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On meromorphic functions with maximal defect sum

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Abstract. The purpose of this article is twofold. The first is to give necessary conditions for the maximality of the defect sum. The second is to show that the class of meromorphic functions with maximal defect sum is very thin in the sense that deformations of meromorphic functions with maximal defect sum by small meromorphic functions are not meromorphic functions with maximal defect sum.

1. Introduction and main results. We set

$$|z| = \Big(\sum_{j=1}^{n} |z_j|^2\Big)^{1/2}, \quad \forall z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$
$$S_n(r) = \{z \in \mathbb{C}^n : |z| = r\}, \quad \overline{B}_n(r) = \{z \in \mathbb{C}^n : |z| \le r\},$$
$$d = \partial + \overline{\partial}, \quad d^c = \frac{1}{4\pi}(\partial - \overline{\partial}),$$
$$\omega_n(z) = dd^c \log |z|^2, \quad \sigma_n(z) = d^c \log |z|^2 \wedge \omega_n^{n-1}(z),$$
$$\nu_n(z) = dd^c|z|^2.$$

Let $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function. For each $a \in \mathbb{P}^1(\mathbb{C})$ with $f^{-1}(a) \ne \mathbb{C}^n$,

$$\begin{cases} Z_f^a \text{ is the } a\text{-divisor of } f, \\ Z_f^a(r) = \overline{B}_n(r) \cap Z_f^a. \end{cases}$$

Define

$$n_f(r, a) = r^{2-2n} \int_{Z_f^a(r)} \nu_n^{n-1}(z).$$

We define the *counting function* of f by

$$N_f(r, a) = \int_1^r \frac{n_f(t, a)}{t}\, dt.$$

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The *proximity function* of f is defined by

$$m_f(r,a) = \begin{cases} \int\limits_{S_n(r)} \log^+ \dfrac{1}{|f(z)-a|}\sigma_n(z), & a \neq \infty, \\ \int\limits_{S_n(r)} \log^+ |f(z)|\sigma_n(z), & a = \infty. \end{cases}$$

The *characteristic function* of f is defined by

$$T_f(r) = m_f(r,\infty) + N_f(r,\infty).$$

Then the first main theorem in value distribution theory states that

$$T_f(r) = m_f(r,a) + N_f(r,a) + O(1).$$

We call the quantity

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m_f(r,a)}{T_f(r)} = 1 - \limsup_{r\to\infty} \frac{N_f(r,a)}{T_f(r)}$$

the *defect* (or deficiency) of a with respect to f. Then $0 \leq \delta(a,f) \leq 1$. The quantity

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

is said to be the *order* of f, and the quantity

$$\gamma_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

is the *lower order* of f.

For each $z \in \mathbb{C}^n$, we define

$$D_f(z) = \sum_{j=1}^{n} z_j f_{z_j}(z),$$

where f_{z_j} is the partial differential of f with respect to z_j.

The classical Nevanlinna theorem on the defect relation states that if $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ is a meromorphic function, then $\sum_{a\in\mathbb{P}^1(\mathbb{C})} \delta(a,f) \leq 2$.

There is a natural question: What can we say about the class of meromorphic functions f such that $\sum_{a\in\mathbb{P}^1(\mathbb{C})} \delta(a,f) = 2$? Much attention has been given to this problem and several theorems on meromorphic mappings with maximal defect sum have been obtained by various authors [JY], [TD], [T1], [T2], [T3] (see the references therein for related subjects).

The purpose of this article is twofold. The first is to give necessary conditions for the maximality of the defect sum. The second is to show that the class of meromorphic functions with maximal defect sum is very thin in the sense that deformations of meromorphic functions with maximal defect sum by small meromorphic functions are not meromorphic functions with maximal defect sum. Namely, we prove the following

THEOREM 1.1. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function of finite order. For each $m \geq 1$ and $z \in \mathbb{C}$, define $g_m(z) = f(z^m)$ and $h_m(z) = f^m(z)$. Suppose that one of the following conditions is satisfied:*

(i) *There exists $m_0 \geq 2$ such that $\sum_{a\in\overline{\mathbb{C}}} \delta(a, g_{m_0}) = 2$.*
(ii) *There exists a sequence $\{m_i\}_{i=1}^{\infty} \subset \mathbb{Z}^+$ such that $\sum_{a\in\overline{\mathbb{C}}} \delta(a, h_{m_i}) = 2$ for all $i \geq 1$.*

Then $\lambda := \rho_f \in \mathbb{Z}^+$ and λ equals the lower order of f.

THEOREM 1.2. *Let $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function of finite order satisfying*

$$\lambda := \rho_f \notin \mathbb{Z} \quad \text{and} \quad \sum_{a\in\overline{\mathbb{C}}} \delta(a, f) = 2.$$

Denote by $\mathcal{A}$ the set of all nonconstant meromorphic functions $h : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ such that $T_h(r) = o(T_f(r))$ and $T_{D_h}(r) = o(T_{D_f}(r))$. Then, for each $h \in \mathcal{A}$, we have

$$\sum_{a\in\overline{\mathbb{C}}} \delta(a, f + h) \leq 2 - 2k(\lambda) < 2,$$

where $k(\lambda)$ is a positive constant which depends only on λ.

In [Ne1, p. 83] (see also [EF, p. 299]), R. Nevanlinna gave examples of meromorphic functions f on $\mathbb{C}$ of finite order such that $\lambda := \rho_f \notin \mathbb{Z}$ and $\sum_a \delta(a, f) = 2$.

2. Lemmas

LEMMA 2.1 ([Y, Lemma 6]). *Let $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function. Then, for each $1 \leq j \leq n$, we have*

$$m_{f_{z_j}/f}(r, \infty) = \int\limits_{S_n(r)} \log^+ \left|\frac{f_{z_j}}{f}(z)\right| \sigma_n(z) = O(\log rT_f(r))$$

for all r outside a finite Lebesgue measure set. Moreover, if $\rho_f < \infty$, then $m_{f_{z_j}/f} = O(\log r)$.

LEMMA 2.2. *Let $f, g : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be nonconstant meromorphic functions of finite order. Assume that $\rho_f = \lambda$, $\rho_g = \lambda'$ and $\lambda > \lambda'$. Then*

(i) $\rho_{f+g} = \lambda$.
(ii) $\rho_{f\cdot g} = \lambda$.

Proof. (i) Fix $\varepsilon > 0$. Since $\rho_f = \lambda$, we have $\log T_f(r)/\log r < \lambda + \varepsilon$ for r large enough. Hence $T_f(r) < r^{\lambda+\varepsilon}$ for r large enough. Similarly, $T_g(r) < r^{\lambda'+\varepsilon}$ for r large enough. This yields

$$T_{f+g}(r) \leq T_f(r) + T_g(r) + O(1) < r^{\lambda+\varepsilon} + r^{\lambda'+\varepsilon} + O(1).$$

This implies that $\log T_{f+g}(r)/\log r < \lambda+2\varepsilon$ for r large enough. Hence $\rho_{f+g} \leq \lambda + 2\varepsilon$ for each $\varepsilon > 0$, i.e.,

$$\rho_{f+g} \leq \lambda. \tag{2.1}$$

Take $0 < \varepsilon < \frac{1}{2}(\lambda - \lambda')$. Since $\limsup_{r\to\infty} \log T_f(r)/\log r = \lambda$, there exists a sequence $\{r_n\}$ such that $\lim_{n\to\infty} \log T_f(r_n)/\log r_n = \lambda$. Hence there exists n_0 such that $\log T_f(r_n)/\log r_n > \lambda - \varepsilon$ for all $n > n_0$, and so $T_f(r_n) > r_n^{\lambda-\varepsilon}$ for all $n > n_0$. On the other hand, we have

$$T_f(r) - T_g(r) + O(1) < T_{f+g}(r).$$

Hence $T_f(r_n) - T_g(r_n) < T_{f+g}(r) + O(1)$, i.e. $r_n^{\lambda-\varepsilon} - r_n^{\lambda'+\varepsilon} < T_{f+g}(r) + O(1)$. This yields $\log T_{f+g}(r_n)/\log r_n \geq \lambda - \varepsilon$ for all $n > n_0$. We get

$$\limsup_{n\to\infty} \frac{\log T_{f+g}(r_n)}{\log r_n} \geq \lambda - \varepsilon.$$

Hence $\rho_{f+g} \geq \lambda - \varepsilon$ for all $\varepsilon > 0$, i.e,

$$\rho_{f+g} \geq \lambda. \tag{2.2}$$

Combining (2.1) with (2.2) proves the assertion.

(ii) By the same argument, we also get $\rho_{f\cdot g} = \lambda$. ∎

Lemma 2.3. *Let $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function of finite order. Then $T_{D_f}(r) \leq 2T_f(r) + O(\log rT_f(r))$, and hence $\rho_{D_f} \leq \rho_f$.*

Proof. We show that

$$m_{D_f}(r,\infty) \leq m_f(r,\infty) + O(\log rT_f(r)). \tag{2.3}$$

Indeed, we have

$$m_{D_f} \leq m_{D_f/f}(r,\infty) + m_f(r,\infty).$$

On the other hand,

$$\frac{D_f}{f} = \frac{\sum z_j f_{z_j}}{f} = \sum z_j \cdot \frac{f_{z_j}}{f}.$$

Hence

$$\begin{aligned} m_{D_f/f}(r,\infty) &\leq \sum_{j=1}^{n} (m_{z_j}(r,\infty) + m_{f_{z_j}/f}(r,\infty)) + O(1) \\ &\leq O(\log rT_f(r)) \quad \text{(by Lemma 2.1).} \end{aligned}$$

We now show that

$$N_{D_f}(r,\infty) \leq 2N_f(r,\infty). \tag{2.4}$$

Indeed, since $f = g/h$ (g, h are holomorphic on $\mathbb{C}^n$),

$$D_f = \frac{hD_g - gD_h}{h^2}.$$

This yields

$$N_{D_f}(r,\infty) \le N_{h^2}(r,0) = 2N_h(r,0) \le 2N_f(r,\infty).$$

From (2.3) and (2.4) we get

$$\begin{aligned} T_{D_f}(r) &= m_{D_f}(r,\infty) + N_{D_f}(r,\infty) \\ &\le m_f(r,\infty) + 2N_f(r,\infty) + O(\log rT_f(r)) \le 2T_f(r) + O(\log rT_f(r)). \ \blacksquare \end{aligned}$$

LEMMA 2.4. *Let $f, g : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be nonconstant meromorphic functions of finite order. Then one of the following two assertions holds:*

(i) $\rho_{D_f} = \rho_f$.
(ii) $\rho_{D_{1/f}} = \rho_{1/f}$.

Proof. By Lemma 2.3, we have $\rho_{D_f} \le \rho_f$. If $\rho_{D_f} = \rho_f$, then the assertion is proved.

Assume that $\rho_{D_f} < \rho_f$. Put $f_1 = 1/f$. Then $D_{f_1} = -D_f/f^2$. On the other hand, we have

$$\rho_{f^2} = \rho_f \quad \text{and hence} \quad \rho_{1/f^2} = \rho_f$$

and

$$\rho_{-D_f} = \rho_{D_f} < \rho_f.$$

By Lemma 2.2, we have $\rho_{-D_f/f^2} = \rho_f = \rho_{1/f}$. Hence $\rho_{D_{f_1}} = \rho_{f_1}$. ■

LEMMA 2.5. *The following mappings do not change the defect sum:*

$$\alpha : f \mapsto 1/f \quad \textit{and} \quad \beta_a : f \mapsto f + a, \quad \forall a \in \mathbb{C}.$$

LEMMA 2.6 ([H]). *Let $a_1, \ldots, a_q$ be q distinct points in $\mathbb{C}$. Define*

$$F(z) = \sum_{j=1}^{q} \frac{1}{z - a_j} \quad \textit{and} \quad \delta = \frac{1}{3} \min_{j<k} |a_j - a_k|.$$

Then

$$\log^+ |F(z)| \ge \sum_{j=1}^{q} \log^+ \frac{1}{|z - a_j|} - q \log^+ \frac{3q}{\delta} - \log 3.$$

LEMMA 2.7 ([JY]). *Let $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function and $a_1, \ldots, a_q$ be distinct points in $\mathbb{C}$. Then $\sum_{j=1}^{q} m_f(r, a_j) \le m_{D_f}(r, 0)$.*

LEMMA 2.8. *Let $f : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function such that $\delta(\infty, f) = 0$. Then*

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, f) = \sum_{a \in \mathbb{C}} \delta(a, f) \le 2\delta(0, D_f).$$

Proof. By Lemma 2.7, for each $\{a_j\}_{j=1}^q \subset \mathbb{C}$, we have

$$\sum_{j=1}^{q} m_f(r, a_j) \le m_{D_f}(r, 0).$$

By Lemma 2.3, we have $T_{D_f}(r) \le 2T_f(r) + O(\log rT_f(r))$. This yields

$$\sum_{j=1}^{q} \frac{m_f(r, a_j)}{T_f(r) + O(\log rT_f(r))} \le 2 \cdot \frac{m_{D_f}(r, 0)}{T_{D_f}(r)}.$$

Hence $\sum_{j=1}^q \delta(a_j, f) \le 2\delta(0, D_f)$, i.e. $\sum_{a\in\mathbb{C}} \delta(a_j, f) \le 2\delta(0, D_f)$. Thus,

$$\sum_{a\in\overline{\mathbb{C}}} \delta(a_j, f) \le 2\delta(0, D_f) \quad (\text{as } \delta(\infty, f) = 0). \ \blacksquare$$

By the same argument as in Lemma 2.2, we have the following

LEMMA 2.9. *Let $f, g : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be nonconstant meromorphic functions of finite order satisfying $\rho_f = \lambda$ and $T_g(r) = o(T_f(r))$. Then*

(i) $\rho_{f+g} = \lambda$.
(ii) $\rho_{f.g} = \lambda$.

LEMMA 2.10. *Let $f, h : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be nonconstant meromorphic functions satisfying $\delta(\infty, f) = 0$ and $T_h(r) = o(T_f(r))$. Put $g = f + h$. Then $\delta(\infty, g) = 0$.*

Proof. Since $T_h(r) = o(T_f(r))$, it follows that

$$m_g(r, \infty) = m_f(r, \infty) + o(T_f(r)),$$
$$T_g(r) = T_f(r) + o(T_f(r)).$$

Hence

$$\begin{aligned}\delta(\infty, g) &= \liminf_{r\to\infty} \frac{m_g(r, \infty)}{T_g(r)} = \liminf_{r\to\infty} \frac{m_f(r, \infty) + o(T_f(r))}{T_f(r) + o(T_f(r))} \\ &= \liminf_{r\to\infty} \frac{m_f(r, \infty)}{T_f(r)} = \delta(\infty, f) = 0. \ \blacksquare\end{aligned}$$

LEMMA 2.11 ([No]). *Let $g : \mathbb{C}^n \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function such that $\rho_g = \lambda < \infty$. Then*

(i) *For each $a_1, a_2 \in \mathbb{P}^1(\mathbb{C})$, we have*

$$\limsup_{r\to\infty} \frac{N_g(r, a_1) + N_g(r, a_2)}{T_g(r)} \ge k(\lambda) := \frac{2\Gamma^4(3/4)|\sin \lambda\pi|}{\pi^2\lambda + \Gamma^4(3/4)|\sin \lambda\pi|}.$$

(ii) *If $a_1, a_2 \in \mathbb{P}^1(\mathbb{C})$ are such that $\delta(a_1, g) = \delta(a_2, g) = 1$, then $\lambda \in \mathbb{Z}^+$ and λ equals the lower order of g.*

LEMMA 2.12. *Let $f:\mathbb{C}^n\to\mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function of finite order. Define $g=f^m$, where $m\in\mathbb{Z}^+$. Then $T_g(r)=mT_f(r)$, and hence $\rho_g=\rho_f$.*

Proof. It is easy to see that

$$\begin{aligned} N_g(r,\infty) &= N_{f^m}(r,\infty)=mN_f(r,\infty),\\ m_g(r,\infty) &= \int\limits_{S_n(r)} \log^+|f^m(z)|\sigma_n(z)\\ &= m\int\limits_{S_n(r)} \log^+|f(z)|\sigma_n(z)=m\cdot m_f(r,\infty). \end{aligned}$$

Hence $T_g(r)=N_g(r,\infty)+m_g(r,\infty)=mT_f(r)$. ∎

LEMMA 2.13. *Let $f:\mathbb{C}^n\to\mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function. Define $g=f^m$, where $m\in\mathbb{Z}^+$. Then*

$$T_{D_g}(r)\le \frac{m+1}{m}T_g(r)+O(\log rT_f(r)).$$

Proof. By the same argument as in Lemma 2.3, we get

$$m_{D_g}(r,\infty)\le m_g(r,\infty)+O(\log rT_g(r)).$$

We show that

$$N_{D_g}(r,\infty)\le \frac{m+1}{m}N_g(r,\infty). \tag{2.5}$$

Indeed, assume that $f=f_0/f_1$. Then $g=f_0^m/f_1^m$ and

$$D_g=m\cdot\frac{f_0^{m-1}(f_1D_{f_0}-f_0D_{f_1})}{f_1^{m+1}}.$$

Hence, every pole of D_g is a zero of f_1 and also a pole of g. This implies

$$\frac{\text{the multiplicity of pole of } D_g}{\text{the multiplicity of pole of } g}\le\frac{m+1}{m}.$$

Thus, we have (2.5). ∎

LEMMA 2.14. *Let $f:\mathbb{C}^n\to\mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function of finite order. Then there exists a meromorphic function $f_1:\mathbb{C}^n\to\mathbb{P}^1(\mathbb{C})$ of finite order such that*

$$\sum_{a\in\overline{\mathbb{C}}}\delta(a,f_1)=\sum_{a\in\overline{\mathbb{C}}}\delta(a,f)\quad\text{and}\quad\begin{cases}\rho_{f_1}=\rho_{D_{f_1}},\\ \delta(\infty,f_1)=0.\end{cases}$$

Proof. Consider two cases.

CASE 1: $\delta(\infty,f)=0$. If $\rho_f=\rho_{D_f}$, then the assertion is proved. If $\rho_f\ne\rho_{D_f}$, then we choose $a\in\mathbb{C}$ such that $\delta(a,f)=0$. Hence $\rho_{f-a}\ne$

$\rho_{D(f-a)}$. By Lemma 2.4, we have $\rho_{\frac{1}{f-a}} = \rho_{D_{\frac{1}{f-a}}}$. Put $f_1 = \frac{1}{f-a}$. Then $\rho_{f_1} = \rho_{D_{f_1}}$ and $\delta(\infty, f_1) = \delta(a, f) = 0$.

CASE 2: $\delta(\infty, f) \neq 0$. Choose $a \in \mathbb{C}$ such that $\delta(a, f) = 0$. Replacing f by $\frac{1}{f-a}$, we return to Case 1. Since the transformations in the proof do not change the defect sum, Lemma 2.14 is proved. ∎

3. Proofs of theorems

3.1. Proof of Theorem 1.1. (i) It suffices to prove the case $m_0 = 2$. In fact, we have

$$m_g(r,a) = \begin{cases} \int\limits_{|z|=r} \log^+ |g(z)|\, d^c \log|z|^2 & \text{if } a = \infty \\ \int\limits_{|z|=r} \log^+ \dfrac{1}{|g(z)-a|}\, d^c \log|z|^2 & \text{if } a \neq \infty \end{cases}$$

$$= \begin{cases} \dfrac{1}{2} \int\limits_{|z|=r} \log^+ |f(z^2)|\, d^c \log|z^2|^2 & \text{if } a = \infty \\ \dfrac{1}{2} \int\limits_{|z|=r} \log^+ \dfrac{1}{|f(z^2)-a|}\, d^c \log|z^2|^2 & \text{if } a \neq \infty \end{cases}$$

$$= \frac{1}{2} m_f(r^2, a).$$

On the other hand, since $n_g(r,a) = 2n_f(r^2, a)$, we get

$$N_g(r,a) = \int\limits_1^r \frac{n_g(t,a)}{t}\, dt = \int\limits_1^r \frac{n_f(t^2,a)}{t^2}\, dt^2 = \int\limits_1^{r^2} \frac{n_f(t,a)}{t}\, dt = N_f(r^2, a).$$

Hence

$$\delta(a,g) = \liminf_{r\to\infty} \frac{m_g(r,a)}{m_g(r,a) + N_g(r,a)} = \liminf_{r\to\infty} \frac{1}{1 + \frac{N_g(r,a)}{m_g(r,a)}}$$

$$= \liminf_{r\to\infty} \frac{1}{1 + 2\frac{N_f(r^2,a)}{m_f(r^2,a)}}$$

$$= \frac{1}{1 + 2\big(\frac{1}{\delta(a,f)} - 1\big)} = \frac{\delta(a,f)}{2 - \delta(a,f)} \leq \delta(a,f).$$

Equality holds if and only if $\delta(a,f) = 0$ or $\delta(a,f) = 1$. Hence $\sum_{a\in\overline{\mathbb{C}}} \delta(a,g) \leq \sum_{a\in\overline{\mathbb{C}}} \delta(a,f) \leq 2$. Equality holds if and only if

$$\begin{cases} \sum_{a\in\overline{\mathbb{C}}} \delta(a,f) = 2, \\ \forall a:\ \delta(a,f) = 0 \text{ or } \delta(a,f) = 1. \end{cases}$$

By Lemma 2.11, the assertion is proved.

(ii) Suppose the contrary. By Lemma 2.7, for each $\{a_j\}_{j=1}^q \subset \mathbb{C}$, we get

$$\sum_{j=1}^{q} m_{h_m}(r, a_j) \le m_{D_{h_m}}(r, 0).$$

By using Lemma 2.13, we get $T_{D_{h_m}}(r) \le \frac{m+1}{m} T_{h_m}(r)$. Hence

$$\sum_{j=1}^{q} \frac{m_{h_m}(r, a_i)}{T_{h_m}(r)} \le \frac{m+1}{m} \cdot \frac{m_{D_{h_m}}(r, 0)}{T_{D_{h_m}}(r)}$$

$$\Rightarrow \sum_{j=1}^{q} \delta(a_j, h_m) \le \frac{m+1}{m} \cdot \delta(0, D_{h_m}) \le \frac{m+1}{m}$$

$$\Rightarrow \sum_{a \in \mathbb{C}} \delta(a, h_m) \le \frac{m+1}{m}$$

$$\Rightarrow \sum_{a \in \overline{\mathbb{C}}} \delta(a, h_m) \le \frac{m+1}{m} + \delta(\infty, h_m).$$

Thus, if $\delta(\infty, h_m) < 1$, then there is m_1 large enough such that

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, h_m) \le \frac{m+1}{m} + \delta(\infty, h_m) < 2, \qquad \forall m \ge m_1.$$

This is a contradiction. Hence $\delta(\infty, h_m) = 1$. This implies that $\delta(\infty, f) = \delta(\infty, h_m) = 1$.

By replacing f by $1/f$ and by repeating the above argument, we have $\delta(\infty, 1/f) = 1$, i.e. $\delta(0, f) = 1$, and hence $\delta(\infty, f) = \delta(0, f) = 1$. This contradicts Lemma 2.11.

3.2. Proof of Theorem 1.2. By Lemma 2.14, we only need to consider meromorphic functions $f : \mathbb{C}^n \to \mathbb{P}^1$ satisfying $\delta(\infty, f) = 0$ and $\rho_f = \rho_{D_f}$.

Since $\delta(\infty, f) = 0$ and by Lemma 2.8, we have

$$2 = \sum_{a \in \overline{\mathbb{C}}} \delta(a, f) \le 2\delta(0, D_f) = 2 - 2 \limsup \frac{N_{D_f}(r, 0)}{T_{D_f}(r)}.$$

Hence $\limsup N_{D_f}(r, 0)/T_{D_f}(r) = 0$.

Suppose that $h \in \mathcal{A}$. Put $g = f + h$. Then $D_g = D_f + D_h$. By Lemmas 2.9 and 2.10, we have $\rho_g = \rho_{D_g} = \lambda$ and $\delta(\infty, g) = 0$. Again by Lemma 2.8,

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a, g) \le 2\delta(0, D_g) = 2 - 2 \limsup \frac{N_{D_g}(r, 0)}{T_{D_g}(r, 0)}.$$

We have

$$\begin{aligned} N_{D_g}(r,0) &= N_{D_f+D_h}(r,0) \geq N_{D_f/D_h}(r,-1) - N_{D_h}(r,\infty) \\ &\geq N_{D_f/D_h}(r,-1) - O(T_{D_f}(r)). \end{aligned}$$

On the other hand, since $T_{D_h}(r) = o\big(T_{D_f}(r)\big)$, we get

$$\begin{aligned} T_{D_g}(r) &= T_{D_f}(r) + o(T_{D_f}(r)), \\ T_{D_f/D_h}(r) &= T_{D_f}(r) + o(T_{D_f}(r)). \end{aligned}$$

This yields $T_{D_g}(r) = T_{D_f/D_h}(r) + o(T_{D_f}(r))$. Put $f_1 = D_f/D_h$. Then

$$(*) \qquad \begin{aligned} \limsup \frac{N_{D_g}(r,0)}{T_{D_g}(r,0)} &\geq \limsup \frac{N_{f_1}(r,-1) + o(T_{D_f}(r))}{T_{f_1}(r) + o(T_{D_f}(r))} \\ &= \limsup \frac{N_{f_1}(r,-1)}{T_{f_1}(r)}. \end{aligned}$$

We see that

$$\begin{aligned} N_{f_1}(r,0) &\leq N_{D_f}(r,0) + N_{D_h}(r,\infty) \leq N_{D_f}(r,0) + o(T_{D_f}(r)), \\ T_{f_1}(r) &= T_{D_f}(r) + o(T_{D_f}(r)). \end{aligned}$$

Hence

$$\begin{aligned} \limsup \frac{N_{f_1}(r,0)}{T_{f_1}(r)} &\leq \limsup \frac{N_{D_f}(r,0) + o(T_{D_f}(r))}{T_{D_f}(r) + o(T_{D_f}(r))} \\ &= \limsup \frac{N_{D_f}(r,0)}{T_{D_f}(r)} = 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} \limsup \frac{N_{f_1}(r,-1)}{T_{f_1}(r)} &= \limsup \frac{N_{f_1}(r,-1)}{T_{f_1}(r)} + \limsup \frac{N_{f_1}(r,0)}{T_{f_1}(r)} \\ &\geq \limsup \frac{N_{f_1}(r,-1) + N_{f_1}(r,0)}{T_{f_1}(r)} \geq k(\lambda) \end{aligned}$$

(by Lemma 2.9 we have $\rho_{f_1} = \lambda$). Combining this with $(*)$ we obtain $\limsup N_{D_g}(r,0)/T_{D_g}(r) \geq k(\lambda)$. Since $\lambda \notin \mathbb{Z}$, this implies that

$$k(\lambda) = \frac{2\Gamma^4(3/4)|\sin \lambda\pi|}{\pi^2\lambda + \Gamma^4(3/4)|\sin \lambda\pi|} > 0.$$

Hence

$$\sum_{a \in \overline{\mathbb{C}}} \delta(a,g) \leq 2 - 2\limsup \frac{N_{D_g}(r,0)}{T_{D_g}(r)} \leq 2 - 2k(\lambda) < 2.$$

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