) is weaker than (H1). In fact, if (H1) holds, then setting

$$\alpha = \begin{cases} (b_0 + c_0)/2 & \text{if } f \text{ satisfies the first inequality of (1.4)}, \\ -(b_0 + c_0)/2 & \text{if } f \text{ satisfies the second inequality of (1.4)}, \end{cases}$$

$\beta = (c_0 - b_0)/2$ and $M = \sup\{|f(t, x_0, x_1, \ldots, x_{2n})| : t \in [0, 2\pi];\ |x_i| \leq a_i,\ i = 0, 1, \ldots, 2n\} + 1$ (then $0 < \beta < |\alpha|$), from (1.3) and (1.4), we can easily deduce (1.6). Moreover, we shall weaken (H2) to the following simple condition:

(F2) the constants α, β and $c_1, \ldots, c_{2n}$ satisfy the inequality

$$\beta/|\alpha| + c_1 + \cdots + c_{2n} < 1. \tag{1.7}$$

In fact, under conditions (H1) and (H2), defining α and β as above, we have

$$\begin{aligned} \frac{1}{1 - \beta/|\alpha|}(c_1 + \cdots + c_{2n}) &= \frac{1}{2}\left(1 + \frac{c_0}{b_0}\right)(c_1 + \cdots + c_{2n}) \\ &< \left(1 + \frac{c_0}{b_0}\right)(c_1 + \cdots + c_{2n}) \\ &< \left(1 + \frac{c_0}{b_0}\right)\sum_{i=1}^{2n} 2^i c_i < 1. \end{aligned} \tag{1.8}$$

This implies that the inequality (1.7) holds. Therefore condition (F2) is much weaker than (H2). Under condition (H1) or (H3), condition (F2) is equivalent to

(H2)* the constants b_0 and $c_0, c_1, \ldots, c_{2n}$ satisfy the inequality

$$\frac{1}{2}\left(1 + \frac{c_0}{b_0}\right)(c_1 + \cdots + c_{2n}) < 1.$$

The main results of this paper are as follows:

THEOREM 1. *If conditions* (F1) *and* (F2) *hold, then equation* (1.1) *has a 2π-periodic solution.*

Clearly, Theorem 1 is an extension of Theorem A. Likewise, Theorem B is improved by the following result.

THEOREM 2. *If the partial derivative* f_{x_0} *of* $f(t, x_0, x_1, \ldots, x_{2n})$ *with respect to* x_0 *exists and conditions* (H3) *and* (H2)* *hold, then equation* (1.1) *has a unique* 2π*-periodic solution.*

If the partial derivatives $f_{x_0}, f_{x_1}, \ldots, f_{x_{2n}}$ exist, then from Theorem 2 and the differential mean value theorem, we have

COROLLARY 1. *Suppose the partial derivatives* $f_{x_0}, f_{x_1}, \ldots, f_{x_{2n}}$ *exist. If there exist positive constants* $b_0, c_0, c_1, \ldots, c_{2n}$ *such that*

$$b_0 \le |f_{x_0}| \le c_0, \qquad |f_{x_i}| \le c_i \quad \text{for } i = 1, \ldots, 2n,$$

and the constants $b_0, c_0, c_1, \ldots, c_{2n}$ *satisfy condition* (H2)*, *then equation* (1.1) *has a unique* 2π*-periodic solution.*

In Theorem 1, if condition (F1) is modified as

(F1)* there exist constants $\alpha \neq 0$, $\beta \in (0, |\alpha|)$ and $c_1, \ldots, c_{2n}, M$ such that

$$|f(t, x_0, x_1, \ldots, x_{2n}) - \alpha x_0| \le \sqrt{\beta^2 x_0^2 + \sum_{i=1}^{2n} c_i^2 x_i^2 + M}$$

for any $(t, x_0, x_1, \ldots, x_{2n}) \in \mathbb{R} \times \mathbb{R}^{2n+1}$,

then condition (F2) can be further weakened to

(F2)* the constants $c_1, \ldots, c_{2n}$ satisfy the inequality

$$c_1 + \cdots + c_{2n} < 1.$$

THEOREM 3. *If conditions* (F1)* *and* (F2)* *hold, then equation* (1.1) *has a* 2π*-periodic solution.*

Condition (F1)* is slightly stronger than (F1), but condition (F2)* is independent of the constants α and β or b_0 and c_0, and it is much weaker than (H2) and (F2). Similarly, (F1)* is also weaker than (H1). By Theorem 3, we have

COROLLARY 2. *If conditions* (H1) *and* (F2)* *hold, then equation* (1.1) *has a* 2π*-periodic solution.*

Corollary 2 is another extension of Theorem A.

Theorems 1–3 will be proved in the next section by using fixed-point theorems in the Sobolev space $W^{2n,2}(I)$, where $I = [0, 2\pi]$. We shall choose an equivalent norm in $W^{2n,2}(I)$ such that the Leray–Schauder fixed-point theorem can be applied to the periodic problem for equation (1.1).

2. Proof of the main results. Let $I = [0, 2\pi]$ and $H = L^2(I)$ be the usual Hilbert space with the norm $\|u\|_2 = (\int_0^{2\pi} |u(t)|^2\, dt)^{1/2}$. For $m \in \mathbb{N}$, let $W^{m,2}(I)$ be the usual Sobolev space with the norm $\|u\|_{m,2} = \sqrt{\sum_{i=0}^{m} \|u^{(i)}\|_2^2}$; $u \in W^{m,2}(I)$ means that $u \in C^{m-1}(I)$, $u^{(m-1)}(t)$ is absolutely continuous on I and $u^{(m)} \in L^2(I)$.

Taking $\alpha \neq 0$ and $h \in L^2(I)$, we consider the linear periodic boundary value problem (LPBVP)

$$\begin{cases} u^{(2n+1)}(t) - \alpha u(t) = h(t), & 0 \le t \le 2\pi, \\ u^{(i)}(0) = u^{(i)}(2\pi), & i = 0, 1, \ldots, 2n. \end{cases} \tag{2.1}$$

From [8, Lemma 1] or a direct calculation, we know that LPBVP (2.1) has a unique solution $u := Sh \in W^{2n+1,2}(I)$ in the Carathéodory sense. If $h \in C(I)$, the solution is in $C^{2n+1}(I)$ and it is a classical solution. Moreover, the solution operator of LPBVP (2.1), $S : L^2(I) \to W^{2n+1,2}(I)$, is a bounded linear operator.

It is well-known that $\|u\|_{\alpha,2n,2} = \sqrt{\alpha^2 \|u\|_2^2 + \|u^{(2n)}\|_2^2}$ is an equivalent norm in $W^{2n,2}(I)$. For convenience, we use X to denote the Banach space $W^{2n,2}(I)$ endowed with the norm $\|u\|_X = \|u\|_{\alpha,2n,2}$. By the compactness of the Sobolev embedding $W^{2n+1,2}(I) \hookrightarrow C^{2n}(I)$ and the continuity of $C^{2n}(I) \hookrightarrow W^{2n,2}(I)$, we see that S maps H into X and $S : H \to X$ is a completely continuous operator.

LEMMA 1. *Let $\alpha \neq 0$. Then*

(a) *The norm of the solution operator $S : H \to X$ of LPBVP* (2.1) *satisfies $\|S\|_{\mathcal{L}(H,X)} \le 1$.*
(b) *For every $h \in H$, the unique solution $u \in W^{2n+1,2}(I)$ of LPBVP* (2.1) *satisfies the inequality*

$$\|u^{(m)}\|_2 \le \|u^{(2n)}\|_2, \qquad m = 1, \ldots, 2n-1. \tag{2.2}$$

Proof. It is well-known that $\{e^{\mathrm{i}kt} \mid k \in \mathbb{Z}\}$ is a completely orthogonal system in H, where i is the imaginary unit. Hence every $h \in H$ has the Fourier series expansion

$$h(t) = \sum_{k=-\infty}^{\infty} a_k e^{\mathrm{i}kt},$$

where $a_k = (2\pi)^{-1} \int_0^{2\pi} h(t) e^{-\mathrm{i}kt}\, dt$ $(k = 0, \pm 1, \pm 2, \ldots)$.

Let $u = Sh$. Then $u \in W^{2n+1,2}(I)$ is the unique solution of LPBVP (2.1), and u and each $u^{(m)}$ $(m = 1, \ldots, 2n+1)$ can be expressed by the Fourier series expansion. Let

$$u(t) = \sum_{k=-\infty}^{\infty} b_k e^{\mathrm{i}kt}.$$

Then by the formula for Fourier coefficients we have

$$u^{(m)}(t) = \sum_{k=-\infty}^{\infty} (k\mathrm{i})^m b_k e^{\mathrm{i}kt}, \qquad m = 1, \ldots, 2n+1. \tag{2.3}$$

Therefore, h can also be expressed by

$$h(t) = u^{(2n+1)}(t) - \alpha u(t) = \sum_{k=-\infty}^{\infty} ((k\mathrm{i})^{2n+1} - \alpha) b_k e^{\mathrm{i}kt}.$$

By the uniqueness of Fourier series expansion,

$$((k\mathrm{i})^{2n+1} - \alpha) b_k = a_k, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Now by the Parseval equality, we have

$$\begin{aligned}
\|S\,h\|_X^2 = \alpha^2\|u\|_2^2 + \|u^{(2n)}\|_2^2 &= 2\pi \sum_{k=-\infty}^{\infty} (\alpha^2|b_k|^2 + |(k\mathrm{i})^{2n} b_k|^2) \\
&= 2\pi \sum_{k=-\infty}^{\infty} (\alpha^2 + |(k\mathrm{i})^{2n}|^2) \left| \frac{a_k}{(k\mathrm{i})^{2n+1} - \alpha} \right|^2 \\
&= 2\pi \sum_{k=-\infty}^{\infty} \frac{\alpha^2 + k^{4n}}{\alpha^2 + k^{4n+2}} |a_k|^2 \\
&\le 2\pi \sum_{k=-\infty}^{\infty} |a_k|^2 = \|h\|_2^2.
\end{aligned}$$

This means that $\|S\|_{\mathcal{L}(H,X)} \le 1$, so (a) holds.

For every $1 \le m \le 2n$, by (2.3) and the Parseval equality, we have

$$\begin{aligned}
\|u^{(m)}\|_2^2 = 2\pi \sum_{k=-\infty}^{\infty} |(k\mathrm{i})^m b_k|^2 &= 2\pi \sum_{k=-\infty}^{\infty} k^{2m} |b_k|^2 \\
&\le 2\pi \sum_{k=-\infty}^{\infty} k^{4n}|b_k|^2 = 2\pi \sum_{k=-\infty}^{\infty} |(k\mathrm{i})^{2n} b_k|^2 = \|u^{(2n)}\|_2^2.
\end{aligned}$$

This shows that conclusion (b) holds. ■

Proof of Theorem 1. Consider the $(2n+1)$th-order periodic boundary value problem (PBVP)

$$\begin{cases} u^{(2n+1)}(t) = f(t, u(t), u'(t), \ldots, u^{(2n)}(t)), & 0 \le t \le 2\pi, \\ u^{(i)}(0) = u^{(i)}(2\pi), & i = 0, 1, \ldots, 2n. \end{cases} \tag{2.4}$$

If PBVP (2.4) has a solution $u \in C^{2n+1}(I)$, then the 2π-periodic extension of u is a 2π-periodic solution of (1.1). We now prove that PBVP (2.4) has at least one solution in $C^{2n+1}(I)$ under conditions (F1) and (F2).

We define a mapping $F : X \to H$ by

$$F(u)(t) := f(t, u(t), u'(t), \dots, u^{(2n)}(t)) - \alpha\, u(t), \qquad u \in X. \tag{2.5}$$

From (1.6) and the properties of Carathéodory mappings it follows that $F : X \to H$ is continuous and it maps every bounded set in X into a bounded set in H. Hence, the composite mapping $S \circ F : X \to X$ is completely continuous. We use the Leray–Schauder fixed point theorem to show that $S \circ F$ has a fixed point. For this, we consider the homotopic family of operator equations

$$u = \lambda\, (S \circ F)(u), \qquad 0 < \lambda < 1. \tag{2.6}$$

We need to prove that the set of solutions of all equations (2.6) is bounded in X.

Let $u \in X$ be a solution of (2.6) for some $\lambda \in (0, 1)$. Set $h = \lambda\, F(u)$. Then by the definition of S, $u = Sh \in W^{2n+1,2}(I)$ is the unique solution of LPBVP (2.1). By Lemma 1(a), we have

$$\|u\|_X = \|Sh\|_X \le \|S\|_{\mathcal{L}(X,H)} \|h\|_2 \le \|h\|_2 \le \|F(u)\|_2. \tag{2.7}$$

From (2.5), (1.6) and Lemma 1(b), we obtain

$$\begin{aligned} \|F(u)\|_2 &\le \beta\|u\|_2 + c_1\|u^{(1)}\|_2 + c_2\|u^{(2)}\|_2 + \dots + c_{2n}\|u^{(2n)}\|_2 + \sqrt{2\pi}M \\ &\le \beta\|u\|_2 + (c_1 + \dots + c_{2n})\, \|u^{(2n)}\|_2 + \sqrt{2\pi}M \\ &\le (\beta/|\alpha| + c_1 + \dots + c_{2n})\|u\|_X + \sqrt{2\pi}M. \end{aligned}$$

Combining this inequality with (2.7), it follows that

$$\|u\|_X \le \frac{\sqrt{2\pi}M}{1 - (\beta/|\alpha| + c_1 + \dots + c_{2n})} =: C_0.$$

This means that the set of solutions of equations (2.6) is bounded in X. Therefore, by the Leray-Schauder fixed-point theorem [6], $S \circ F$ has a fixed point $u_0 \in X$. Let $h_0 = F(u_0)$. By the definition of S, $u_0 = Sh_0 \in W^{2n+1,2}(I)$ is a solution of LPBVP (2.1) for $h = h_0$. Since $W^{2n+1,2}(I) \hookrightarrow C^{2n}(I)$, from (2.5) it follows that $h_0 \in C(I)$. Hence $u_0 \in C^{2n+1}(I)$ is a classical solution of LPBVP (2.1), and by (2.5), it is also a solution of PBVP (2.4) in $C^{2n+1}(I)$.

This completes the proof of Theorem 1. ■

Proof of Theorem 2. First, we show that (H3) implies (F1). From the inequality $b_0 \le |f_{x_0}| \le c_0$ in (H3) we know that f_{x_0} does not change sign. Set $\alpha = \frac{b_0 + c_0}{2}\,\mathrm{sgn}(f_{x_0})$, $\beta = (c_0 - b_0)/2$ and $M = \max\{|f(t, 0, \dots, 0)| : t \in \mathbb{R}\}$.

For $(t, x_0, x_1, \ldots, x_{2n}) \in \mathbb{R} \times \mathbb{R}^{2n+1}$, by (H3) we have

$$|f(t, x_0, x_1, \ldots, x_{2n}) - f(t, 0, x_1, \ldots, x_{2n}) - \alpha x_0| = |f_{x_0} - \alpha| \cdot |x_0| \le \beta |x_0|,$$
$$|f(t, 0, x_1, \ldots, x_{2n}) - f(t, 0, 0, \ldots, 0)| \le \sum_{i=1}^{2n} c_i |x_i|,$$
$$|f(t, 0, 0, \ldots, 0)| \le M.$$

Summing these inequalities we obtain

$$|f(t, x_0, x_1, \ldots, x_{2n}) - \alpha x_0| \le \beta |x_0| + \sum_{i=1}^{2n} c_i |x_i| + M. \tag{2.8}$$

Hence, (F1) holds. From (H2)* and (1.8) we see that (F2) holds. By Theorem 1, PBVP (2.4) has a solution.

Now, let $u_1, u_2 \in C^{2n+1}(I)$ be two solutions of PBVP (2.4). Then $u_i = S(F(u_i))$, $i = 1, 2$. From (H3) and (2.5), using a similar argument to (2.8), we obtain

$$|F(u_2)(t) - F(u_1)(t)| \le \beta |u_2(t) - u_1(t)| + \sum_{i=1}^{2n} c_i |u_2^{(i)}(t) - u_1^{(i)}(t)| \tag{2.9}$$

for $t \in I$. Since $u_2 - u_1$ is the solution of LPBVP (2.1) for $h = F(u_2) - F(u_1)$, by (2.9) and Lemma 1(b) we have

$$\begin{aligned} \|F(u_2) - F(u_1)\|_2 &\le \beta \|u_2 - u_1\|_2 + \sum_{i=1}^{2n} c_i \|u_2^{(i)} - u_1^{(i)}\|_2 \\ &\le \beta \|u_2 - u_1\|_2 + \Big(\sum_{i=1}^{2n} c_i\Big) \|u_2^{(2n)} - u_1^{(2n)}\|_2 \\ &\le (\beta/|\alpha| + c_1 + \cdots + c_{2n}) \|u_2 - u_1\|_X. \end{aligned}$$

From this and Lemma 1(a), it follows that

$$\begin{aligned} \|u_2 - u_1\|_X &\le \|S(F(u_2) - F(u_1))\|_X \\ &\le \|S\|_{\mathcal{L}(X,H)} \|F(u_2) - F(u_1)\|_2 \\ &\le (\beta/|\alpha| + c_1 + \cdots + c_{2n}) \|u_2 - u_1\|_X. \end{aligned} \tag{2.10}$$

Since $\beta/|\alpha| + c_1 + \cdots + c_{2n} < 1$, from (2.10) we see that $\|u_2 - u_1\|_X = 0$, that is, $u_2 = u_1$. Therefore, PBVP (2.4) has only one solution. Equivalently, equation (1.1) has a unique 2π-periodic solution.

The proof of Theorem 2 is complete. ∎

Proof of Theorem 3. Since (F1)* implies (F1), by the argument in the proof of Theorem 1 the mapping $F : X \to H$ defined by (2.5) is continuous and it maps every bounded set in X into a bounded set in H. Hence the mapping $S \circ F : X \to X$ is completely continuous.

Consider the family of operator equations (2.6). Let $u \in X$ be a solution of (2.6) for some $\lambda \in (0,1)$. Set $h = \lambda F(u)$. Then (2.7) is valid. Let $r = \max\{\beta/|\alpha|, c_1 + \cdots + c_{2n}\}$. By (F2)*, $r \in (0,1)$. Since $u = S\,h \in W^{2n+1,2}(I)$ is the unique solution of LPBVP (2.1), using (2.5), (F1)* and Lemma 1(b), we have

$$\begin{aligned}\|F(u)\|_2 &\le \sqrt{\beta^2\|u\|_2^2 + c_1^2\|u^{(1)}\|_2^2 + \cdots + c_{2n}^2\|u^{(2n)}\|_2^2} + \sqrt{2\pi}M \\ &\le \sqrt{\beta^2\|u\|_2^2 + (c_1^2 + \cdots + c_{2n}^2)\|u^{(2n)}\|_2^2} + \sqrt{2\pi}M \\ &\le \sqrt{\beta^2\|u\|_2^2 + (c_1 + \cdots + c_{2n})^2\,\|u^{(2n)}\|_2^2} + \sqrt{2\pi}M \\ &\le \sqrt{r^2(\alpha^2\|u\|_2^2 + \|u^{(2n)}\|_2^2)} + \sqrt{2\pi}M \\ &= r\|u\|_X + \sqrt{2\pi}M.\end{aligned}$$

From this inequality and (2.7), we obtain

$$\|u\|_X \le \frac{\sqrt{2\pi}M}{1-r} := C_1.$$

That is, the set of solutions for equations (2.6) is bounded in X. Therefore, by the Leray–Schauder fixed-point theorem, $S \circ F$ has a fixed point, whose 2π-periodic extension is a 2π-periodic solution of (1.1).

The proof of Theorem 3 is complete. ■

Acknowledgements. This research was partly supported by NNSF of China (10871160), the NSF of Gansu Province (0710RJZA103) and Project of NWNU-KJCXGC-3-47.

References

[1] A. Cabada, *The method of lower and upper solutions for nth-order periodic boundary value problems*, J. Appl. Math. Stoch. Anal. 7 (1994), 33–47.

[2] A. Cabada and S. Lois, *Maximum principles for fourth and sixth order periodic boundary value problems*, Nonlinear Anal. 29 (1997), 1161–1171.

[3] F. Z. Cong, *Periodic solutions for 2kth-order ordinary differential equations with nonresonance*, ibid. 32 (1998), 787–793.

[4] —, *Existence of periodic solutions of* $(2n+1)$*th-order ordinary differential equations*, Appl. Math. Lett. 17 (2004), 727–732.

[5] F. Z. Cong, Q. D. Huang and S. Y. Shi, *Existence and uniqueness of periodic solutions for* $(2n+1)$*th-order differential equations*, J. Math. Anal. Appl. 241 (2000), 1–9.

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985.

[7] Y. X. Li, *Positive solutions of higher order periodic boundary value problems*, Comput. Math. Appl. 48 (2004), 153–161.

[8] —, *Existence and uniqueness for higher order periodic boundary value problems under spectral separation conditions*, J. Math. Anal. Appl. 322 (2006), 530–539.

[9] Y. X. Li, *On the existence and uniqueness for higher order periodic boundary value problems*, Nonlinear Anal. 70 (2009), 711–718.
[10] V. Seda, J. J. Nieto and M. Gera, *Periodic boundary value problems for nonlinear higher order ordinary differential equations*, Appl. Math. Comput. 48 (1992), 71–82.

Yongxiang Li, He Yang
Department of Mathematics
Northwest Normal University
Lanzhou 730070, People's Republic of China
E-mail: liyxnwnu@163.com
heyang256@163.com

Received 8.6.2009
and in final form 30.5.2010 (2027)